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Chow polynomials of simplicial posets with positive h-vector are real-rooted

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Abstract. We prove that a finite graded simplicial poset with a top element added has real-rooted Chow and augmented Chow polynomials whenever it has a positive h-vector. This class of posets includes Cohen-Macaulay simplicial posets and in particular lattices of flats of uniform matroids.

Keywords: Chow polynomials, real-rootedness, simplicial posets

1 Main Results

Let P be a finite graded simplicial poset. That is, P is a finite poset with $\hat{0}$ for which all maximal intervals are boolean of the same rank n. Its h-vector $(h_0, \ldots, h_n)$ is given by $h_0 + h_1 x + \cdots + h_n x^n = \sum_{i=0}^{n} f_{i-1} x^i (1-x)^{n-i}$ where f_{i-1} is the number of elements of rank i [18, p. III.6]. We say that P has a ***positive h-vector*** if $h_i \geq 0$ for all i. Let $\widehat{P}$ be obtained from P by adding a top element $\hat{1}$. Following [10], the Chow and augmented Chow polynomials of $\widehat{P}$ are

$$\mathrm{H}_{\widehat{P}}(x) = \sum_{\substack{S \subseteq \{2,\ldots,n\} \\ S \text{ isolated}}} \beta_{\widehat{P}}(S) \cdot x^{|S|} \cdot (1+x)^{n-2\cdot|S|}, \tag{1.1}$$

$$\mathrm{H}^{\mathrm{aug}}_{\widehat{P}}(x) = \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ S \text{ isolated}}} \beta_{\widehat{P}}(S) \cdot x^{|S|} \cdot (1+x)^{n+1-2\cdot|S|}. \tag{1.2}$$

Here, a set $S \subset \mathbb{Z}$ is ***isolated***[1] if $i \in S$ implies $i+1 \notin S$, and $\beta_{\widehat{P}}(S)$ of $S \subseteq \{1, \ldots, n\}$ denotes the ***flag h-vector***

$$\beta_{\widehat{P}}(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha_{\widehat{P}}(T),$$

for $\alpha_{\widehat{P}}(T)$ being the ***flag f-vector*** counting maximal chains in the subposet of $\widehat{P}$ with only the ranks in T selected.

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[1] This property has been called *good* in [10] and $\mathrm{Stab}(\mathbb{Z})$ in [1].

Chow polynomials originate as Hilbert–Poincaré series of Chow rings defined by Feichtner and Yuzvinsky [8]. Ferroni, Matherne and Vecchi [10] extended the definition to finite graded bounded posets and showed that combinatorial properties such as unimodality, palindromicity and γ-positivity are preserved for Cohen–Macaulay posets. Whether Chow polynomials are real-rooted is an active discussion. We give more insight into the status of research later.

Theorem 1.1. *Let P be a finite graded simplicial poset with positive h-vector. Then*

$$\mathrm{H}_{\widehat{P}}(x) \text{ and } \mathrm{H}^{\mathrm{aug}}_{\widehat{P}}(x) \text{ are real-rooted.}$$

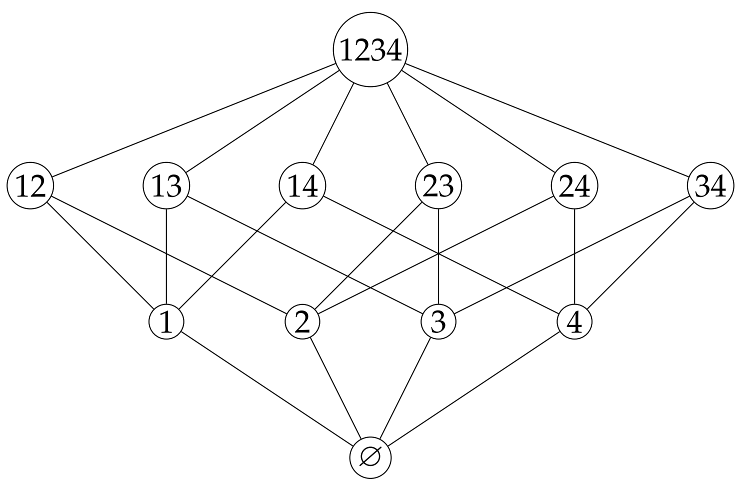
For the Chow polynomials of the dual poset $\widehat{P}^*$ of $\widehat{P}$, we even have the analogous statement and in addition the interlacingness.

Theorem 1.2. *Let P be a finite graded simplicial poset with positive h-vector. Then*

$$\mathrm{H}_{\widehat{P}^*}(x) \text{ is real-rooted.}$$

Moreover, the roots of $\mathrm{H}_{\widehat{P}^*}(x)$ interlace the roots of $\mathrm{H}^{\mathrm{aug}}_{\widehat{P}^*}(x) = \mathrm{H}^{\mathrm{aug}}_{\widehat{P}}(x)$.

Example 1.3. Let $M := U_{3,4}$ be the uniform matroid of rank 3 on $E = \{1, 2, 3, 4\}$. Denote by $\mathcal{L}(M)$ its lattice of flats



and let $P = \mathcal{L}(M) \setminus \{E\}$, so that $\widehat{P} = \mathcal{L}(M)$. Then P is clearly simplicial of rank $n = 2$ with the desired properties and we obtain the flag f-vector

$$\alpha_{\widehat{P}}(\varnothing) = 1, \quad \alpha_{\widehat{P}}(\{1\}) = 4, \quad \alpha_{\widehat{P}}(\{2\}) = 6, \quad \alpha_{\widehat{P}}(\{1,2\}) = 12$$

and the flag h-vector

$$\beta_{\widehat{P}}(\varnothing) = 1, \quad \beta_{\widehat{P}}(\{1\}) = 3, \quad \beta_{\widehat{P}}(\{2\}) = 5, \quad \beta_{\widehat{P}}(\{1,2\}) = 3\,.$$

Moreover, the f-vector of P is $(1,4,6)$ and the h-vector is $(1,2,3)$. The flag h-vector of $\widehat{P}^*$ is given by $\beta_{\widehat{P}^*}(S) = \beta_{\widehat{P}}(\{3-s \mid s \in S\})$. Considering the isolated subsets of $\{2\}$ and of $\{1,2\}$, respectively, we get

$$\begin{aligned}
\mathrm{H}_{\widehat{P}}(x) &= \beta_{\widehat{P}}(\varnothing)\cdot(1+x)^2 + \beta_{\widehat{P}}(\{2\})\,x = (1+x)^2 + 5x = x^2+7x+1 \\
&= \left(x + \tfrac{7-3\sqrt{5}}{2}\right)\cdot\left(x + \tfrac{7+3\sqrt{5}}{2}\right), \\
\mathrm{H}_{\widehat{P}^*}(x) &= \beta_{\widehat{P}^*}(\varnothing)\cdot(1+x)^2 + \beta_{\widehat{P}^*}(\{2\})\,x = (1+x)^2 + 3x = x^2+5x+1 \\
&= \left(x + \tfrac{5-\sqrt{21}}{2}\right)\cdot\left(x + \tfrac{5+\sqrt{21}}{2}\right), \\
\mathrm{H}^{\mathrm{aug}}_{\widehat{P}}(x) &= \beta_{\widehat{P}}(\varnothing)\cdot(1+x)^3 + \Big(\beta_{\widehat{P}}(\{1\}) + \beta_{\widehat{P}}(\{2\})\Big)x(1+x) \\
&= (1+x)^3 + 3x(1+x) + 5x(1+x) = x^3 + 11x^2 + 11x + 1 \\
&= (x+1)\cdot(x+5-2\sqrt{6})\cdot(x+5+2\sqrt{6})\,.
\end{aligned}$$

We finally observe that

$$-5-2\sqrt{6} < \tfrac{-7-3\sqrt{5}}{2} < \tfrac{5-\sqrt{21}}{2} < -1 < \tfrac{5+\sqrt{21}}{2} < \tfrac{-7+3\sqrt{5}}{2} < -5+2\sqrt{6}\,,$$

so the roots of $\mathrm{H}_{\widehat{P}^*}(x)$ indeed interlace the roots of $\mathrm{H}^{\mathrm{aug}}_{\widehat{P}}(x)$. Moreover, we observe that also the roots of $\mathrm{H}_{\widehat{P}}(x)$ interlace the roots of $\mathrm{H}^{\mathrm{aug}}_{\widehat{P}}(x)$, compare Conjecture 1.5 below.

For later usage, we record a relationship between the flag h-vector of $\widehat{P}$ and the h-vector of P that holds for simplicial posets P, namely

$$\beta_{\widehat{P}}(S) = \sum_{k=0}^{n} h_k \cdot \#\big\{w \in \mathfrak{S}_{n+1} \mid w(1) = k+1,\ \mathrm{Des}(w) = n+1-S\big\}\,, \tag{1.3}$$

where $n+1-S = \{n+1-s \mid s \in S\}$.

Example 1.4. We continue the above example, where we have $n=2$. We thus obtain

$$\begin{aligned}
\beta_{\widehat{P}}(\varnothing) &= 1\cdot\#\{123\} = 1 \\
\beta_{\widehat{P}}(\{1\}) &= 1\cdot\#\{132\} + 2\cdot\#\{231\} = 3 \\
\beta_{\widehat{P}}(\{2\}) &= 2\cdot\#\{213\} + 3\cdot\#\{312\} = 5 \\
\beta_{\widehat{P}}(\{1,2\}) &= 3\cdot\#\{321\} = 3
\end{aligned}$$

The relation (1.3) will later ensure the real-rootedness of Chow polynomials when the h-vector is positive. However, we were not able to find a counterexample otherwise.

Conjecture 1.5. *Let P be a simplicial poset. Then*

$$\mathrm{H}_{\widehat{P}}(x),\ \mathrm{H}_{\widehat{P}^*}(x) \text{ and } \mathrm{H}^{\mathrm{aug}}_{\widehat{P}}(x) \text{ are real-rooted.}$$

Moreover, the roots of both $\mathrm{H}_{\widehat{P}}(x)$ and $\mathrm{H}_{\widehat{P}^*}(x)$ interlace the roots of $\mathrm{H}^{\mathrm{aug}}_{\widehat{P}}(x) = \mathrm{H}^{\mathrm{aug}}_{\widehat{P}^*}(x)$.

Conjectures about the interlacing property had also been made by other authors, see e.g. [10, Remark 4.28]. We conclude this section by relating our main result to existing results in the literature.

Chow polynomials for uniform matroids The lattice of flats of the uniform matroid $U_{k,n}$ has coatoms given by all $(k-1)$-subsets of $\{1,\ldots,n\}$. It is thus obtained from a graded, simplicial poset by adding a top element. Our main result thus recovers [9, Theorem 1.10] where it was shown that its augmented Chow polynomial is real-rooted and [5] where it was shown that its Chow polynomial is real-rooted.

Chain polynomials of simplicial posets In [2] it was shown that the chain polynomial of $\widehat{P}$ is real-rooted, if P is a finite graded simplicial poset with positive h-vector. Our argument for the real-rootedness of the Chow polynomials is based on the same initial rewriting of the flag h-vector in Equation (1.3), but the families of interlacing polynomials we use are rather different from theirs. Our approach relies on a decomposition of Chow polynomials into palindromic polynomials using the γ-expansion. In particular, for the Boolean lattice $\widehat{P} = B_n$ of rank n, this yields a decomposition of both the Eulerian polynomials and the binomial Eulerian polynomials into interlacing palindromic polynomials, which provides an alternative proof of their real-rootedness.

Chow polynomials of Cohen-Macaulay posets It is conjectured in [10, Conjecture 4.26] that the Chow polynomials of Cohen-Macaulay posets are real-rooted. Our main results proves this conjecture in the case of Cohen-Macaulay simplicial posets as Cohen-Macaulay posets are known to be h-positive [18, Theorem 6.4]. It is moreover conjectured in [10, Remark 4.28] that the Chow polynomial of a Cohen-Macaulay poset interlaces the augmented Chow polynomial. Our main result again proves this conjecture in the case of the dual of Cohen-Macaulay simplicial posets.

Chow and chain polynomials of geometric lattices Geometric lattices are the lattices of flats of matroids. It is conjectured that Chow polynomials of geometric lattices are real-rooted [11, Conjecture 8.18] and that chain polynomials of geometric lattices are real-rooted [3, Conjecture 1.2]. Our main result covers only very special cases of the first conjecture.

2 Properties of real-rooted polynomials

A polynomial is ***real-rooted*** if it has only real roots. This section recalls properties of real-rooted polynomials from [19], [12], and [4]. Let

$$f = \prod(x - \alpha_i), \quad g = \prod(x - \beta_i) \in \mathbb{R}_{\geq 0}[x]$$

be two real-rooted polynomials with nonnegative coefficients[2] and weakly decreasing roots

$$\alpha_1 \geq \alpha_2 \geq \dots \qquad \text{and} \qquad \beta_1 \geq \beta_2 \geq \dots .$$

Then f ***interlaces*** g, denoted by $f \preceq g$, if $\deg g \in \{\deg f, \deg f + 1\}$ and the roots of f interlace the roots of g,

$$\beta_1 \geq \alpha_1 \geq \beta_2 \geq \alpha_2 \geq \dots .$$

By convention, any two polynomials of degree 0 or 1 interlace, $0 \preceq 0$, $0 \preceq f$, and $f \preceq 0$ for all real-rooted polynomials f. We collect the following elementary properties for sums of interlacing polynomials, see for example [12, Chapter 3].

Lemma 2.1. *Let $f, g, h \in \mathbb{R}_{\geq 0}[x]$ be real-rooted polynomials.*

1. *If $f, g \preceq h$, then $f + g \preceq h$.*
2. *If $f \preceq g, h$, then $f \preceq g + h$.*
3. *If $f \preceq g$, then $g \preceq x \cdot f$.*

The relation of interlacing polynomials is not transitive, and a sequence $(f_1, \dots, f_n)$ is an ***interlacing sequence*** if all polynomials are real-rooted and $f_i \preceq f_j$ for $i < j$.

Lemma 2.2 ([19])**.** *Let $f_1, f_2, \dots, f_n \in \mathbb{R}_{\geq 0}[x]$ be real-rooted polynomials, $f_i \not\equiv 0$ for all i. Then $(f_1, \dots, f_n)$ is an interlacing sequence if and only if $f_i \preceq f_{i+1}$ for $1 \leq i < n$ and $f_1 \preceq f_n$.*

Interlacing sequences play a fundamental role in proving real-rootedness. For later usage, we list further essential properties for sums of interlacing polynomials, following from [4, Theorem 8.5].

Lemma 2.3. *Let $(f_1, \dots, f_n)$ be an interlacing sequence. Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{\geq 0}$, then $\sum_i \lambda_i f_i$ is real-rooted and*

$$f_1 \ \preceq \ \sum_{i=1}^{n} \lambda_i f_i \ \preceq \ f_n \, .$$

Each of the following four sequences forms an interlacing sequence:

[2] Many of the properties referenced in this paper originally only ask for a positive leading term.

1. *The sequence* $(p_1, \dots, p_n)$ *of lower partial sums,*

$$p_k(x) = \sum_{i=1}^{k} f_i(x) \quad \text{for } 1 \le k \le n\,.$$

2. *The sequence* $(q_1, \dots, q_n)$ *of upper partial sums,*

$$q_k(x) = \sum_{i=k}^{n} f_i(x) \quad \text{for } 1 \le k \le n\,.$$

3. *For any* $0 \le \ell < n$*, the sequence* $(r_1, \dots, r_{n-\ell+1})$ *defined by*

$$r_k(x) = f_k(x) + \cdots + f_{k+\ell}(x) \quad \text{for } 1 \le k \le n-\ell\,.$$

4. *The sequence* $(t_1, t_2, \dots, t_{n+1})$ *defined by*

$$t_k(x) = x \cdot \sum_{i=1}^{k-1} f_i(x) + \sum_{i=k}^{n} f_i(x) \quad \text{for } 1 \le k \le n+1\,.$$

A polynomial $f(x) = \lambda_0 + \lambda_1 x + \cdots + \lambda_d x^d \in \mathbb{R}_{\ge 0}[x]$ of degree d is ***palindromic*** if $\lambda_i = \lambda_{d-i}$ for all $0 \le i \le d$. We next turn to the γ***-expansion*** of such a palindromic polynomial f given by

$$f(x) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i \cdot x^i \cdot (1+x)^{d-2i}$$

and its associated γ***-polynomial***

$$\gamma(f;x) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i \cdot x^i\,.$$

Real-rootedness interacts naturally with γ-expansions.

Lemma 2.4 ([16, Observation 4.2])**.** *Let* $f \in \mathbb{R}_{\ge 0}[x]$ *be palindromic. Then*

$$f \text{ real-rooted} \quad \Longleftrightarrow \quad \gamma(f) \text{ real-rooted}\,.$$

The proof of Lemma 2.4 extends to interlacingness, and yields the following proposition.

Proposition 2.5. *Let* $f, g \in \mathbb{R}_{\ge 0}[x]$ *be palindromic with* $\deg g = \deg f + 1$*. Then*

$$f \preceq g \quad \Longleftrightarrow \quad \gamma(f) \preceq \gamma(g)\,.$$

Example 2.6. We continue the running example for $\widehat{P} = \mathcal{L}(U_{3,4})$ and compute

$$\begin{aligned} \mathrm{H}_{\widehat{P}}(x) &= 1 \cdot x^0 \cdot (1+x)^2 + 5 \cdot x \cdot (1+x)^0 \\ \mathrm{H}_{\widehat{P}^*}(x) &= 1 \cdot x^0 \cdot (1+x)^2 + 3 \cdot x \cdot (1+x)^0 \\ \mathrm{H}^{\mathrm{aug}}_{\widehat{P}}(x) &= 1 \cdot x^0 \cdot (1+x)^3 + 8 \cdot x \cdot (1+x)^1 \end{aligned}$$

so we obtain

$$\gamma(\mathrm{H}_{\widehat{P}}; x) = 1 + 5 \cdot x\,, \quad \gamma(\mathrm{H}_{\widehat{P}^*}; x) = 1 + 3 \cdot x\,, \quad \text{and} \quad \gamma(\mathrm{H}^{\mathrm{aug}}_{\widehat{P}}; x) = 1 + 8 \cdot x\,,$$

which interlace as $\gamma(\mathrm{H}_{\widehat{P}}; x) \preceq \gamma(\mathrm{H}^{\mathrm{aug}}_{\widehat{P}}; x)$ and $\gamma(\mathrm{H}_{\widehat{P}^*}; x) \preceq \gamma(\mathrm{H}^{\mathrm{aug}}_{\widehat{P}}; x)$.

3 A family of interlacing polynomials

In this section, we introduce a family of interlacing polynomials that play the crucial role in the proof of Theorem 1.1. Let $S \subseteq T \subseteq \{1, \dots, n\}$. Define the polynomial $p^{S\subseteq T}_{n,k} \in \mathbb{R}_{\geq 0}[x]$ for $0 \leq k \leq n$ by

$$p^{S\subseteq T}_{n,k}(x) = \sum_{\substack{w \in \widehat{\mathfrak{S}}_{n+1,k+1} \\ S \subseteq \mathrm{Des}(w) \subseteq T}} x^{\mathrm{des}(w)}\,, \tag{3.1}$$

where $\widehat{\mathfrak{S}}_{n+1,k+1} = \{w \in \mathfrak{S}_{n+1} \mid w(1) = k+1\,,\ \mathrm{Des}(w) \text{ isolated}\}$. When the set S is isolated, this polynomial can be described recursively by

$$p^{S\subseteq T}_{1,k}(x) = \begin{cases} 1 & k = 0 \text{ and } S = \varnothing\,, \\ x & k = 1 \text{ and } T = \{1\}\,, \\ 0 & \text{else}\,, \end{cases}$$

and

$$p^{S\subseteq T}_{n,k}(x) = \mathbb{1}_{\{1 \in T\}} \cdot x \cdot \sum_{j=0}^{k-1} p^{S-1 \subseteq (T-1)\setminus\{1\}}_{n-1,j}(x) + \mathbb{1}_{\{1 \notin S\}} \cdot \sum_{j=k}^{n-1} p^{S-1\subseteq T-1}_{n-1,j}(x)\,. \tag{3.2}$$

where $S - 1 = \{i - 1 \mid i \in S\} \setminus \{1\}$ and $T - 1$ analogously. For $s \in S \subseteq T$, we have

$$p^{S\setminus\{s\}\subseteq T}_{n,k}(x) = p^{S\subseteq T}_{n,k}(x) + p^{S\setminus\{s\}\subseteq T\setminus\{s\}}_{n,k}(x)\,,$$

since the left side polynomial counts permutations in which s **can be a descent**, while the polynomials on the right side count permutations in which s **must be a descent** and in which s **must not be a descent**, respectively.

We write $p^T_{n,k} = p^{\varnothing\subseteq T}_{n,k}$ and $[a, b] = \{a, a+1, \dots, b\}$. The following lemma describes Chow polynomials in terms of the polynomials $p^{S\subseteq T}_{n,k}$ for a particular choice of subsets.

Lemma 3.1. *Let P be a graded simplicial poset of rank $n \geq 2$ with h-vector $(h_0, \dots, h_n)$. Let $\widehat{P}^*$ be the dual poset of $\widehat{P}$. Then,*

$$\begin{aligned}
\gamma(\mathrm{H}_{\widehat{P}}; x) &= \sum_{k=0}^{n} h_k \cdot p_{n,k}^{[1,n-1]}(x)\,, \\
\gamma(\mathrm{H}^{\mathrm{aug}}_{\widehat{P}^*}; x) &= \sum_{k=0}^{n} h_k \cdot p_{n,k}^{[2,n]}(x)\,, \quad \textit{and} \\
\gamma(\mathrm{H}^{\mathrm{aug}}_{\widehat{P}}; x) = \gamma(\mathrm{H}^{\mathrm{aug}}_{\widehat{P}^*}; x) &= \sum_{k=0}^{n} h_k \cdot p_{n,k}^{[1,n]}(x)\,.
\end{aligned}$$

Proof. We use Equation (1.3) to rewrite the flag h-vectors $\beta_{\widehat{P}}(S)$ in terms of the h-vector of P. The γ-polynomial of the Chow polynomial then becomes

$$\begin{aligned}
\gamma(\mathrm{H}_{\widehat{P}}; x) &= \sum_{\substack{S \subseteq \{2,\dots,n\} \\ S \text{ isolated}}} \beta_{\widehat{P}}(S) \cdot x^{|S|} \\
&= \sum_{k=0}^{n} h_k \sum_{\substack{S \subseteq \{2,\dots,n\} \\ S \text{ isolated}}} \#\{w \in \mathfrak{S}_{n+1} \mid w(1) = k+1\,,\ \mathrm{Des}(w) = n+1-S\} \cdot x^{|S|} \\
&= \sum_{k=0}^{n} h_k \sum_{\substack{S \subseteq \{1,\dots,n-1\} \\ S \text{ isolated}}} \#\{w \in \mathfrak{S}_{n+1} \mid w(1) = k+1\,,\ \mathrm{Des}(w) = S\} \cdot x^{|S|} \\
&= \sum_{k=0}^{n} h_k \cdot p_{n,k}^{[1,n-1]}(x)\,.
\end{aligned}$$

The calculation for the augmented case is completely analogous. The equalities for the dual poset $\widehat{P}^*$ now follow immediately with $\beta_{\widehat{P}^*}(n+1-S) = \beta_{\widehat{P}}(S)$. □

Example 3.2. Let $\widehat{P}$ be the lattice of flats of $U_{3,4}$ from our running example. The h-vector of P is $(h_0, h_1, h_2) = (1, 2, 3)$. The polynomials $p_{2,k}^{T}$ for the sets $T = \{1\}$ and $T = \{1,2\}$ are

T	$p_{2,0}^{T}$	$p_{2,1}^{T}$	$p_{2,2}^{T}$
$\{1\}$	1	x	x
$\{1,2\}$	$1+x$	$2x$	x

Thus Lemma 3.1 yields

$$\begin{aligned}
\gamma(\mathrm{H}_{\widehat{P}}; x) &= 1 \cdot \quad 1 \quad + 2 \cdot \ \ x + 3 \cdot x = 1 + 5x\,, \\
\gamma(\mathrm{H}^{\mathrm{aug}}_{\widehat{P}^*}; x) &= 1 \cdot (1+x) + 2 \cdot \ \ x + 3 \cdot 0 = 1 + 3x\,, \\
\gamma(\mathrm{H}^{\mathrm{aug}}_{\widehat{P}}; x) &= 1 \cdot (1+x) + 2 \cdot 2x + 3 \cdot x = 1 + 8x\,.
\end{aligned}$$

The following theorem is the heart of the argument.

Theorem 3.3. *Let $n \geq 2$ and let $T = [1,n]$ or $T = [1,n-1]$. The following diagram is an interlacing diagram, that is, any path $(\cdot \to \cdot \to \cdots \to \cdot \to \cdot)$ forms an interlacing sequence.*

$$
\begin{tikzcd}
p_{n,0}^{T\setminus\{1\}} \arrow[r] \arrow[d] & p_{n,1}^{T\setminus\{1\}} \arrow[r, dash] \arrow[d] & \cdots \arrow[r] & p_{n,k}^{T\setminus\{1\}} \arrow[r, dash] \arrow[d] & \cdots \arrow[r] & p_{n,n-1}^{T\setminus\{1\}} \arrow[r] \arrow[d] & p_{n,n}^{T\setminus\{1\}} \arrow[d] \arrow[ddllllll, bend right] \\
p_{n,0}^{T} \arrow[r] \arrow[d] & p_{n,1}^{T} \arrow[r, dash] \arrow[d] & \cdots \arrow[r] & p_{n,k}^{T} \arrow[r, dash] \arrow[d] & \cdots \arrow[r] & p_{n,n-1}^{T} \arrow[r] \arrow[d] & p_{n,n}^{T} \arrow[d] \\
p_{n,0}^{\{1\}\subseteq T} \arrow[r] & p_{n,1}^{\{1\}\subseteq T} \arrow[r, dash] & \cdots \arrow[r] & p_{n,k}^{\{1\}\subseteq T} \arrow[r, dash] & \cdots \arrow[r] & p_{n,n-1}^{\{1\}\subseteq T} \arrow[r] & p_{n,n}^{\{1\}\subseteq T}
\end{tikzcd}
\tag{3.3}
$$

Shortly after the first preprint of this paper, the interlacing diagram in Theorem 3.3 was further generalized in [6] to prove that Chow polynomials of *UMEL-shellable* posets are real-rooted, a class that in particular contains h-positive simplicial posets.

The proof of Theorem 3.3 is based on a successive application of properties of interlacing polynomials. Before we sketch some details in Section 3.1, we deduce Theorems 1.1 and 1.2, the main theorems of this paper.

Proof of Theorems 1.1 and 1.2. Let P be a graded, simplicial poset of rank n with positive h-vector $(h_0, \ldots, h_n)$. For $n = 1$, the Chow polynomials are given by

$$\mathrm{H}_{\widehat{P}}(x) = \mathrm{H}_{\widehat{P}^*}(x) = 1 + x \quad \text{and} \quad \mathrm{H}^{\mathrm{aug}}_{\widehat{P}}(x) = 1 + (m+2)x + x^2\,,$$

where $m = \#\{p \in P \mid \operatorname{rk}(p) = 1\}$. Both polynomials are real-rooted. For $n \geq 2$ and $T \in \{[1,n-1], [2,n], [1,n]\}$, the sequence $\left(p^T_{n,0}\,, p^T_{n,1}\,,\ \ldots\ , p^T_{n,n}\right)$ is interlacing by Theorem 3.3. By Lemma 2.1, any nonnegative linear combination of these polynomials, in particular

$$\sum_{k=1}^{n} h_k \cdot p^T_{n,k}(x) \qquad \text{is real-rooted.}$$

As shown in Lemma 3.1, this polynomial equals

$$\sum_{k=1}^{n} h_k \cdot p^T_{n,k}(x) = \begin{cases} \gamma(\mathrm{H}_{\widehat{P}}; x) & \text{if } T = [1,n-1]\,, \\ \gamma(\mathrm{H}_{\widehat{P}^*}; x) & \text{if } T = [2,n]\,, \\ \gamma(\mathrm{H}^{\mathrm{aug}}_{\widehat{P}}; x) & \text{if } T = [1,n]\,. \end{cases}$$

Lemma 2.4 completes the proof for the real-rootedness.

Using the pink diagonal arrow in Theorem 3.3, we also get the interlacing sequence

$$\left(h_0 \cdot p^{[2,n]}_{n,0}\,, h_1 \cdot p^{[2,n]}_{n,1}\,,\ \ldots\ , h_n \cdot p^{[2,n]}_{n,n}\,, h_0 \cdot p^{\{1\}\subseteq[1,n]}_{n,0}\,, h_1 \cdot p^{\{1\}\subseteq[1,n]}_{n,1}\,,\ \ldots\ , h_n \cdot p^{\{1\}\subseteq[1,n]}_{n,n}\right).$$

The sum of the first half of this sequence interlaces the sum of the second half. Moreover,

$$\sum_{k=0}^{n} h_k \cdot p_{n,k}^{[2,n]} \quad \preceq \quad \sum_{k=0}^{n} h_k \cdot \left(p_{n,k}^{[2,n]} + p_{n,k}^{\{1\}\subseteq[2,n]}\right) = \sum_{k=0}^{n} h_k \cdot p_{n,k}^{[1,n]} \quad \preceq \quad \sum_{k=0}^{n} h_k \cdot p_{n,k}^{\{1\}\subseteq[2,n]},$$

that is, $\gamma(\mathrm{H}_{\widehat{P}*}; x)$ interlaces $\gamma(\mathrm{H}^{\mathrm{aug}}_{\widehat{P}}; x)$. Using Proposition 2.5, this concludes the proof. □

Remark 3.4 (Eulerian polynomials)**.** The Eulerian polynomial $A_n(x)$ and the binomial Eulerian polynomial $\widetilde{A}_n(x)$ are defined by

$$A_n(x) = \sum_{w\in\mathfrak{S}_n} x^{\mathrm{des}(w)} \quad \text{and} \quad \widetilde{A}_n(x) = \sum_{i=0}^{n} \binom{n}{i} A_i(x)\,.$$

The γ-expansion of $A_n(x)$ was first described in [13] and the one of $\widetilde{A}_n(x)$ was given in [17] as

$$A_n(x) = \sum_{i\geq 0} \gamma_{n,i}\; x^i(1+x)^{n-1-2i} \quad \text{and} \quad \widetilde{A}_n(x) = \sum_{i\geq 0} \tilde{\gamma}_{n,i}\; x^i(1+x)^{n-2i},$$

where $\gamma_{n,i}$ is the number of permutations $w \in \mathfrak{S}_n$, such that $\mathrm{Des}(w) \subset \{1,\ldots,n-2\}$ is an isolated set of size i, and where $\tilde{\gamma}_{n,i}$ is the number of permutations $w \in \mathfrak{S}_n$, such that $\mathrm{Des}(w) \subset \{1,\ldots,n-1\}$ is isolated and has size i.

The polynomials $p^T_{n,k}$ thus provide a natural decomposition of A_{n+1} and $\widetilde{A}_{n+1}$ into palindromic polynomials, that is

$$A_{n+1}(x) = \sum_{k=0}^{n} (1+x)^n \cdot p_{n,k}^{[1,n-1]}\left(\frac{x}{(1+x)^2}\right), \text{ and}$$
$$\widetilde{A}_{n+1}(x) = \sum_{k=0}^{n} (1+x)^{n+1} \cdot p_{n,k}^{[1,n]}\left(\frac{x}{(1+x)^2}\right).$$

The real-rootedness of A_{n+1} and $\widetilde{A}_{n+1}$ has been proven in various ways, see, for example, [14] for a uniform approach. Theorem 3.3 provides yet another proof for their real-rootedness. For the Boolean lattice B_{n+1} of rank $n+1$, we moreover recover

$$\mathrm{H}_{B_{n+1}}(x) = A_{n+1}(x) \quad \text{and} \quad \mathrm{H}^{\mathrm{aug}}_{B_{n+1}}(x) = \widetilde{A}_{n+1}(x)$$

since the h-vector of $B_{n+1} - \{\hat{1}\}$ is $(1,\ldots,1)$. The first identity was proven in [15], the second in [7].

3.1 Sketch of the proof of Theorem 3.3

For $n \geq 2$ and $1 \in T \subseteq \{1, \ldots, n\}$, let $D_n(T)$ denote the diagram given in Theorem 3.3. Theorem 3.3 claims that $D_n([1, n])$ as well as $D_n([1, n-1])$ are interlacing diagrams. We proceed by induction on n. To clearly present the arguments, we split them into their two base cases (Lemma 3.5), and five statements for the induction step (Lemma 3.6).

Lemma 3.5 (Base cases for $T = \{1\}$ and $T = \{1, 2\}$)**.** *The diagrams $D_2(\{1\})$ and $D_2(\{1, 2\})$ are both interlacing diagrams.*

Proof. The diagram $D_2(\{1\})$ (left) and the diagram $D_2(\{1, 2\})$ (right) are

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[ddll, bend right] \\ 1 \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & x \arrow[d] \\ 0 \arrow[r] & x \arrow[r] & x \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x+1 \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[ddll, bend right] \\ x+1 \arrow[r] \arrow[d] & 2x \arrow[r] \arrow[d] & x \arrow[d] \\ 0 \arrow[r] & x \arrow[r] & x \end{tikzcd}$$

and the roots of the polynomials interlace along every path. □

Lemma 3.6. *Let $T = [1, n]$ or $T = [1, n-1]$ for $n \geq 3$. For the diagram $D_n(T)$, as given as in* (3.3)*, the following five statements hold:*

1. $\left(p_{n,0}^{T\setminus\{1\}}, p_{n,1}^{T\setminus\{1\}}, \ldots, p_{n,n}^{T\setminus\{1\}}\right)$ *is an interlacing sequence,*
2. $\left(p_{n,0}^{T}, p_{n,1}^{T}, \ldots, p_{n+1,n+1}^{T}\right)$ *is an interlacing sequence,*
3. $\left(p_{n+1,0}^{\{1\}\subseteq T}, p_{n+1,1}^{\{1\}\subseteq T}, \ldots, p_{n+1,n+1}^{\{1\}\subseteq T}\right)$ *is an interlacing sequence,*
4. $\left(p_{n,k}^{T\setminus\{1\}}, p_{n,k}^{T}, p_{n,k}^{\{1\}\subseteq T}\right)$ *is an interlacing sequence for all* $0 \leq k \leq n$*, and*
5. $p_{n,n-1}^{T\setminus\{1\}} \preceq p_{n,0}^{\{1\}\subseteq T}$.

Sketch of the proof. We successively apply Lemmas 2.1 and 2.3 to the recursive formula for $p_{n,k}^{S\subseteq T}$ given in (3.2). □

Proof of Theorem 3.3. We proceed by induction on n. The two base cases for $n = 1$ are given in Lemma 3.5, the induction step follows from Lemma 3.6 with Lemma 2.2. □

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