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Properties of the quilt lattice

Hans Höngesberg*[1], Matjaž Konvalinka†[1,2] and Martin Rubey‡[3]

[1] *Fakulteta za matematiko in fiziko, Univerza v Ljubljani, Slovenia*
[2] *Inštitut za matematiko, fiziko in mehaniko, Ljubljana, Slovenia*
[3] *Fakultät für Mathematik und Geoinformation, TU Wien, Vienna, Austria*

Abstract. Quilts were recently introduced as a generalization of the concept of alternating sign matrices. While their enumeration is hard, it turns out that they form a lattice with beautiful properties, in many cases generalizing and expanding on the properties of the lattices of alternating sign matrices. In this extended abstract, we present several such results.

Keywords: quilts, alternating sign matrices, join-irreducible elements, lattices

1 Introduction

In [2], Billey and the second author introduced *quilts of alternating sign matrices* as a broad generalization of alternating sign matrices and their corresponding corner sum matrices. An *alternating sign matrix* (ASM) is a square matrix with entries in $\{-1, 0, 1\}$ such that in each row and in each column the non-zero entries alternate in sign and sum to 1.

ASMs are in bijective correspondence with corner sum matrices. A *corner sum matrix* (CSM), first introduced in [19], is an $(n+1) \times (n+1)$ matrix with integer entries such that the first column and the bottom row only contain zeros, the first row (from left to right) and the last column (from bottom to top) read as $(0, 1, \ldots, n)$, and adjacent entries differ by at most one. Figure 1 shows an ASM with its corresponding CSM.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 1: An ASM of size 4×4 and its corresponding CSM of size 5×5.

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*hans.hoengesberg@fmf.uni-lj.si. H. Höngesberg was partially supported by the Austrian Science Fund (FWF) grant 10.55776/J4810.

†matjaz.konvalinka@fmf.uni-lj.si. M. Konvalinka was partially supported by ERC AdG KARST and by projects and programs J1-2452, N1-0218, P1-0294, P1-0297 of the Slovenian Research Agency.

‡martin.rubey@tuwien.ac.at

The bijection works as follows: Given an $n \times n$ ASM, we append a row of 0s at the bottom and a column of 0s on the left to obtain an $(n+1) \times (n+1)$ matrix. We then map every entry to the sum of entries weakly left of it and weakly below it. This construction ensures that the rows are weakly increasing from left to right and the columns from bottom to top.

Quilts generalize CSMs from a poset perspective.

Definition 1. For two bounded graded posets P and Q, a *quilt of type* (P, Q) is a map $f\colon P \times Q \longrightarrow \mathbb{N}$ satisfying:

- $f(x, y) = 0$ whenever $x = \hat{0}_P$ or $y = \hat{0}_Q$,
- $f(\hat{1}_P, \hat{1}_Q) = \min\{\operatorname{rank} P,\ \operatorname{rank} Q\}$, and
- if $(x, y) \lessdot (x', y')$ in $P \times Q$, then $f(x', y') \in \{f(x, y), f(x, y) + 1\}$.

We refer to the last condition as the *Boolean growth rule*. The set of all quilts of type (P, Q) will be denoted by $\operatorname{Quilts}(P, Q)$. We order the set of quilts by setting $f \leqslant g$ if $f(x, y) \leqslant g(x, y)$ for all $x \in P$, $y \in Q$. If one of the posets is a chain poset C_n of any rank n, we refer to the quilts as *chain quilts*.

We can identify chain quilts of type (C_n, C_n) with $(n+1) \times (n+1)$ corner sum matrices, and the order on $\operatorname{Quilts}(C_n, C_n)$ coincides with the well-known distributive lattice on $n \times n$ ASMs. More generally, quilts of type (C_k, C_n) lead to a natural notion of *rectangular ASMs*.

We call $\operatorname{Quilts}(P, Q)$ the *quilt lattice*. This is justified by the following theorem.

Theorem 2. *[2, Theorem 3.14] For two bounded graded posets P and Q, the poset* $\operatorname{Quilts}(P, Q)$ *is a distributive lattice. Its least and its greatest elements are*

$$\begin{aligned} f_{\hat{0}}(x, y) &= \max\{0, \operatorname{rank} x + \operatorname{rank} y - \max\{\operatorname{rank} P, \operatorname{rank} Q\}\} \quad \textit{and} \\ f_{\hat{1}}(x, y) &= \min\{\operatorname{rank} x,\ \operatorname{rank} y\}, \end{aligned}$$

respectively. The rank function of $\operatorname{Quilts}(P, Q)$ *is*

$$\operatorname{rank} f = \sum_{x \in P,\, y \in Q} f(x, y) - \sum_{x \in P,\, y \in Q} f_{\hat{0}}(x, y).$$

In general, we cannot expect to find exact formulas for quilt enumeration, since computing $|\operatorname{Quilts}(P, Q)|$ for general P and Q is a $\#P$-complete problem [2, Theorem 3.18]. However, studying the lattice structure remains worthwhile, as it yields interesting insights and generalizations of several known results concerning the poset of ASMs.

In particular, after recalling some basic notions in Section 2, we study the poset of join-irreducibles of quilt lattices in Section 3. We provide a simple description (Definition 3 and Theorem 4) and show that it is indeed graded (Proposition 6). Furthermore, Proposition 7 yields a formula for the number of join-irreducible quilts and their rank generating function. At the end of this section, we use these results to describe rowmotion on quilt lattices (Section 3.1) and to compute the dimension of the quilt lattices of chains (Section 3.2), thus generalizing a result of Reading [18]. We then provide the skeletal decomposition of quilt lattices via pseudocomplements in Section 4. Finally, we define a weak order on chain quilts in Section 5 and show in Section 6 that certain posets with maximal chains corresponding to chain quilts are shellable, extending results of Hamaker and Reiner [10]. This extended abstract is based on papers [12] and [11].

2 Preliminaries

A partially ordered set (*poset*) P is *bounded* if it has a least and a greatest element, which we denote by $\hat{0}_P$ and $\hat{1}_P$, respectively. It is *graded* if all the maximal chains have the same length. In this case, its *rank function* $\operatorname{rank}\colon P \to \mathbb{N}$ is recursively defined as $\operatorname{rank}\hat{0}_P = 0$ and, if $x \lessdot y$, $\operatorname{rank} y = \operatorname{rank} x + 1$. The rank of P is the rank of its greatest element $\hat{1}_P$. We finally define the *Cartesian product* $P \times Q$ of posets P and Q to have cover relations $(x, y) \lessdot (x', y)$ if $x \lessdot x'$ and $(x, y) \lessdot (x, y')$ if $y \lessdot y'$.

Some well-known examples of posets that appear in this extended abstract are the following:

- For $n \geqslant 0$, the *chain C_n of rank n* is the poset on $\{0, 1, \ldots, n\}$ with the usual order $\leqslant$.
- For $n \geqslant 1$, the *Boolean lattice B_n of rank n* is the poset of subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion.
- For $n \geqslant 2$, the *zigzag poset Z_n* is the poset of rank 3 with elements $\{\hat{0}, x_1, \ldots, x_n, \hat{1}\}$, where $\hat{0} \lessdot x_i$ for i odd, $x_i \lessdot \hat{1}$ for i even, and $x_i \lessdot x_{i\pm1}$ for i odd.

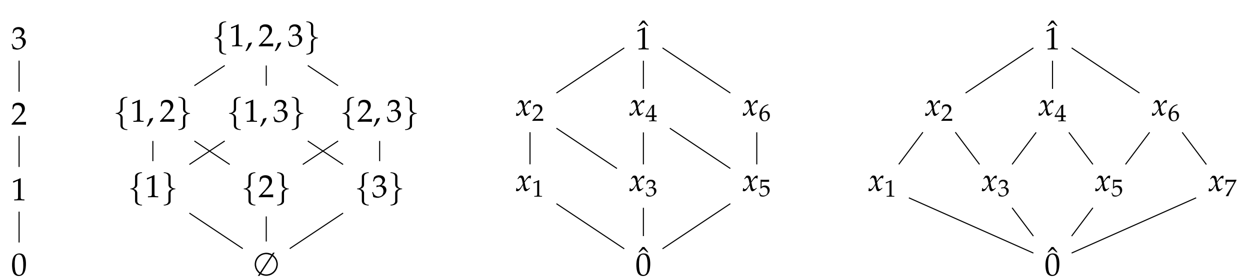

Figure 2: Hasse diagrams for the posets C_3, B_3, Z_6, and Z_7.

All posets in Figure 2 are in fact *lattices*, that is, every two elements x and y have a unique greatest common lower bound (*meet*) $x \wedge y$ and a unique least common upper bound (*join*) $x \vee y$. In contrast to the chain and the Boolean lattice, the zigzag poset Z_n is not *distributive* for $n \geqslant 5$ since we have $(x_1 \vee x_5) \wedge x_2 = x_2 \neq x_1 = (x_1 \wedge x_2) \vee (x_5 \wedge x_2)$.

3 The poset of join-irreducibles

An element of a poset P is called *join-irreducible* if it is not the join of two other distinct elements. The bottom element $\hat{0}_P$ is never join-irreducible, whereas the top element $\hat{1}_P$ may be. In Figure 2, only $\hat{1}_{C_3}$ is join-irreducible since it covers exactly one element. An *order ideal* (also *lower set* or *down set*) is a subset $\mathcal{O} \subseteq P$ such that $x \in \mathcal{O}$ whenever $x \leqslant y$ for some $y \in \mathcal{O}$. Birkhoff's representation theorem [3], sometimes called the *fundamental theorem for finite distributive lattices*, states that a finite distributive lattice is isomorphic to the lattice of order ideals of the poset of its join-irreducible elements.

In this section, we show that the poset $\operatorname{Irr}(\operatorname{Quilts}(P, Q))$ of join-irreducible elements of the quilt lattice $\operatorname{Quilts}(P, Q)$ is remarkably well behaved. It turns out to be isomorphic to the poset $\mathcal{I}(P, Q)$, defined as follows.

Definition 3. For two bounded graded posets P and Q, the poset $\mathcal{I}(P, Q)$ is the set of triples $(x, y, m) \in P \times Q \times \mathbb{N}$ such that $f_{\hat{0}}(x, y) < m \leqslant f_{\hat{1}}(x, y)$, with the cover relation

$$(x, y, m) \lessdot (x', y', m') \iff m = m', (x, y) \gtrdot (x', y') \quad \text{or} \quad m + 1 = m', (x, y) \lessdot (x', y').$$

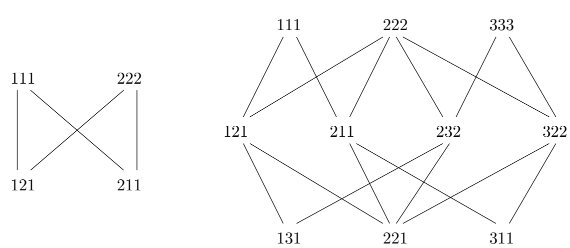

Figure 3: The posets $\mathcal{I}(C_3, C_3)$ and $\mathcal{I}(C_4, C_4)$.

The main result of this section is the following.

Theorem 4. *The poset* $\operatorname{Irr}(\operatorname{Quilts}(P, Q))$ *of join-irreducible elements of the lattice* $\operatorname{Quilts}(P, Q)$ *is isomorphic to* $\mathcal{I}(P, Q)$.

In the following we make the isomorphism explicit. For two elements x and x' in a bounded graded poset P, the *up-distance* $u_P(x, x')$ *from* x *to* x' is the number of $\lessdot$'s in a shortest possible sequence $x = x_0 R_1 x_1 R_2 \dots R_\ell x_\ell = x'$, where $R_i \in \{\lessdot, \gtrdot\}$. In other words, $u_P(x, x')$ is the number of up steps on any shortest path from x to x' in the Hasse diagram of P. This is well defined: if the shortest path from x to x' has u up steps and d down steps, we have $u - d = \operatorname{rank} x' - \operatorname{rank} x$, and so $u_P(x, x') =$

$(d_P(x,x') + \operatorname{rank} x' - \operatorname{rank} x)/2$, where $d_P(x,x')$ is the usual graph distance from x to x' in the Hasse diagram of P.

For $(x_0, y_0, m_0) \in \mathcal{I}(P,Q)$, we define the map $\mathcal{J}_{x_0,y_0,m_0} \colon P \times Q \longrightarrow \mathbb{N}$,

$$\mathcal{J}_{x_0,y_0,m_0}(x,y) = \max\{m_0 - u_P(x,x_0) - u_Q(y,y_0), f_{\hat{0}}(x,y)\}.$$

Proposition 5. *Let P and Q be two bounded graded posets. Then the join-irreducible elements of* $\operatorname{Quilts}(P,Q)$ *are precisely $\mathcal{J}_{x_0,y_0,m_0}$ for $(x_0,y_0,m_0) \in \mathcal{I}(P,Q)$. Moreover, for $(x_1,y_1,m_1) \in \mathcal{I}(P,Q)$ we have the following order and covering relations:*

1. $\mathcal{J}_{x_0,y_0,m_0} \leqslant \mathcal{J}_{x_1,y_1,m_1} \iff u_P(x_0,x_1) + u_Q(y_0,y_1) \leqslant m_1 - m_0.$
2. $\mathcal{J}_{x_0,y_0,m_0} \lessdot \mathcal{J}_{x_1,y_1,m_1} \iff m_0 = m_1$ *and* $(x_0,y_0) \gtrdot (x_1,y_1)$, *or* $m_0 + 1 = m_1$ *and* $(x_0,y_0) \lessdot (x_1,y_1)$.

The number of join-irreducible quilts is the rank of the quilt lattice, which holds if and only if the lattice is distributive according to the Avann–Markowsky theorem [1, 16]:

$$|\mathcal{I}(P,Q)| = \sum_{x\in P, y\in Q} (f_{\hat{1}}(x,y) - f_{\hat{0}}(x,y)) = \operatorname{rank}\operatorname{Quilts}(P,Q).$$

Curiously, $\mathcal{I}(P,Q)$ is itself a graded poset.

Proposition 6. *The poset $\mathcal{I}(P,Q)$ is graded. Its rank function is*

$$\operatorname{rank}(x,y,m) = -\operatorname{rank} x - \operatorname{rank} y + 2m + \max\{\operatorname{rank} P, \operatorname{rank} Q\} - 2.$$

Moreover, $\operatorname{rank}\mathcal{I}(P,Q) = \max\{\operatorname{rank} P, \operatorname{rank} Q\} - 2$*, unless P and Q both have rank at most* 1.

We can thus deduce expressions for the number of join-irreducible quilts and even the rank generating function of $\mathcal{I}(P,Q)$.

Proposition 7. *Let $r = \operatorname{rank} P$, $s = \operatorname{rank} Q$ and, without loss of generality, let $r \leqslant s$. Furthermore, let c_j be the number of elements of rank j in P and let d_j be the number of elements of rank j in Q. Then*

$$|\mathcal{I}(P,Q)| = \sum_{i=1}^{r} \sum_{j=i}^{s-i} c_j \sum_{j=i}^{s-i} d_j.$$

More generally, the rank generating function of $\mathcal{I}(P,Q)$, as a polynomial in q, is the matrix product

$$(c_0, \ldots, c_r) \left(\frac{q^{|i+j-s|} - q^{s-|i-j|}}{1-q^2} \right)_{\substack{0\le i\le r\\ 0\le j\le s}} \begin{pmatrix} d_0 \\ \vdots \\ d_s \end{pmatrix}.$$

It is not hard to see that the proposition implies that $|\mathcal{I}(P,P)|$ has to be a sum of non-consecutive squares, and that any sum of non-consecutive squares is equal to $|\mathcal{I}(P,P)|$ for some P.

Denote by $\mathrm{Antichains}(P)$ the set of antichains of the poset P. Every order ideal is determined by the antichain of its maximal elements, and an antichain determines an order ideal of all elements below any element of the antichain. Birkhoff's representation theorem implies $|\mathrm{Quilts}(P,Q)| = |\mathrm{Antichains}(\mathcal{I}(P,Q))|$. Using Proposition 5, we can also describe the bijection between $\mathrm{Quilts}(P,Q)$ and antichains of $\mathcal{I}(P,Q)$ explicitly.

For a quilt $f \in \mathrm{Quilts}(P,Q)$, we say that $(x,y) \in P \times Q$ is a *decreasable pair for* f if $(x',y') \lessdot (x,y)$ implies $f(x',y') = f(x,y) - 1$ and $(x,y) \lessdot (x',y')$ implies $f(x',y') = f(x,y)$. In other words, the pair (x,y) is decreasable for f if and only if the map f' that differs from f only for (x,y) and satisfies $f'(x,y) = f(x,y) - 1$ is also a quilt. Denote by $\mathrm{dec}(f)$ the set of all decreasable pairs for f.

Theorem 8. *Let P and Q be two bounded graded posets. The map*

$$\Phi\colon\ \mathrm{Quilts}(P,Q) \longrightarrow \mathrm{Antichains}(\mathcal{I}(P,Q)),$$
$$\Phi(f) = \{(x,y,f(x,y))\colon (x,y) \in \mathrm{dec}(f)\}$$

is a bijection. Its inverse takes an antichain A to

$$\Psi(A)(x,y) = \max\left(\{m' - u_P(x,x') - u_Q(y,y') : (x',y',m') \in A\} \cup \{f_{\hat{0}}(x,y)\}\right).$$

The join-irreducible elements of the ASM lattice have long been known to be bi-Grassmannian permutations, see for example [9] and [15]. We can use Theorem 8 and a simple translation to find another encoding of these join-irreducibles, giving an interesting interpretation of the *Robbins numbers* that enumerate ASMs.

Definition 9. Define the *tower* T_n as the set of cells $(i,j,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ with $i,j,k \geqslant 1$, $i + j + k \leqslant n + 1$. If we place an *archer* in the cell (i,j,k), we say that she *attacks* all cells (i',j',k') satisfying $i' \geqslant i$, $j' \geqslant j$, $k' \leqslant k$, $k - k' \leqslant (i' - i) + (j' - j)$.

In other words, the archer is turned toward the north-east and can hit any cell that is positioned lower than her, but not too low.

Corollary 10. *The total number of ways to place non-attacking archers in the tower T_n is the n-th Robbins number.*

For example, the generating function for $n = 7$ is $1 + 56q + 924q^2 + 6664q^3 + 24781q^4 + 51576q^5 + 62748q^6 + 45640q^7 + 19952q^8 + 5176q^9 + 768q^{10} + 60q^{11} + 2q^{12}$, whose coefficients sum to the seventh Robbins number 218348.

This construction has already appeared implicitly elsewhere, see for instance [20]. Surprisingly, the number of archers is a statistic on ASMs that does not seem to have been studied before.[1]

[1] It is now www.findstat.org/St001965.

3.1 Quilts of zigzag posets and rowmotion on quilt lattices

Apart from chain posets, we will study another family of posets whose quilts yield a classical lattice. We consider matrices with entries $\{0, 1\}$ and no horizontally or vertically adjacent 1s. In different terms, these are independent sets of a rectangular grid graph, as studied, for example, by Calkin and Wilf [7]. Curiously, as with ASMs, these objects also arise in statistical mechanics.

We call the position of a 1 *even* if the sum of its coordinates is even and *odd* otherwise.

Corollary 11. *The quilt lattice* $\mathrm{Quilts}(Z_k, Z_n)$ *is isomorphic to the poset* $\{0, 1\}^{k\times n}$ *without horizontally or vertically adjacent* 1*s with respect to the following cover relation: a matrix* M_1 *covers matrix* M_2 *if* M_2 *is obtained from* M_1 *by either deleting a* 1 *at an odd position or by deleting a* 1 *at an even position and simultaneously introducing* 1*s at horizontally and vertically adjacent positions if possible.*

The number of quilts in $\mathrm{Quilts}(Z_k, Z_n)$ is given by [17, A089934]. The poset of join-irreducibles has rank 1. Its elements correspond to matrices with a single entry equal to 1. If this entry is at an even position, the rank is 1, otherwise it is 0.

An important action on the order ideals of a poset is *rowmotion*, also known as the *Panyushev complement*. For a poset P and an order ideal $\mathcal{O} \subseteq P$, $\mathrm{Row}(\mathcal{O})$ is the order ideal generated by the minimal elements in $P \setminus \mathcal{O}$. This map was most likely first introduced in 1974 by Brouwer and Schrijver [6]. A brief account of the history is given by Striker and Williams in [20], who were, curiously, also the first citing the former.

Since we can interpret every quilt $f \in \mathrm{Quilts}(P, Q)$ as an order ideal in $\mathcal{I}(P, Q)$ due to Birkhoff's representation theorem, we obtain a natural notion of rowmotion on quilts. Explicitly, we say that a pair $(x, y) \in P \times Q$ of a quilt $f \in \mathrm{Quilts}(P, Q)$ is *increasable* if the map f' that differs from f only for (x, y) and satisfies $f'(x, y) = f(x, y) + 1$ is also in $\mathrm{Quilts}(P, Q)$. Let $\mathrm{inc}(f)$ denote the set of all increasable pairs of the quilt f.

Theorem 12. *For two bounded graded posets* P *and* Q *and a quilt* $f \in \mathrm{Quilts}(P, Q)$*, the rowmotion* $g = \mathrm{Row}(f)$ *of* f *is given by*

$$g(x, y) = \max\left(\{f(x_0, y_0) + 1 - u_P(x, x_0) + u_Q(y, y_0) : (x_0, y_0) \in \mathrm{inc}(f)\} \cup \{f_{\hat{0}}(x, y)\}\right).$$

In other words, rowmotion is the join of irreducible quilts $\mathcal{J}_{x_0,y_0,f(x_0,y_0)+1}$, where (x_0, y_0) goes over all increasable pairs.

In general it is a hard problem to determine the orbit structure of rowmotion. However, in the case of $\mathrm{Quilts}(C_1, Z_n)$, that is, the independent sets of a path graph, this has been achieved in [14, Corollary 4.3].

3.2 Dimension of quilt lattices

The *dimension* (also called the *order dimension* or the *Dushnik–Miller dimension*) of a poset $(P, \leqslant)$ is the smallest number of linear extensions of $\leqslant$ whose intersection is $\leqslant$. It is equal to the smallest integer m for which there exists an order-preserving embedding from P to $\mathbb{R}^m$. If P is a distributive lattice, this number is equal to the *width* (the size of the largest antichain) of the poset of irreducibles, see [8].

Reading proved that the dimension of the lattice of $n \times n$ ASMs, which is isomorphic to the quilt lattice $\mathrm{Quilts}(C_n, C_n)$, is $\lfloor n^2/4 \rfloor$, see [18, Theorem 1]. We generalize this to $\mathrm{Quilts}(C_k, C_n)$.

Theorem 13. *The dimension of the quilt lattice* $\mathrm{Quilts}(C_k, C_n)$ *is*

$$\binom{\min\{k,n\}+1}{2} - \max\{0, \lfloor (2\min\{k,n\} - \max\{k,n\}+1)^2/4 \rfloor\}.$$

A natural question is whether we can compute the dimension of the general quilt lattice $\mathrm{Quilts}(P, Q)$; however, $\mathrm{Irr}(\mathrm{Quilts}(P, Q)) \cong \mathcal{I}(P, Q)$ does not, in general, satisfy the *Sperner property*, that is, the width can be larger than the largest rank number. For example, if Q is the poset on the right of Figure 4, then $\mathcal{I}(Q, Q)$ has rank 1, with 18 elements on each rank, but its width is 20, so $\mathcal{I}(Q, Q)$ is not Sperner. Note that if P is the poset on the left of Figure 4, the rank numbers of $\mathcal{I}(P, P)$ are $18, 16, 18$, so the rank sequence is not necessarily unimodal even if the rank sequence of P is.

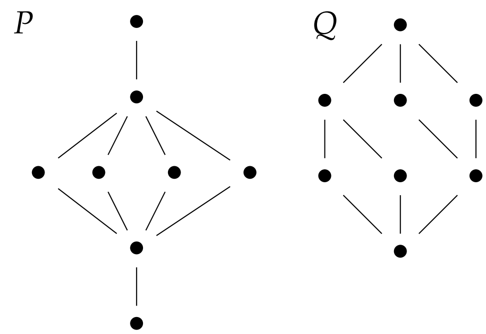

Figure 4: Posets P and Q for which $\mathcal{I}(P, P)$ is not unimodal and $\mathcal{I}(Q, Q)$ is not Sperner.

However, the lattice of rectangular ASMs has the Sperner property.

Theorem 14. *The poset* $\mathrm{Irr}(\mathrm{Quilts}(C_k, C_n))$ *satisfies the Sperner property. Furthermore, its sequence of rank numbers is symmetric and unimodal.*

We leave computing the dimension of the general quilt lattice as an open problem.

4 Skeletal decomposition via pseudocomplements

For a lattice L with a bottom element $\hat{0}_L$, we say that $x \in L$ has a *pseudocomplement* if there exists a largest element $x^* \in L$ satisfying $x \wedge x^* = \hat{0}_L$. A lattice is *pseudocomplemented* if every element has a pseudocomplement. Every distributive lattice is pseudocomplemented, and $x^* = \bigvee_{y:\, x \wedge y = \hat{0}_L} y$. The unary operation $x \mapsto x^*$ has many useful properties, for example:

- $x^{***} = x^*$ for all $x \in L$;

- if $x \leqslant y$ for $x, y \in L$, then $x^* \geqslant y^*$;
- $(x \wedge y)^{**} = x^{**} \wedge y^{**}$ for all $x, y \in L$;
- $(x \vee y)^* = x^* \wedge y^*$ for all $x, y \in L$;

see [5, Theorem 7.1]. The set $S(L) = \{x^* \colon x \in L\}$ is called the *skeleton*. Note that if $a \in S(L)$, then $a = x^*$ for some $x \in L$, and therefore $a^{**} = x^{***} = x^* = a$. The skeleton $S(L)$ is a boolean algebra for the operations $\sqcap$, $\sqcup$ and the complement *, where $x \sqcap y := x \wedge y$, $x \sqcup y := (x \vee y)^{**} = (x^* \wedge y^*)^*$, see [5, Theorem 7.2]. An element $x \in L$ is called *dense* if $x^* = \hat{0}_L$.

The following simple statement implies that the lattice L decomposes into certain intervals. We call this the *skeletal decomposition*.

Lemma 15. *Let L be a pseudocomplemented lattice. Then, for $a \in S(L)$, the set of elements with pseudocomplement a is an interval with bottom element $a_* = \bigwedge_{x \colon x^* = a} x$ and top element a^*.*

We now make pseudocomplementation and skeletal decomposition for the lattice $\mathrm{Quilts}(P, Q)$ explicit. Define

$$A(P, Q) = \{(x, y) \in P \times Q \colon x \neq \hat{0}_P, y \neq \hat{0}_Q, \operatorname{rank} x + \operatorname{rank} y = \max\{\operatorname{rank} P, \operatorname{rank} Q\}\}$$

and, for $T \subseteq A(P, Q)$, maps s_T and $r_T \colon P \times Q \to \mathbb{N}$ such that

$$\begin{aligned} s_T(x, y) &= \min\left(\{u_P(x_0, x) + u_Q(y_0, y) \colon (x_0, y_0) \in T\} \cup \{\operatorname{rank} x, \operatorname{rank} y\}\right), \\ r_T(x, y) &= \begin{cases} 1 & \text{if } (x, y) \in A(P, Q) \setminus T, \\ f_{\hat{0}}(x, y) & \text{otherwise.} \end{cases} \end{aligned}$$

By construction and by the definition of $f_{\hat{0}}(x, y)$, $s_T(x, y) = r_T(x, y) = 0$ for $(x, y) \in T$. Moreover, $s_T(x, y) > 0$ and $r_T(x, y) > 0$ for $(x, y) \in A(P, Q) \setminus T$: we clearly have $\operatorname{rank} x > 0$ and $\operatorname{rank} y > 0$. Finally, $u_P(x_0, x) + u_Q(y_0, y) = 0$ would imply that $x \leqslant x_0$ and $y \leqslant y_0$ and therefore $\operatorname{rank} x + \operatorname{rank} y < \operatorname{rank} x_0 + \operatorname{rank} y_0$, contradicting $(x, y) \in A(P, Q)$.

Theorem 16. *The maps s_T, r_T are in $\mathrm{Quilts}(P, Q)$. Furthermore, $\{s_T \colon T \subseteq A(P, Q)\}$ with operations*

$$s_T \sqcap s_{T'} = s_{T \cup T'}, \quad s_T \sqcup s_{T'} = s_{T \cap T'}, \quad s_T^* = s_{A(P,Q) \setminus T}$$

is precisely the skeleton lattice of $\mathrm{Quilts}(P, Q)$. In particular, $|S(\mathrm{Quilts}(P, Q))| = 2^{|A(P,Q)|}$.

For $f \in \mathrm{Quilts}(P, Q)$, we have $f^* = s_T$, where $T = \{(x, y) \in A(P, Q) \colon f(x, y) > 0\}$. Furthermore, for $T \subseteq A(P, Q)$, the set $\{f \in \mathrm{Quilts}(P, Q) \colon f^* = s_T\}$ is equal to the interval $[r_{A(P,Q) \setminus T}, s_{A(P,Q) \setminus T}]$. In particular, the dense quilts are precisely those $f \in \mathrm{Quilts}(P, Q)$ that satisfy $f \geqslant r_\varnothing$; equivalently, precisely the ones that satisfy $f(x, y) > 0$ for all $(x, y) \in A(P, Q)$.

5 Weak order on chain quilts

Hamaker and Reiner [10] generalized the weak (Bruhat) order on permutations to ASMs through an action of the 0-Hecke monoid. We further generalize this weak order for chain quilts.

Definition 17. The 0-*Hecke monoid* $\mathcal{H}_n(0)$ *for the symmetric group* $\mathfrak{S}_n$ is the monoid generated by $\pi_1,\ldots,\pi_{n-1}$ subject to the *braid relations* (i) $\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}$ for $1 \leqslant i \leqslant n-2$, (ii) $\pi_i\pi_j = \pi_j\pi_i$ for $1 \leqslant i,j \leqslant n-1$ such that $|i-j| \geqslant 2$, and the *idempotency relations* (iii) $\pi_i^2 = \pi_i$ for $1 \leqslant i \leqslant n-1$.

Chain quilts $f \in \mathrm{Quilts}(P, C_k)$ can be identified with tuples $(f_0, \ldots, f_m)$, where $m = \min\{\operatorname{rank} P, k\}$ and $f_i : P \to \mathbb{N}$ is $x \mapsto f(x,i)$. For $1 \leqslant i \leqslant m-1$ we define π_i to be the map on $\mathrm{Quilts}(P, C_k)$ that replaces $f_i(x)$ with $\min\{f_{i-1}(x)+1, f_{i+1}(x)\}$ for all $x \in P$.

Proposition 18. *The operators π_i induce an action of the* 0*-Hecke monoid on* $\mathrm{Quilts}(P, C_k)$.

For every permutation $w \in \mathfrak{S}_m$, there is a reduced factorization into adjacent transpositions $\{s_1, \ldots, s_{m-1}\}$ with $s_i = (i \quad i+1)$ for all $1 \leqslant i \leqslant m-1$. Consider a permutation w with reduced factorization $w = s_{k_1} \cdots s_{k_l}$, we then define the operator $\pi_w = \pi_{k_1} \cdots \pi_{k_l}$ acting on $\mathrm{Quilts}(P, C_k)$.

The action of $\mathcal{H}_n(0)$ on the symmetric group $\mathfrak{S}_n$ gives rise to the classical weak (Bruhat) order on $\mathfrak{S}_n$. By our extension of the $\mathcal{H}_n(0)$ action, we thus obtain a weak order on chain quilts which generalizes the weak order on the symmetric group and the weak order on ASMs, as defined in [10].

Compared to the weak order on the symmetric group, the weak order on ASMs loses a lot of its nice properties, such as being a bounded and graded lattice. However, the Möbius function conjecturally keeps its simple behavior.

Conjecture 1. *The Möbius function for the weak order on chain quilts has values in* $\{-1, 0, +1\}$.

6 Shellability of Dedekind posets

A *Dedekind map of rank k on P* is a surjective map $f : P \to C_k$ exhibiting Boolean growth, that is, if $x \lessdot y$, then $f(x) \leqslant f(y) \leqslant f(x)+1$ for all $x, y \in P$. We denote the set of all Dedekind maps of rank k on P by $D_k(P)$ and their union by $D(P)$. Note that a Dedekind map $f : P \to C_k$ satisfies $f(\hat{0}_P) = 0$ and $f(\hat{1}_P) = k$, implying that $D_k(P)$ is empty if k is larger than $\operatorname{rank} P$.

Dedekind maps in this general form were first studied in [2] and are closely related to the classical notion of Dedekind numbers [17, A000372]. Let $d_k(P)$ denote the number of all Dedekind maps of rank k on P. Then $d_1(B_k) + 2$ equals the *k-th Dedekind number*. Dedekind numbers are notoriously difficult to compute; in fact, the ninth Dedekind number has only recently been determined [13, 22].

Definition 19. The *Dedekind poset* on P is the bounded and graded poset on $D(P)$ ordered as follows: given $f, g \in D(P)$, we say $f \leqslant_{D(P)} g$ if

$$\forall x \in P : f(x) \leqslant g(x) \leqslant f(x) + \operatorname{rank} g - \operatorname{rank} f.$$

Consequently, we have the cover relation $f \lessdot_{D(P)} g$ if

$$\operatorname{rank} g = \operatorname{rank} f + 1 \quad \text{and} \quad \forall x \in P : f(x) \leqslant g(x) \leqslant f(x) + 1.$$

A chain quilt $f \in \operatorname{Quilts}(P, C_n)$ for $n \leqslant \operatorname{rank} P$ is a walk in the Dedekind poset on P (that is, a saturated chain) from the least element $D_0(P)$ to an element in $D_n(P)$. Furthermore, the Dedekind poset on C_n is isomorphic to the poset Φ_n defined by Terwilliger [21]. Thus, the maximal chains of Φ_n are in bijection with $n \times n$ ASMs.

A poset is *shellable* if its maximal chains can be ordered in such a way that each new chain intersects the union of the previous ones in a pure subcomplex of the order complex, ensuring that the complex is homotopy equivalent to a bouquet of spheres.

Hamaker and Reiner [10, Theorem 1.2] showed that Φ_n is shellable by constructing a suitable edge labeling of Φ_n, called *edge lexicographic labeling* or *EL-labeling* for short, that provides the required chain ordering due to a result by Björner [4, Theorem 2.3]. We generalize their result as follows.

Theorem 20. *For a bounded graded poset P, the Dedekind poset $D(P)$ admits an EL-labeling and is therefore shellable.*

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