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Combinatorial Mutation Equivalence of Restricted Chain-Order Polytopes

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Abstract. We study restricted chain-order polytopes associated with Young diagrams via combinatorial mutations. Each such polytope is realized as the intersection of a chain-order polytope with a certain hyperplane. For a fixed Young diagram, we demonstrate that all restricted chain-order polytopes are connected via a sequence of combinatorial mutations. As the phenomenon of period collapse remains invariant under these combinatorial mutations, our results yield a wide class of rational polytopes having period collapse.

Résumé. Notre objectif est d'étudier des polytopes dits "restricted chain-order" associés aux diagrammes de Young, à travers des mutations combinatoires. Chacun de ces polytopes est défini comme l'intersection d'un polytope chain-order avec un certain hyperplan. Pour un diagramme de Young fixé, nous montrons que tous les polytopes de ce type sont reliés par une séquence de mutations combinatoires. Le phénomène de réduction de la période étant invariant sous ces mutations, nos résultats donnent une vaste classe de polytopes rationnels ayant la propriété de réduction de la période.

Keywords: chain-order polytopes, Young diagram, combinatorial mutations, Ehrhart quasi-polynomials, period collapse

1 Introduction

1.1 Background of restricted chain-order polytopes

This manuscript focuses on a class of poset-related polytopes, including Gelfand–Tsetlin polytopes and Birkhoff polytopes [2, 3, 6, 14]. The Gelfand–Tsetlin polytopes $\mathrm{GT}_{\alpha,\beta}$ arise in representation theory, where their lattice points form a basis for the irreducible

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representations of the Lie algebra $\mathfrak{sl}_n$. Each Gelfand–Tsetlin polytope can be realized as the intersection of an *order polytope* with certain affine hyperplanes. The order and chain polytopes, introduced by Stanley [14], are fundamental polytopes associated with a poset. Chain-order polytopes interpolate between them through *transfer maps* [14, Definition 3.1], which preserve the Ehrhart polynomials.

Recently, it is shown in [2] that restricted order and restricted chain polytopes corresponding to certain Gelfand–Tsetlin and Birkhoff polytopes, respectively, are related by a sequence of piecewise-linear maps. These maps, originally defined by Pak [12], are closely related to the Robinson–Schensted–Knuth (RSK) correspondence [13]. However, as observed in [2, Section 5], the transfer maps do not commute with intersections by the hyperplanes defining restricted order and restricted chain polytopes. One of the aims of our manuscript is to address this issue by decomposing Pak's piecewise-linear map into smaller transformations, called *combinatorial mutations*.

1.2 Combinatorial mutations and Ehrhart quasi-polynomials

Combinatorial mutations [1, 8] were originally introduced in the study of mirror symmetry for Fano varieties. They admit two complementary descriptions, dual to each other; here we adopt the one defined on the so-called M-lattice, viewing a combinatorial mutation as a piecewise-linear transformation. See Section 2.1.

Let $P \subset \mathbb{R}^N$ be a d-dimensional rational polytope. The enumerative function $L_P(n) := |nP \cap \mathbb{Z}^N|$ is a *quasi-polynomial* of degree d, that is,

$$L_P(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_0(n),$$

whose coefficients $c_i(n)$ are periodic in n [4]. The least common multiple of their periods is called the *period* of P. The *denominator* of P is the smallest positive integer m such that all vertices of mP lie in $\mathbb{Z}^N$. Classical Ehrhart's theorem implies that the period divides the denominator. Hence, every lattice polytope has period one, although the converse does not hold. We say that P has *period collapse* if its period is strictly smaller than its denominator. The study of period collapse is becoming a central topic in Ehrhart theory (see, e.g., [5, 7, 11]).

Rational polytopes related by a combinatorial mutation share the same Ehrhart polynomial (see Proposition 2.1). In particular, P has period collapse if P is a rational polytope that is *combinatorial mutation equivalent* to a lattice polytope.

1.3 Main Results

The main goal of our manuscript is to introduce a new family of polytopes having period collapse. From the perspective of the representation theory, it is known that the Ehrhart quasi-polynomials of both Gelfand–Tsetlin and restricted Birkhoff polytopes are in fact polynomials [10]. In particular, these polytopes have period collapse and are related via a piecewise-linear map, known as Pak's map (see Section 3.1). By decomposing Pak's map into a sequence of combinatorial mutations, we construct intermediate polytopes, called *restricted chain-order polytopes*. Our main result shows that restricted order and restricted chain polytopes are equivalent under combinatorial mutations.

Theorem 1.1. *The restricted order polytope $\mathcal{O}(\lambda)^k_{\underline{d}}$ and the restricted chain polytope $\mathcal{C}(\lambda)^k_{\underline{d}}$ are combinatorial mutation equivalent.*

Their precise definitions will be given in Section 2.3. As a consequence, we obtain new examples of polytopes whose Ehrhart quasi-polynomials are polynomials.

Corollary 1.2. *Let λ be a rectangular Young diagram. Then, for any $\underline{d}$, k, and up-set $C \subseteq \lambda$, the Ehrhart quasi-polynomial of the restricted chain-order polytope $\mathcal{O}_C(\lambda)^k_{\underline{d}}$ is a polynomial.*

Corollary 1.3. *Let λ be a Young diagram. If $(\underline{d})_\ell = |\{$boxes in the ℓth diagonal of $\lambda\}|$, then the Ehrhart quasi-polynomial of $\mathcal{O}(\lambda)^k_{\underline{d}}$ is a polynomial.*

2 Preliminaries

The goal of this section is to fix notation and recall the main definitions used throughout the manuscript. For the principal setup, see Setup 2.6 and Definition 2.8.

2.1 Combinatorial mutations

Before introducing our main objects, the restricted chain-order polytopes, we briefly recall the notion of combinatorial mutation. Fix a natural number N. For $x, y \in \mathbb{R}^N$, denote their usual dot product by $x \cdot y$. Let $w \in \mathbb{Z}^N$ be a primitive lattice vector and F a lattice polytope satisfying $F \subset w^\perp = \{x \in \mathbb{R}^N \mid x \cdot w = 0\}$. Define the *tropical map*

$$\varphi_{w,F} : \mathbb{R}^N \to \mathbb{R}^N, \qquad x \mapsto x - x_{\min} w,$$

where $x_{\min} = \min\{x \cdot f \mid f \in F\}$.

Let $P \subset \mathbb{R}^N$ be a polytope. The image $\varphi_{w,F}(P)$ is not always convex; if it is, we call it a *combinatorial mutation* of P. Two polytopes P and Q in $\mathbb{R}^N$ are said to be *combinatorial mutation equivalent* if there exists a finite sequence of combinatorial mutations

$$P_1 = \varphi_{w_1,F_1}(P_0),\, P_2 = \varphi_{w_2,F_2}(P_1), \ldots, P_k = \varphi_{w_k,F_k}(P_{k-1}),$$

with $P_0 = P$, $P_k = Q$ and each $\varphi_{w_i,F_i}(P_{i-1})$ is convex. We refer to the polytopes $P_2, P_3, \ldots, P_{k-1}$ as the *intermediate polytopes* of the sequence of combinatorial mutations.

Since unimodular maps preserve Ehrhart quasi-polynomials, we extend this notion to include them as well; two polytopes are combinatorial mutation equivalent if they are related by a sequence of combinatorial mutations and unimodular transformations. In what follows, "combinatorial mutation equivalence" will always refer to this extended notion, and all results are stated in this sense.

Proposition 2.1 (cf. [1, Proposition 4])**.** *Let P and Q be rational polytopes that are combinatorial mutation equivalent. Then P and Q have the same Ehrhart quasi-polynomial.*

Despite preserving their Ehrhart quasi-polynomials, many other salient features of polytopes are not invariant under combinatorial mutations. For example, the denominator and the number of vertices may change after a combinatorial mutation.

Example 2.2. Let $P = \operatorname{conv}((1,0),(0,-1),(-1,0),(0,\frac{1}{2})) \subset \mathbb{R}^2$. Let $\varphi = \varphi_{w,F}$, where $w = (0,1)$ and $F = \operatorname{conv}((0,0),(-1,0))$. Then for $(x,y) \in \mathbb{R}^2$, we have $\min\{(x,y)\cdot f \mid f \in F\} = \min\{0,-x\}$. Hence,

$$\varphi(x,y) = (x,y) - \min\{0,-x\}(0,1) = \begin{cases} (x,y) & \text{if } x \leq 0, \\ (x,x+y) & \text{if } x \geq 0. \end{cases}$$

We have $\varphi(P) = Q$. Since P and Q are combinatorial mutation equivalent, they have the same Ehrhart quasi-polynomial. Moreover, the polytope Q is a lattice polytope, so we deduce that P has period collapse.

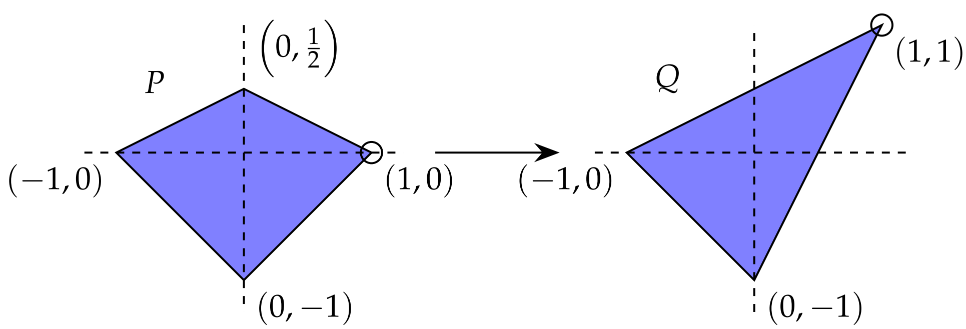

Figure 1: The map φ acts by the identity on the half-space $\{(x,y) \mid x \leq 0\}$ and acts by a shear in the direction $(0,1)$ on $\{(x,y) \mid x \geq 0\}$. The circled vertex $(1,0) \in P$ is mapped to the circled vertex $(1,1) \in Q$ by φ.

2.2 Poset polytopes associated with Young diagrams

Let $(\mathcal{P},<)$ be a poset, which we usually denote as $\mathcal{P}$. A subset $U \subseteq \mathcal{P}$ (resp. $D \subseteq \mathcal{P}$) is called an *up-set* (resp. a *down-set*) if for all $x \in U$ (resp. $x \in D$) and $y \in \mathcal{P}$, we have that $x < y$ implies that $y \in U$ (resp. $y < x$ implies $y \in D$). A subset $S \subseteq \mathcal{P}$

(resp. $A \subseteq \mathcal{P}$) is called a *chain* (resp. an *antichain*) if each pair of elements in S is comparable (resp. incomparable). Given a subset $S \subseteq \mathcal{P}$, the collections of minimal elements (resp. maximal elements) are $\min(S) = \{s \in S \mid x \not< s \text{ for all } x \in S\}$ (resp. $\max(S) = \{s \in S \mid x \not> s \text{ for all } x \in S\}$).

Let $[n] = \{1, 2, \ldots, n\}$ for any $n \in \mathbb{Z}_{>0}$. We equip $\mathbb{Z}^2_{>0}$ with the component-wise partial order: $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$. We identify a finite down-set $\lambda \subseteq \mathbb{Z}^2_{>0}$ with a Young diagram. As usual, Young diagrams of n boxes are regarded as partitions of n, which is naturally associated to the down-set

$$\lambda = \{(1,1), (1,2), \ldots, (1, \lambda_1),\ (2,1), (2,2), \ldots, (2, \lambda_2), \ldots\ , (s,1), \ldots, (s, \lambda_s)\} \subseteq \mathbb{Z}^2_{>0},$$

where $\lambda_1 + \cdots + \lambda_s = n$ with $\lambda_1 \geq \cdots \geq \lambda_s > 0$.

We say that a box $r \in \lambda$ is a *corner* if $r \in \max(\lambda)$ is a maximal element.

We write $\mathbb{R}^\lambda$ for the real vector space with distinguished basis $\{e_p \mid p \in \lambda\}$. For all $x \in \mathbb{R}^\lambda$ and $p \notin \lambda$, we take the convention that $x_p = 0 \in \mathbb{R}$ and $e_p = 0 \in \mathbb{R}^\lambda$.

We recall the definition of two polytopes classically associated to a poset. The *order polytope* of the Young diagram λ is the polytope

$$\mathcal{O}(\lambda) = \left\{ x \in \mathbb{R}^\lambda \mid 0 \leq x_p \leq x_q \leq 1 \text{ for all } p \leq q \text{ in } \lambda \right\}.$$

The *chain polytope* of λ is the polytope

$$\mathcal{C}(\lambda) = \left\{ x \in \mathbb{R}^\lambda \mid 0 \leq x_{p_1} + x_{p_2} + \cdots + x_{p_k} \leq 1 \text{ for all } p_1 < p_2 < \cdots < p_k \text{ in } \lambda \right\}.$$

The vertices of these polytopes have the following descriptions.

Proposition 2.3 ([14, Corollary 1.3 and Theorem 2.2])**.** *Fix a Young diagram λ and for each subset $S \subseteq \lambda$ define the characteristic vector $\chi(S) \in \mathbb{R}^\lambda$ by $\chi(S)_p = 1$ if $p \in S$ and $\chi(S)_p = 0$ if $p \notin S$. The vertices of $\mathcal{O}(\lambda)$ and $\mathcal{C}(\lambda)$ are*

$$V(\mathcal{O}(\lambda)) = \{\chi(S) \mid S \text{ an up-set of } \lambda\} \quad \text{and} \quad V(\mathcal{C}(\lambda)) = \{\chi(S) \mid S \text{ an antichain of } \lambda\}.$$

The order polytope and chain polytope are known to be combinatorial mutation equivalent [8]. The sequence of combinatorial mutations can be realized by a decomposition of a piecewise-linear map called the *transfer map*, introduced in [14], which interpolates between the order polytope and the chain polytope. Intermediate polytopes of this sequence of combinatorial mutations are given by, so-called, the chain-order polytope.

Definition 2.4 (Chain-order polytopes)**.** Let λ be a Young diagram and let $C \subsetneq \lambda$ be a proper up-set. The *chain-order polytope* of λ with respect to C is the polytope

$$\mathcal{O}_C(\lambda) = \left\{ x \in \mathbb{R}^\lambda \;\middle|\; \begin{array}{ll} 0 \leq x_p \leq 1 & \text{for all } p \in \lambda, \\ x_p \leq x_q & \text{for all } p \leq q \text{ in } \lambda \backslash C, \\ x_p + x_{q_1} + \cdots + x_{q_n} \leq 1 & \text{for all } p \in \lambda \backslash C \text{ and } q_1, \ldots, q_n \in C \\ & \qquad \text{such that } p < q_1 < \cdots < q_n. \end{array} \right\}.$$

- If $C = \emptyset$, then $\mathcal{O}_C(\lambda)$ coincides with the order polytope $\mathcal{O}(\lambda)$.
- If $C = \lambda \backslash \{(1,1)\}$, then $\mathcal{O}_C(\lambda)$ coincides with the chain polytope $\mathcal{C}(\lambda)$.

Example 2.5. Consider the Young diagram λ given by the partition $(3,2)$ and the up-set $C = \{(1,2),(1,3),(2,2)\}$ corresponding to the highlighted boxes below. It is possible to show that the chain-order polytope of λ with respect to C has vertices:

$$\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&0\\\cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 0&0&0\\\hline 1&0\\\cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&0&0\\\hline 1&0\\\cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 0&1&0\\\hline 0&0\\\cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 0&1&0\\\hline 1&0\\\cline{1-2}\end{array}\ ,$$

$$\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&1\\\cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 0&0\\\cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 1&0\\\cline{1-2}\end{array}\ \text{and}\ \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 0&1\\\cline{1-2}\end{array}\ .$$

2.3 Restricted chain-order polytopes

We now consider restricted versions of the order and chain polytopes; that is, the intersection of these polytopes with certain hyperplanes. To define them, we fix the following setup that will be used throughout the rest of the manuscript.

Setup 2.6. Fix a Young diagram λ. Let $m_1, m_2 \in \mathbb{Z}_{>0}$ be the smallest natural numbers such that $\lambda \subseteq [m_1] \times [m_2]$. Namely, if λ is given by the partition $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s)$, then $m_1 = s$ and $m_2 = \lambda_1$.

Fix a vector $\underline{d} = (d_{m_1-1}, d_{m_1-2}, \ldots, d_0, d_{-1}, \ldots, d_{1-m_2}) \in \mathbb{Z}_{>0}^{m_1+m_2-1}$ and $k \in \mathbb{Z}_{>0}$. Define the set of integers $\mathrm{Diag}(\lambda) := \{1-m_2, 2-m_2, \ldots, m_1-1\}$, which index the *diagonals* of λ. The *ℓth diagonal of* λ is the set of boxes $\{(i,j) \in \lambda \mid i-j=\ell\}$. For each $\ell \in \mathrm{Diag}(\lambda)$, let r_ℓ be the maximal element in the ℓth diagonal of λ. Visually, the box r_ℓ is the bottom-right-most box of the Young diagram on the ℓth diagonal. See Figure 2.

Definition 2.7. Fix Setup 2.6. The *restricted order polytope of* λ with respect to $\underline{d}$ and k is the restriction of the kth dilate of the order polytope of λ defined by:

$$\mathcal{O}(\lambda)_{\underline{d}}^k = \left\{ x \in k\,\mathcal{O}(\lambda) \;\middle|\; \sum_{i-j=\ell} x_{(i,j)} = d_\ell \text{ for all } \ell \in \mathrm{Diag}(\lambda) \right\}.$$

The *restricted chain polytope of* λ with respect to $\underline{d}$ and k is the restriction of the kth dilate of the chain polytope of λ defined by:

$$\mathcal{C}(\lambda)_{\underline{d}}^k = \left\{ x \in k\,\mathcal{C}(\lambda) \;\middle|\; \sum_{p \leq r_\ell} x_p = d_\ell \text{ for all } \ell \in \mathrm{Diag}(\lambda) \right\}.$$

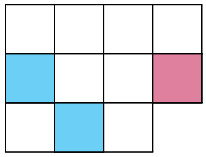

Figure 2: Highlighted in blue, the 1st diagonal given by the boxes $(2,1)$ and $r_1 = (3,2)$. Highlighted in purple, the box $r_{-2} = (2,4)$.

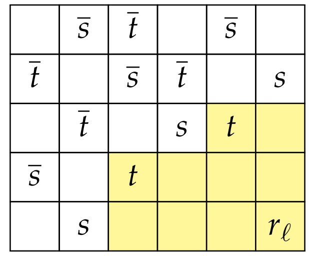

Figure 3: The set S_ℓ is given by the boxes labelled with s, the set $\overline{S}_\ell$ are the boxes s and $\bar{s}$, the set T_ℓ are the boxes t, and the set $\overline{T}_\ell$ are the boxes $\bar{t}$.

Now, we introduce the main object of this manuscript.

Definition 2.8. Fix Setup 2.6 and let $C \subseteq \lambda$ be an up-set. For each $\ell \in \mathrm{Diag}(\lambda)$, we define the following:

$$\begin{aligned}
R_\ell &= \{(a,b) \in \lambda \mid (a,b) \leq r_\ell\} : \text{the rectangular sub-Young diagram of } \lambda;\\
S_\ell &= \max((\lambda \backslash C) \cap R_\ell) : \text{the corners of } (\lambda \backslash C) \cap R_\ell;\\
\overline{S_\ell} &= \{(a,b) \in R_\ell \mid (a+i, b+i) \in S_\ell \text{ for some } i \geq 0\} : \text{the diagonals of } R_\ell \text{ ending in } S_\ell;\\
T_\ell &= \min(C \cap R_\ell) : \text{the minimal elements of } C \cap R_\ell;\\
\overline{T_\ell} &= \{(a,b) \in R_\ell \mid (a+i, b+i) \in T_\ell \text{ for some } i \geq 1\} \ :\\
&\qquad\qquad \text{the diagonals in } R_\ell \text{ that end one step before } T_\ell.
\end{aligned}$$

The *restricted chain-order polytope* with respect to $\underline{d} \in \mathbb{Z}_{>0}^{m_1+m_2-1}$ and $k \in \mathbb{Z}_{>0}$ is

$$\mathcal{O}_C(\lambda)^k_{\underline{d}} = \left\{ x \in k\,\mathcal{O}_C(\lambda) \;\middle|\; \sum_{s \in \overline{S_\ell}} x_s - \sum_{t \in \overline{T_\ell}} x_t + \sum_{u \in C \cap R_\ell} x_u = d_\ell \text{ for each } \ell \in \mathrm{Diag}(\lambda) \right\}.$$

Example 2.9. Consider the Young diagram $\lambda = [5] \times [6]$, i.e., λ is a partition $(6,6,6,6,6)$ of 30. Fix $\ell = -1$. Note that $R_\ell = \lambda$ in this case. Figure 3 depicts an up-set C represented by the highlighted boxes along with the sets S_ℓ, $\overline{S}_\ell$, T_ℓ, and $\overline{T}_\ell$.

Remark 2.10. If $\lambda = \{(1,1), \ldots, (n,n)\}$ is square and $\underline{d} = (1, 2, \ldots, n-1, n, n-1, \ldots, 1)$, then the restricted order polytope is equal to the restricted Gelfand-Tsetlin polytope and the restricted chain polytope is equal to the restricted Birkhoff polytope [2, Proposition 5.2]. The period-collapse phenomenon of these polytopes is studied in [2]. Corollaries 1.2 and 1.3 show that period collapse also occurs for some other families of polytopes.

3 Main results and Examples

The goal of this section is to explain the idea of the proof of Theorem 1.1 as well as Corollaries 1.2 and 1.3.

3.1 Idea of Proof: Pak's piecewise-linear map

We construct a sequence of polytopes that interpolate between the restricted order polytope and the restricted chain polytope. Fix Setup 2.6. In [12, Theorem 3], Pak shows that $\mathcal{O}(\lambda)^k_{\underline{d}}$ and $\mathcal{C}(\lambda)^k_{\underline{d}}$ have the same number of lattice points for $k \geq \max(\underline{d})$. The theorem is proved in [12, Section 4] by constructing a piecewise-linear map $\xi_\lambda : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$, *Pak's map*, which gives a bijection between the lattice points.

Pak's map ξ_λ. Let $r \in \lambda$ in the ℓth diagonal of λ. We recall the convention that for any $x \in \mathbb{R}^\lambda$, we have $x_p = 0$ if $p \notin \lambda$. Define the map $\chi_r : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$ as follows:

$$\chi_r(x)_{(i,j)} = \begin{cases} x_{(i,j)} - \max\{x_{(i-1,j)}, x_{(i,j-1)}\} & \text{if } (i,j) = r, \\ \max\{x_{(i-1,j)}, x_{(i,j-1)}\} & \\ \quad + \min\{x_{(i+1,j)}, x_{(i,j+1)}\} - x_{(i,j)} & \text{if } i-j = \ell \text{ and } (i,j) < r, \\ x_{(i,j)} & \text{otheriwse.} \end{cases} \tag{3.1}$$

Then Pak's map is described as a composition:

$$\xi_\lambda = \chi_{r_1} \circ \chi_{r_2} \circ \cdots \circ \chi_{r_n}$$

for any permutation $(r_1, \ldots, r_n)$ of the elements of λ such that whenever $r_i < r_j$ the corresponding indices satisfy $i < j$. With this notation, [12, Theorem 4] is equivalent to the statement that ξ_λ does not depend on the choice of permutation $(\chi_{r_1}, \ldots, \chi_{r_n})$ under the condition. We also note that each map $\chi_r : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$ is a bijection, which follows from [12, Section 4] where an explicit inverse is constructed, and so ξ_λ is also a bijection.

A decomposition of Pak's map. We give a "further" decomposition of Pak's map ξ_λ. Fix $r = (a,b) \in \lambda$ in the ℓth diagonal of λ. We express the map χ_r as the composition of the following tropical maps and a unimodular map:

- For each $0 \leq i \leq i_{\max}(r) := \min\{a,b\} - 1$, let

  $$w_i = e_{(a-i-1,b-i-1)} - e_{(a-i,b-i)} \text{ and } F_i = \operatorname{conv}\left(e_{(a-i-1,b-i)}, e_{(a-i,b-i-1)}\right).$$

  Define the tropical map $\varphi_i := \varphi_{w_i,F_i}$.
- Let ψ be the unimodular map:

  $$\psi(x)_{(i,j)} = \begin{cases} x_{(i,j)} - (x_{i-1,j} + x_{i,j-1}) & \text{if } (i,j) = r, \\ -x_{(i,j)} + (x_{i-1,j} + x_{i,j-1}) & \text{if } i-j = \ell \text{ and } (i,j) \neq r, \\ x_{(i,j)} & \text{if } i-j \neq \ell. \end{cases}$$

Proposition 3.1. *With the notation above, we have*

$$\chi_r = \psi \circ \varphi_{i_{\max}(r)} \circ \cdots \circ \varphi_0. \tag{3.2}$$

Each constituent φ_i is a combinatorial mutation. For the proof of Theorem 1.1, it suffices to show that each φ_i is indeed a combinatorial mutation by Proposition 3.1. We do this inductively. Namely, for each $0 \le i \le i_{\max}(r)$, we may show that the image of the polytope under φ_i is convex. More concretely, for each $1 \le s \le n$, the image $\chi_{r_s} \circ \chi_{r_{s+1}} \circ \cdots \circ \chi_{r_n}(\mathcal{O}(\lambda)^k_{\underline{d}})$ coincides with the restricted chain-order polytope $\mathcal{O}_C(\lambda)^k_{\underline{d}}$ with respect to the up-set $C = \{r_s, r_{s+1}, \ldots, r_n\}$. This follows from the following lemma.

Lemma 3.2. *Fix Setup 2.6 and let $C \subsetneq \lambda$ be an up-set. Let $r = (a, b)$ be a corner of $\lambda \backslash C$ and $\ell = a - b$. Let $\chi_r : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$ be as in* (3.1)*. Then $\chi_r : \mathcal{O}_C(\lambda)^k_{\underline{d}} \to \mathcal{O}_{C\cup\{r\}}(\lambda)^k_{\underline{d}}$ is a bijection. Moreover, for every $i \in \{0, \ldots, i_{\max}(r)\}$, the tropical map φ_i, which appears in the decomposition* (3.2)*, gives a combinatorial mutation of the polytope $(\varphi_{i-1} \circ \cdots \circ \varphi_0)(\mathcal{O}_C(\lambda)^k_{\underline{d}})$.*

Now, we are ready to prove Theorem 1.1. Consider a maximal flag of up-sets

$$\emptyset = C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_u = \lambda.$$

Each difference $C_i \setminus C_{i-1}$ consists of a single box r, a corner of $\lambda \setminus C_{i-1}$. By Lemma 3.2, the map $\chi_r : \mathcal{O}_{C_{i-1}}(\lambda)^k_{\underline{d}} \to \mathcal{O}_{C_i}(\lambda)^k_{\underline{d}}$ is a bijection between the corresponding restricted polytopes, and each φ_j appearing in (3.2) induces a combinatorial mutation, so consecutive polytopes $\mathcal{O}_{C_{i-1}}(\lambda)^k_{\underline{d}}$ and $\mathcal{O}_{C_i}(\lambda)^k_{\underline{d}}$ are combinatorial mutation equivalent. This proves the theorem.

Here, we illustrate a part of the above discussions by the following example:

Example 3.3. Let $\lambda = \{(1,1), \ldots, (5,6)\}$ be the same Young diagram as in Example 2.9. Fix $\ell = -1$, then $r_\ell = (5,6)$. We have $i_{\max}(r) = 4$ and we express χ_r as a composition of five tropical maps $\varphi_0, \ldots, \varphi_4$ and a unimodular map ψ. As an example, the map φ_0 will only affect the entries $x_{(4,5)}, x_{(5,6)}$ of a point $x \in \mathbb{R}^\lambda$ as follows:

$$\varphi_0(x_{(4,5)}) = x_{(4,5)} - \min\{x_{(4,6)}, x_{(5,5)}\}, \quad \varphi_0(x_{(5,6)}) = x_{(5,6)} + \min\{x_{(4,6)}, x_{(5,5)}\}.$$

Applying φ_1 to $\varphi_0(x)$ will further change the entry $(4,5)$ to

$$\varphi_1 \circ \varphi_0(x_{(4,5)}) = x_{(4,5)} - \min\{x_{(4,6)}, x_{(5,5)}\} + \min\{x_{(3,5)}, x_{(4,4)}\}.$$

3.2 Period collapse for restricted chain-order polytopes

We focus on the phenomenon of period collapse and prove Corollaries 1.2 and 1.3 by Theorem 1.1. First, we observe the following:

Proposition 3.4 (cf. [2, Theorem 2.1 and Lemma 2.2])**.** *If λ is rectangular, then the Ehrhart quasi-polynomial of $\mathcal{O}(\lambda)^k_{\underline{d}}$ is a polynomial.*

By Proposition 2.1, combinatorial mutations preserve the Ehrhart (quasi-)polynomial, so we immediately obtain Corollary 1.2 by Theorem 1.1.

Next, we observe that some non-rectangular Young diagrams give rise to restricted chain-order polytopes with period collapse. Let λ be a Young diagram and fix a pair of adjacent diagonals ℓ and $\ell+1$ where $\ell \geq 0$ (resp. ℓ and $\ell-1$ where $\ell \leq 0$). Assume the diagonals have the same length. We define $\lambda \backslash \ell$ as follows:

$$\lambda \backslash \ell := \begin{cases} \{(i,j) \in \lambda \mid i-j < \ell\} \cup \{(i-1,j) \mid (i,j) \in \lambda \text{ and } i-j > \ell\} & \text{if } \ell \geq 0, \\ \{(i,j) \in \lambda \mid i-j > \ell\} \cup \{(i,j-1) \mid (i,j) \in \lambda \text{ and } i-j < \ell\} & \text{if } \ell \leq 0. \end{cases}$$

Given a vector $\underline{d}$, for the Young diagram $\lambda \subseteq [m_1] \times [m_2]$, we also define

$$\underline{d} \backslash \ell = (d_{m_1-1}, \ldots, d_{\ell+1}, d_{\ell-1}, \ldots, d_{1-m_2}).$$

We think of $\underline{d} \backslash \ell$ as the corresponding vector for the Young diagram $\lambda \backslash \ell$. See Example 3.6.

Proposition 3.5. *If λ contains two adjacent diagonals of the same length, say ℓ and $\ell+1$ with $\ell \geq 0$ (resp. ℓ and $\ell-1$ with $\ell \leq 0$), and $d_\ell = d_{\ell+1}$ (resp. $d_\ell = d_{\ell-1}$), then the polytopes $\mathcal{O}(\lambda)^k_{\underline{d}}$ and $\mathcal{O}(\lambda \backslash \ell)^k_{\underline{d} \backslash \ell}$ are unimodularly equivalent. Also if $d_{\ell+1} < d_\ell$ (resp. $d_{\ell-1} < d_\ell$) then $\mathcal{O}(\lambda)^k_{\underline{d}}$ is empty.*

So for any Young diagram λ, we may apply Propositions 3.4 and 3.5 to construct restricted order polytopes having period collapse. Hence, we obtain Corollary 1.3.

3.3 Examples

Example 3.6. Let λ be the Young diagram corresponding to the partition $(4,4,3)$ of 11 and let $k=2$ and $\underline{d} = (1,2,3,2,2,1)$. Then the vertices of restricted order $\mathcal{O}(\lambda)^k_{\underline{d}}$ are

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 1&1&1&1\\\hline 1&1&1\\\cline{1-3}\end{array}\,,\quad \begin{array}{|c|c|c|c|}\hline 0&1&1&1\\\hline 1&1&1&1\\\hline 1&1&2\\\cline{1-3}\end{array}\,,\quad \begin{array}{|c|c|c|c|}\hline 0&1&1&1\\\hline 0&1&1&1\\\hline 1&2&2\\\cline{1-3}\end{array}\,,\quad \begin{array}{|c|c|c|c|}\hline 0&0&0&1\\\hline 1&1&2&2\\\hline 1&1&2\\\cline{1-3}\end{array}\,,\quad \begin{array}{|c|c|c|c|}\hline 0&0&0&1\\\hline 0&1&2&2\\\hline 1&2&2\\\cline{1-3}\end{array}\,,$$

$$\begin{array}{|c|c|c|c|}\hline 0&\frac{1}{2}&\frac{1}{2}&1\\\hline \frac{1}{2}&\frac{3}{2}&\frac{3}{2}&\frac{3}{2}\\\hline 1&\frac{3}{2}&\frac{3}{2}\\\cline{1-3}\end{array}\,,\quad \text{and} \quad \begin{array}{|c|c|c|c|}\hline \frac{1}{2}&\frac{1}{2}&\frac{1}{2}&1\\\hline \frac{1}{2}&\frac{1}{2}&\frac{3}{2}&\frac{3}{2}\\\hline 1&\frac{3}{2}&2\\\cline{1-3}\end{array}\,.$$

The shaded adjacent diagonals $\ell = -1$ (in blue) and $\ell - 1 = -2$ (in purple) have the same values since $d_{-1} = d_{-2} = 2$. Let $\lambda' = \lambda \backslash \ell$ be the square Young diagram given by the partition $(3,3,3)$ and $\underline{d}' = \underline{d} \backslash \ell = (1,2,3,2,1)$. By Proposition 3.5, the vertices of the restricted order polytope $\mathcal{O}(\lambda')^k_{\underline{d}'}$ can be obtained from the above vertices by applying the projection:

$$\begin{array}{|c|c|c|c|}\hline a & & b & c \\ \hline d & e & & f \\ \hline g & h & i \\ \cline{1-3}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & i \\ \hline\end{array}.$$

This projection does not change the Ehrhart quasi-polynomial of this polytope, which is the polynomial given by:

$$\frac{1}{12}t^4 + \frac{1}{2}t^3 + \frac{17}{12}t^2 + 2t + 1. \tag{3.3}$$

Example 3.7. We give an example of a restricted order polytope for some non-rectangular Young diagram λ to which cannot be applied Proposition 3.5. Consider the partition $\lambda = (4,4,3)$, $k = 3$ and $\underline{d} = (1,2,3,2,3,1)$. We index the coordinates of $\mathbb{R}^\lambda$ as follows:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 \\ \cline{1-3}\end{array} \text{ and let } V = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 3 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 & 3 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 & 2 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 3 & 1 & 2 & 3 \\ 0 & \frac{1}{2} & 1 & 1 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & 2 & 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{5}{2} & 1 & \frac{3}{2} & \frac{3}{2} \end{pmatrix} \in \mathbb{R}^{11\times\lambda}$$

The eleven vertices of the restricted order polytope $\mathcal{O}(\lambda)^k_{\underline{d}}$ are the rows of V. The Ehrhart quasi-polynomial of this polytope is the polynomial given by:

$$\frac{7}{120}t^5 + \frac{1}{2}t^4 + \frac{41}{24}t^3 + 3t^2 + \frac{41}{15}t + 1.$$

Remark 3.8. In general, two rational polytopes with the same Ehrhart (quasi-)polynomial are not necessarily combinatorial mutation equivalent. The simplest example is

$$\operatorname{conv}((1,1),(-1,1),(-1,-1),(1,-1)) \quad \text{and} \quad \operatorname{conv}((1,0),(1,1),(-1,1),(-1,-2));$$

see [9, Example 6]. Regarding the restricted order polytope $\mathcal{O}(\lambda)^k_{\underline{d}}$ in Example 3.6, we expect that this rational polytope is combinatorial mutation equivalent to a lattice polytope sharing the same Ehrhart polynomial. Indeed, the lattice polytopes with the Ehrhart polynomial given in Equation (3.3) are completely characterized: each such polytope is unimodularly equivalent to

$$P = \operatorname{conv}((0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(1,1,1,2)).$$

However, it remains unclear whether this P is combinatorial mutation equivalent to $\mathcal{O}(\lambda)^k_{\underline{d}}$ in Example 3.6.

This remark motivates the following question.

Question 3.9. *Which non-lattice restricted order polytopes are combinatorial mutation equivalent to lattice polytopes? More generally, does there exist a non-lattice restricted chain-order polytope that is combinatorial mutation equivalent to a lattice polytope?*

There is a rational polytope with period collapse that is not combinatorial mutation equivalent to any lattice polytope, though no such example is known for $\mathcal{O}_C(\lambda)^k_{\underline{d}}$.

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