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# A descent basis for the Δ-Springer module

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**Abstract.** We give a descent monomial basis of Δ-Springer modules $R_{n,\lambda,s}$, first defined by Griffin (2021). Our construction simultaneously generalizes the descent basis for the Garsia-Procesi module $R_\lambda$ (Carlsson–C. (2024), H.(2025)), as well as the descent basis for the generalized coinvariant algebras $R_{n,k}$ studied by Haglund-Rhoades-Shimozono (2018). We highlight the representation theoretic properties of this monomial basis by using it to give a direct combinatorial proof of the formula for graded Frobenius character of $R_{n,\lambda,s}$ in terms of battery-powered tableaux, a fact which has only the geometric proof of Gillespie-Griffin (2024).

**Keywords:** Δ-Springer module, descent monomials, catabolizability, shuffles, generalized coinvariant rings, battery-powered tableaux

# 1 Introduction

The coinvariant ring $R_n = \mathbb{C}[x_1, \ldots, x_n]/\langle e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})\rangle$ is well-studied due to its rich combinatorial, geometric, and representation theoretic significance. As an representation of $S_n$, it is isomorphic to the cohomology ring of the flag variety, where $S_n$ acts by permuting the variables. Its graded Frobenius character can be expressed using the **cocharge** statistic on standard tableaux:

$$\mathrm{Frob}_q(R_n) = \sum_{S \in \mathrm{SYT}_n} q^{\mathrm{cocharge}(S)} s_{\mathrm{sh}(S)}.$$

The **descent basis** is defined to be $\{g_\sigma(\mathbf{x}) = \prod_{\sigma_i > \sigma_{i+1}} x_{\sigma_1} \cdots x_{\sigma_i} : \sigma \in S_n\}$. We can see that $\deg(g_\sigma(\mathbf{x})) = \mathrm{maj}(\sigma)$. Although this basis does not form a standard monomial basis, it is useful when studying representation theoretic properties of $R_n$: for instance, they are closely related to certain Demazure atoms that appear in the study of the 0-Hecke algebra $H_n(0)$-action on the coinvariant ring [9].

In this work, we consider different generalizations of the coinvariant ring that are $S_n$ modules under the same action. The first is the **Garsia–Procesi ring** $R_\mu$, indexed by $\mu \vdash n$, a further quotient of $R_n$ that corresponds to the cohomology ring of the

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Springer fiber indexed by $\mu$. When $\mu = 1^n$, we recover the coinvariant ring. We have that $\mathrm{Frob}_q(R_\mu)$ is given by the **modified Hall-Littlewood polynomial**, where $\mathrm{SSYT}(\mu)$ denotes the set of all semistandard Young tableaux of weight $\mu$ [11]:

$$\widetilde{H}_\mu[\mathbf{x};q] = \sum_{T\in \mathrm{SSYT}(\mu)} q^{\mathrm{cocharge}(T)} s_{\mathrm{sh}(T)}. \tag{1.1}$$

There is a monomial basis of $R_\mu$ consisting of a subset of the descent basis of $R_n$, due to Carlsson and the first author [2]. The second author provided an alternative construction of the basis which is compatible with the combinatorics of $\widetilde{H}_\mu[X;q]$ [8].

The second generalization comes from the Delta conjecture. The **Haglund-Rhoades-Shimozono generalized coinvariant ring** $R_{n,k}$ (for $k \leq n$) is a partial representation theoretic model for $\Delta'_{e_{k-1}} e_n$ (up to a twist). When $k = n$ we recover the coinvariant ring. The following combinatorial formula for $\mathrm{Frob}_q(R_{n,k})$ is due to Haglund–Rhoades–Shimozono [7], where $\mathrm{SYT}_n$ denotes the set of all standard Young tableaux of size $n$.

$$\mathrm{Frob}_q(R_{n,k}) = \sum_{\substack{S\in \mathrm{SYT}_n \\ \mathrm{des}(S)<k}} \begin{bmatrix} n-\mathrm{des}(S)-1 \\ n-k \end{bmatrix}_q q^{\mathrm{maj}(S)} s_{\mathrm{sh}(S)}. \tag{1.2}$$

They also provide a generalized descent basis of $R_{n,k}$, in terms of the major index statistic on ordered set partitions.

The third generalization due to the work of Griffin [6], interpolates between the previous two. The **Δ-Springer module** $R_{n,\lambda,s}$ depends on parameters $(n, \lambda, s)$ where $\lambda \vdash k \leq n$, $s \geq \ell(\lambda)$. We have that $R_{k,\lambda,\ell(\lambda)} = R_\lambda, R_{n,(1^k),k} = R_{n,k}$. There is a combinatorial formula for $\mathrm{Frob}_q(R_{n,\lambda,s})$ in terms of **battery-powered tableaux** due to Gillespie–Griffin [5].

$$\mathrm{Frob}_q(R_{n,\lambda,s}) = \frac{1}{q^{\binom{s-1}{2}(n-k)}} \sum_{T=(D,B)\in \mathcal{T}^+_{n,\lambda,s}} q^{\mathrm{cocharge}(T)} s_{\mathrm{sh}(D)}. \tag{1.3}$$

Though (1.3) is combinatorial, the only known proof was through the geometry of generalized Springer fibers. In [5], the authors pose the question of whether there is more direct algebraic or combinatorial proof. Though recent results [4] give such explanation for $R_{n,k}$, there is still no known argument for general $R_{n,\lambda,s}$. Furthermore, there was no descent basis analogue for $R_{n,\lambda,s}$.

In this work, we give a construction of a generalized descent basis for $R_{n,\lambda,s}$ (Theorem 3.3). At the appropriate specializations, this recovers the known descent bases for $R_\mu$ [2, 8] and $R_{n,k}$ [7]. There are two different constructions of this basis: in terms of shuffles (Section 3) and tableaux and cocharge (Section 4): we utilize both presentations in our arguments. We then use our basis to give a tableau formula for the graded Frobenius character of $R_{n,\lambda,s}$ (Theorem 6.8). Our formula can be used to recover (1.1) and (1.2) up to combinatorial observations. Moreover, our formula, which arises from the combinatorics and algebraic structure of $R_{n,\lambda,s}$ as a ring, can be shown to be equivalent to the battery-powered tableaux formula (1.3), a geometric result. This gives a direct proof of (1.3) and bridges the gap between the combinatorics, algebra, and geometry of $R_{n,\lambda,s}$. Full details of this work are in [3].

# 2 Background

## 2.1 Permutations, tableaux and cocharge

For any word $w = w_1 w_2 \dots w_n$, we write $\mathrm{rev}(w) := w_n w_{n-1} \dots w_1$ for the reverse word. Throughout, we use French notation for Young diagrams, meaning that the first part corresponds to the bottom row. The **dominance** ordering on partitions, denoted $\trianglerighteq$, is defined by $\mu \trianglerighteq \lambda \Leftrightarrow \mu_1 + \cdots + \mu_k \geq \lambda_1 + \cdots + \lambda_k$ for all $k$. The **Schensted correspondence** gives a bijection from permutations $w \in S_n$ to pairs $(P(w), Q(w))$ of **standard Young tableaux** of size $n$ of the same shape.

**Definition 2.1.** The **cocharge word** of $w \in \mathfrak{S}_n$ (denoted $\mathrm{cc}(w)$) is a word of length $n$ consisting of the labeling of $w$ that we obtain in the following way. Label the 1 of $w$ with 0. Assume we labeled the letter $i$ with $k$. Label $(i+1)$ with a $k$ if it is to the right of $i$. We label $(i+1)$ with a $(k+1)$ if it is to the left of $i$.

This process is equivalent to scanning $w$ from left to right, labeling entries in increasing order. The label of $i$ corresponds to the number of times we need to return to the left end of $w$ before we label $i$. The statistic **cocharge**$(w)$ is the sum of the letters in $\mathrm{cc}(w)$.

**Example 2.2.** Let $w = 7136524$. The corresponding cocharge word is $\mathrm{cc}(w) = 4013201$ , hence $\mathrm{cocharge}(w) = 4+0+1+3+2+0+1 = 11$.

We can define $\mathrm{cc}(S)$ directly on tableaux and set $\mathrm{cocharge}(S)$ to be the sum of the entries in $\mathrm{cc}(S)$. We always have that $\mathrm{cocharge}(P(w)) = \mathrm{cocharge}(w)$.

**Definition 2.3.** The *cocharge tableau* of a standard tableau $S$ (denoted $\mathrm{cc}(S)$) is a filling of the same shape consisting of the labeling of $S$ that we obtain in the following way. Label the 1 of $S$ with 0. Assume we labeled the letter $i$ with $k$. Label $(i+1)$ with a $k$ if it is in the same row or lower than $i$. We label $(i+1)$ with a $(k+1)$ if it is in a higher row than $i$.

**Example 2.4.** If $S = \begin{array}{|c|c|c|c|}\cline{1-1} 5 \\ \cline{1-2} 4 & 6 \\ \hline 1 & 2 & 3 & 7 \\ \hline\end{array}$, the corresponding cocharge tableau is $\mathrm{cc}(S) = \begin{array}{|c|c|c|c|}\cline{1-1} 2 \\ \cline{1-2} 1 & 2 \\ \hline 0 & 0 & 0 & 2 \\ \hline\end{array}$.

We can think of a descent monomial as being a "reversed cocharge monomial", using the fact that $g_\sigma(\mathbf{x}) = \mathbf{x}^{\mathrm{rev}(\mathrm{cc}(w))}$ where $w = \mathrm{rev}(\mathrm{rev}(\sigma)^{-1})$. Let $\mathcal{D}_n = \{\mathbf{a} : \mathbf{x}^{\mathbf{a}} = g_\sigma(\mathbf{x}), \sigma \in S_n\}$ be the collection of exponents of descent monomials.

**Proposition 2.5.** $\mathcal{D}_n = \{\mathrm{rev}(\mathrm{cc}(w)) \mid w \in \mathfrak{S}_n\}$.

## 2.2 Symmetric Functions

We refer to [14, Chapter 7] or [12] for background on symmetric functions and the representation theory of the symmetric group. We follow Macdonald's notation [12] for the the elementary symmetric functions $\{e_\lambda\}$, the complete (homogeneous) symmetric functions $\{h_\lambda\}$, and the Schur functions $\{s_\lambda\}$.

We denote the **Frobenius character** of an $S_n$ representation $V$ by $\mathrm{Frob}(V)$. Recall that $\mathrm{Frob}(V_\lambda) = s_\lambda$, where $V_\lambda$ is the irreducible $S_n$ representation indexed by partition $\lambda$. For a graded vector space $V = \oplus_{d\geq 0} V_d$, the **graded Frobenius character** is $\mathrm{Frob}_q(V) = \sum_{d\geq 0} q^d \,\mathrm{Frob}(V_d)$.

## 2.3 Descent words

The descent monomials do not form a standard monomial basis: we can see that they are not closed under monomial divisibility. However, they can be thought of as leading terms with respect to the **descent order** $<_{\mathrm{des}}$, defined below.

**Definition 2.6.** Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$. Define $\mathrm{sort}(\mathbf{a}) := (a_{i_1} \geq \cdots \geq a_{i_n})$ to be the unique way to write the entries of $\mathbf{a}$ in weakly decreasing order. Then, we say that $\mathbf{a} <_{\mathrm{des}} \mathbf{b}$ if:

$$\begin{cases} \mathrm{sort}(\mathbf{a}) <_{\mathrm{lex}} \mathrm{sort}(\mathbf{b}) \text{ or,} \\ \mathrm{sort}(\mathbf{a}) = \mathrm{sort}(\mathbf{b}) \text{ and } \mathbf{a} <_{\mathrm{lex}} \mathbf{b} \end{cases}$$

Note that this does not induce a monomial order on $\mathbb{Q}[\mathbf{x}_n]$, as we do not have $\mathbf{a} <_{\mathrm{des}} \mathbf{b} \implies \mathbf{a}+\mathbf{c} <_{\mathrm{des}} \mathbf{b}+\mathbf{c}$, but is only a total order on monomials of $\mathbb{Q}[\mathbf{x}_n]$. We have that the descent monomials are exactly the leading terms of $R_n$ with respect to $\leq_{\mathrm{des}}$, meaning they are the monomials that do not appear as leading terms of elements in the ideal $\langle e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})\rangle$ with respect to $\leq_{\mathrm{des}}$. The descent order satisfies the following useful lemma.

**Lemma 2.7.** *Let $A \subset [n]$ with complement $A'$. If $\alpha$ is a composition, denote $\alpha|_A := (\alpha_{i_1}, \ldots, \alpha_{i_s})$, where $A = \{i_1 < \cdots < i_s\}$. Then, we have $\alpha|_A \leq_{\mathrm{des}} \beta|_A, \quad \alpha|_{A'} \leq_{\mathrm{des}} \beta|_{A'} \implies \alpha \leq_{\mathrm{des}} \beta$.*

## 2.4 Catabolizability and the Garsia-Procesi ring

In this subsection, we recall results from [2, 8]. For any partition $\mu \vdash n$, define $\mathcal{OSP}_\mu$ to be the collection of ordered set partitions $(A_1|\ldots|A_{\ell(\mu)})$ of $[n]$ where $|A_i| = \mu_i$. Using this, we can define the following set:

$$\mathcal{D}_\mu := \{\mathbf{a} : \mathbf{a}|_{A_i} \in \mathcal{D}_{\mu'_i} \text{ for } (A_1|\ldots|A_{\mu_1}) \in \mathcal{OSP}_{\mu'}\}. \tag{2.1}$$

Note that we take $\mathcal{OSP}_{\mu'}$, so the sizes of the parts correspond to the lengths of the columns. We have that $\mathcal{D}_\mu \subset \mathcal{D}_n$, hence $\{\mathbf{x}^\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_\mu\} \subset \{g_\sigma(\mathbf{x}) \mid \sigma \in \mathfrak{S}_n\}$. Furthermore, we have that this subset is in fact a basis of $R_\mu$ due to Carlsson and the first author.

**Theorem 2.8** (Carlsson–C.[2])**.** *The collection of monomials* $\{\mathbf{x}^\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_\mu\}$ *descends to a basis of* $R_\mu$.

In [8], the second author reformulates this construction in terms of catabolism. To each standard Young tableau $T$, we associate a partition called its **catabolizability type** $\mathrm{ctype}(T)$ (for details, see [10, 13]). We can translate statements about semistandard tableaux to those about subsets of standard tableaux using the following fact. Let $\Sigma_\mu := \{S \in \mathrm{SYT}_n : \mathrm{ctype}(T) \trianglerighteq \mu\}$.

**Theorem 2.9** (Lascoux [10])**.** *Let* $\mathrm{SSYT}(\mu)$ *denote the set of semistandard tableaux of weight* $\mu$. *There exists a bijection* $\mathrm{std} : \mathrm{SSYT}(\mu) \to \Sigma_\mu$ *which preserves shape and cocharge.*

We refer to this map as the **Lascoux standardization map**. This map can be explicitly described using certain crystal operators [13]: from this, we can extend it to consider any word of partition weight, rather than just reading words of semistandard tableaux. This standardization differs from the common standardization map on semistandard tableaux, which standardizes entries from left to right, demonstrated in the following example.

**Example 2.10.** We have that $\begin{array}{|c|c|cc}\hline 2&4&\multicolumn{2}{c}{}\\\hline 1&1&\multicolumn{1}{c|}{2}&\multicolumn{1}{c|}{3}\\\hline\end{array} \xrightarrow{\mathrm{std}} \begin{array}{|c|c|cc}\hline 4&6&\multicolumn{2}{c}{}\\\hline 1&2&\multicolumn{1}{c|}{3}&\multicolumn{1}{c|}{5}\\\hline\end{array}$, which differs from $\begin{array}{|c|c|cc}\hline 3&6&\multicolumn{2}{c}{}\\\hline 1&2&\multicolumn{1}{c|}{4}&\multicolumn{1}{c|}{5}\\\hline\end{array}$.

Using $\mathrm{std} : \mathrm{SSYT}(\mu) \to \Sigma_\mu$, we can replace the indexing set of (1.1) with $\Sigma_\mu$. This new expression for $\widetilde{H}_\mu[X;q]$ matches the second presentation of the descent basis of $R_\mu$ given below. Note that it is not obvious that the two presentations coincide.

**Theorem 2.11** (H [8])**.** $\{\mathbf{x}^{\mathrm{rev}(\mathrm{cc}(w))} : w \in S_n, P(w) \in \Sigma_\mu\}$ *is a monomial basis of* $R_\mu$. *In fact, it coincides with the basis given in Theorem 2.8.*

## 2.5 Δ-Springer module $R_{n,\lambda,s}$

In this section, we review some properties of $R_{n,\lambda,s}$, originally defined by Griffin [6], that we make use of in our work. The first property is the structure of $R_{n,\lambda,s}$ as an ungraded $S_n$-module. To do this, let $\mathcal{OSP}_{n,\lambda,s}$ be the collection of (weak) ordered set partitions $(B_1|\ldots|B_s)$ of $[n]$ where $|B_i| \geq \lambda_i$.

**Theorem 2.12** (Griffin [6, Theorem 3.21])**.** $R_{n,\lambda,s} \simeq_{S_n} \mathbb{Q}\mathcal{OSP}_{n,\lambda,s}$. *In particular, we have* $\mathrm{Frob}(R_{n,\lambda,s}) = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_s)\models n,\\ \lambda\subset\alpha}} h_\alpha$.

From this, we have a formula for $\dim(R_{n,\lambda,s})$ in terms of semistandard tableaux.

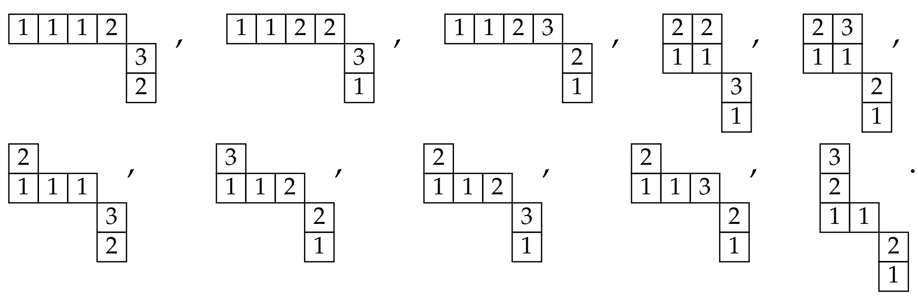
**Corollary 2.13.** $\dim(R_{n,\lambda,s}) = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_s)\models n,\\ \lambda\subset\alpha}} \sum_{T\in \mathrm{SSYT}(\alpha)} |\,\mathrm{SYT}(\mathrm{sh}(T))|.$

We can reformulate Corollary 2.13 in terms of battery-powered tableaux.

**Definition 2.14.** A **battery-powered tableau** of parameters $(n,\lambda,s)$ consists of a pair $T=(D,B)$ of semistandard Young tableaux, where $B$ is of shape $(n-k)\times(s-1)$ and the total content of $(D,B)$ is given by $\Lambda_{n,\lambda,s} := (n-k)\times s+\lambda$.

We refer to $D, B$ as the **device** and **battery** of $T$. Let $\mathcal{T}^{+}_{n,\lambda,s}$ denote the set of all battery-powered tableaux of parameters $(n,\lambda,s)$.

**Example 2.15.** For $(n,\lambda,s) = (4,(2,1),3)$, where $k=|\lambda|=3$, the battery-powered tableaux are:



Note that a battery-powered tableau is uniquely defined by the device: that is, once we know $D$, there is at most one $B$ so that the total tableau has the correct weight. Thus the map $\mathcal{T}^{+}_{n,\lambda,s} \to \bigcup_{\substack{\alpha=(\alpha_1,\ldots,\alpha_s)\models n\\ \lambda\subset\alpha}} \mathrm{SSYT}(\alpha)$ that takes $(D,B)\mapsto D$ is a bijection. Using this, we can restate Corollary 2.13 in terms of $\mathcal{T}^{+}_{n,\lambda,s}$.

**Corollary 2.16.** $\dim(R_{n,\lambda,s}) = \sum_{(D,B)\in\mathcal{T}^{+}_{n,\lambda,s}} |\,\mathrm{SYT}(\mathrm{sh}(D))|.$

# 3 Shuffle Construction

In this section, we will go over the construction used in our main theorem, which gives a descent basis of $R_{n,\lambda,s}$.

**Definition 3.1.** For any $\lambda \vdash k \le n$, define $\mathcal{OSP}_{n,\lambda'}$ to be the collection of ordered set partitions $\sigma = (A_1|\ldots|A_{\lambda_1}\,|\,B_1\,|\ldots|\,B_{n-k}) = (\mathbf{A}_\lambda|\mathbf{B})$ of $[n]$ where $|A_i| = \lambda'_i$ and $|B_j|=1$ for any $i,j$. We denote $\mathbf{B} = \bigsqcup_{i=1}^{n-k} B_i$. We define the $n,\lambda,s$**-generalized descent words** to be

$$\mathcal{D}_{n,\lambda,s} := \{\mathbf{a} : \mathbf{a}|_{A_i} \in \mathcal{D}_{\lambda'_i}, 0 \le \mathbf{a}|_{B_j} < s \text{ for some } \sigma\in\mathcal{OSP}_{n,\lambda'}\}. \tag{3.1}$$

**Example 3.2.** Consider $(n, \lambda, s) = (4, (2, 1), 3)$. Using that $\mathcal{D}_1 = \{0\}, \mathcal{D}_2 = \{00, 01\}$, we have

$$\begin{aligned}\mathcal{D}_{n,\lambda,s} = \{&0000, 1000, 0100, 0010, 0001, 2000, 0200, 0020, 0002,\\ &0101, 0110, 0011, 1001, 1010, 0102, 0120, 0012, 2001, 2010, 0201, 0210\}.\end{aligned}$$

We can state our main theorem using $\mathcal{D}_{n,\lambda,s}$.

**Theorem 3.3.** *The set $\mathcal{B}_{n,\lambda,s} := \{\mathbf{x}^{\mathbf{b}} : \mathbf{b} \in \mathcal{D}_{n,\lambda,s}\}$ is a monomial basis of $R_{n,\lambda,s}$.*

To prove Theorem 3.3, we will show that $|\mathcal{D}_{n,\lambda,s}| = \dim(R_{n,\lambda,s})$ and that $\mathcal{B}_{n,\lambda,s}$ is linearly independent in $R_{n,\lambda,s}$.

**Remark 3.4.** It is difficult to determine $|\mathcal{D}_{n,\lambda,s}|$ using Definition 3.1, since multiple ordered set partitions satisfy the condition in (3.1) for the same word. For $\mathbf{z} = 0110$ in Example 3.2, we have $(12|4|3)$ and $(13|4|2)$ both work.

# 4 Tableaux Construction

To get around the issue described in Remark 3.4 and show that $|\mathcal{D}_{n,\lambda,s}| = \dim(R_{n,\lambda,s})$, we introduce a new indexing set in terms of tableaux and partitions.

$$\Sigma_{n,\lambda,s} := \{(S, \mu) : S \in \mathrm{SYT}_n, S|_k \in \Sigma_\lambda, \mu \subseteq (n-k) \times (s - \mathrm{des}(S) - 1)\}. \tag{4.1}$$

Note that we do not consider $S$ where $s - \mathrm{des}(S) < 0$. Our indexing set specializes naturally to the indexing set in Theorem 2.11: it is immediate that $\Sigma_{k,\lambda,\ell(\lambda)} = \Sigma_\lambda$, where we identify all $S \in \Sigma_\lambda$ with the tuple $(S, \emptyset)$ in $\Sigma_{k,\lambda,\ell(\lambda)}$.

**Example 4.1.** For $(n, \lambda, s) = (4, (2, 1), 3)$, the pairs in $\Sigma_{n,\lambda,s}$ are:

$$\left(\begin{ytableau} 1 & 2 & 3 & 4 \end{ytableau}, \emptyset\right), \quad \left(\begin{ytableau} 1 & 2 & 3 & 4 \end{ytableau}, \ydiagram{1}\right), \quad \left(\begin{ytableau} 1 & 2 & 3 & 4 \end{ytableau}, \ydiagram{2}\right), \quad \left(\begin{ytableau} 3 & 4 \\ 1 & 2 \end{ytableau}, \emptyset\right), \quad \left(\begin{ytableau} 3 & 4 \\ 1 & 2 \end{ytableau}, \ydiagram{1}\right),$$

$$\left(\begin{ytableau} 4 \\ 1 & 2 & 3 \end{ytableau}, \emptyset\right), \quad \left(\begin{ytableau} 4 \\ 1 & 2 & 3 \end{ytableau}, \ydiagram{1}\right), \quad \left(\begin{ytableau} 3 \\ 1 & 2 & 4 \end{ytableau}, \emptyset\right), \quad \left(\begin{ytableau} 3 \\ 1 & 2 & 4 \end{ytableau}, \ydiagram{1}\right), \quad \left(\begin{ytableau} 4 \\ 3 \\ 1 & 2 \end{ytableau}, \emptyset\right).$$

Now, given $(w, \mu)$ where $(P(w), \mu) \in \Sigma_{n,k}$, let $\mathrm{cc}(w, \mu)$ be the word defined by

$$\mathrm{cc}(w, \mu)_i = \begin{cases} \mathrm{cc}(w)_i + \mu_{n-w_i+1} & \text{if } w_i \in \{k+1, \ldots, n\} \\ \mathrm{cc}(w)_i & \text{otherwise} \end{cases} \tag{4.2}$$

In other words, we can think of $\mathrm{cc}(w, \mu)$ as the word we get from "boosting" the cocharge of entries $k+1, \ldots, n$ in $w$ according to $\mu$.

**Example 4.2.** Let $w = 7136524$ and $\mu = (3,2,2)$. Then $\mathrm{cc}(w,\mu) = 7014301$.

We can also define this notion of boosted cocharge directly on $\Sigma_{n,\lambda,s}$. Given $(S,\mu) \in \Sigma_{n,\lambda,s}$, let $\mathrm{cc}(w,\mu)$ be a semistandard tableau of the same shape obtained by taking $\mathrm{cc}(S)$ and adding $\mu_{n-j+1}$ to $\mathrm{cc}(S)_j$ for $j \in \{k+1,\ldots,n\}$.

Now, we construct a bijection between pairs $(w,\mu)$ where $(P(w),\mu) \in \Sigma_{n,\lambda,s}$ and our indexing set $\mathcal{D}_{n,\lambda,s}$, using the notion of boosted cocharge words.

**Proposition 4.3.** *There exists a bijection* $\{(w,\mu) \mid w \in \mathfrak{S}_n, (P(w),\mu) \in \Sigma_{n,\lambda,s}\} \to \mathcal{D}_{n,\lambda,s}$.

In particular, for any $(w,\mu)$, we have that $\mathrm{cc}(w,\mu) = \mathrm{rev}(\mathbf{b})$ for some $\mathbf{b} \in \mathcal{D}_{n,\lambda,s}$. This gives the bijection in Proposition 4.3.

**Example 4.4.** Consider the pair $(1324, \square)$, where $P(1324) = \begin{array}{|c|c|c|}\hline 3 & & \\ \hline 1 & 2 & 4 \\ \hline\end{array}$ and $\left(\left(\begin{array}{|c|c|c|}\hline 3 & & \\ \hline 1 & 2 & 4 \\ \hline\end{array}, \square\right)\right) \in \Sigma_{4,(2,1),3}$ (as seen in Example 4.1). Then, we have $\mathrm{cc}(1324, \square) = 0102 = \mathrm{rev}(2010)$ for $2010 \in \mathcal{D}_{4,(2,1),3}$ (as seen in Example 3.2).

Now, we want to relate the set $\Sigma_{n,\lambda,s}$ to the set of battery-powered tableaux $\mathcal{T}^+_{n,\lambda,s}$ to make use of Corollary 2.16. First, let $\mathrm{std}(\mathcal{T}^+_{n,\lambda,s})$ be the image of the set $\mathcal{T}^+_{n,\lambda,s}$ under Lascoux standardization.

Let $\mathrm{cc}(\mathrm{std}(T))$ denote the cocharge labeling of $\mathrm{std}(T)$ and $\mathrm{cc}(\mathrm{std}(T))|_D, \mathrm{cc}(\mathrm{std}(T))|_B$ denote the device and battery part of $\mathrm{cc}(\mathrm{std}(T))$. Then we have the following fact.

**Lemma 4.5.** *The tableau* $\mathrm{cc}(\mathrm{std}(T))|_B$ *is the superstandard tableau of shape* $(n-k)\times(s-1)$*: that is, the semistandard tableau with row $i$ filled with* $(i-1)$.

The proof of Lemma 4.5 uses properties of catabolizability, as well as an algorithm by Blasiak [1] which computes the catabolizability type of a permutation.

**Remark 4.6.** Lemma 4.5 **does not** hold for $\mathrm{cc}(T)$ for battery-powered tableau $T \in \mathcal{T}^+_{n,\lambda,s}$, as demonstrated in the figures below, where $(n,\lambda,s) = (9,(3,2,1,1),4)$.

$$T = \begin{array}{cccccccc} 4 & & & & & & & \\ 3 & 3 & & & & & & \\ 1 & 1 & 1 & 2 & 2 & 2 & & \\ & & & & & & 4 & 4 \\ & & & & & & 2 & 3 \\ & & & & & & 1 & 1 \end{array} \qquad \mathrm{cc}(T) = \begin{array}{cccccccc} 3 & & & & & & & \\ 2 & 2 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & \\ & & & & & & 1 & 2 \\ & & & & & & 1 & 0 \\ & & & & & & 0 & 0 \end{array}.$$

If we standardize $T$ and then take the cocharge tableau, we get the following.

$$\mathrm{std}(T) = \begin{array}{cccccccc} 15 & & & & & & & \\ 9 & 12 & & & & & & \\ 1 & 2 & 3 & 4 & 5 & 6 & & \\ & & & & & & 13 & 14 \\ & & & & & & 10 & 11 \\ & & & & & & 7 & 8 \end{array} \qquad \mathrm{cc}(\mathrm{std}(T)) = \begin{array}{cccccccc} 3 & & & & & & & \\ 1 & 2 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \\ & & & & & & 2 & 2 \\ & & & & & & 1 & 1 \\ & & & & & & 0 & 0 \end{array}.$$

Now, we construct a map $\Phi$ on $\Sigma_{n,\lambda,s}$. For any $(S,\mu) \in \Sigma_{n,\lambda,s}$ we can construct a boosted cocharge tableau $\mathrm{cc}(S,\mu)$. Let $\Phi(S,\mu)$ be the tableau we get by appending the superstandard tableau of shape $(n-k)\times(s-1)$ to the bottom right corner of $\mathrm{cc}(S,\mu)$.

**Example 4.7.** Consider $(S,\mu) = \left(\begin{ytableau} 3 \\ 1 & 2 & 4 \end{ytableau}, \begin{ytableau} \; \end{ytableau}\right)$. Then we have $\Phi(S,\mu) = \begin{ytableau} 1 \\ 0 & 0 & 2 \\ \none & \none & \none & 1 \\ \none & \none & \none & 0 \end{ytableau}$.

**Lemma 4.8.** *For $(S,\mu) \in \Sigma_{n,\lambda,s}$, we have that $\Phi(S,\mu) = \mathrm{cc}(\mathrm{std}(T))$ for some $T \in \mathcal{T}^{+}_{n,\lambda,s}$.*

Thus $\Phi$ gives us a map $\Sigma_{n,\lambda,s} \to \{\mathrm{cc}(\mathrm{std}(T)) : T \in \mathcal{T}^{+}_{n,\lambda,s}\}$. Using the fact that we can always recover $\mathrm{std}(T)$ from $\mathrm{cc}(\mathrm{std}(T))$, we can think of this as a map $\Sigma_{n,\lambda,s} \to \mathrm{std}(\mathcal{T}^{+}_{n,\lambda,s})$. We can show that this in fact gives a bijection $\Sigma_{n,\lambda,s} \to \mathcal{T}^{+}_{n,\lambda,s}$.

**Proposition 4.9.** *We have a bijection $\Psi : \Sigma_{n,\lambda,s} \xrightarrow{\mathrm{std}} \mathrm{std}(\mathcal{T}^{+}_{n,\lambda,s}) \xrightarrow{\cong} \mathcal{T}^{+}_{n,\lambda,s}$ such that the device of $\Psi(S,\mu)$ is the same shape as $S$ and* $\mathrm{cocharge}(\Psi(S,\mu)) = \mathrm{cocharge}(S) + |\mu|$.

**Example 4.10.** Under $\Psi$, the tableaux in Example 4.1 map to those in Example 2.15 (in the same order).

Combining Proposition 4.9 and Corollary 2.16, we get the following.

**Corollary 4.11.** *We have that* $\dim(R_{n,\lambda,s}) = |\mathcal{D}_{n,\lambda,s}|$.

**Remark 4.12.** Though the map $(D,B) \mapsto D$ gives a more straightforward bijection than $\Phi$ in Proposition 4.9, it does not behave well with respect to cocharge: there is no clear relation between the cocharge of $(D,B) \in \mathcal{T}^{+}_{n,\lambda,s}$ and $D \in \mathrm{SSYT}(\alpha)$ as demonstrated in Remark 4.6. Lemma 4.5 resolves this issue: by standardization, the cocharge values in the battery are forced to be given by the superstandard with $i-1$'s in row $i$.

## 5 Linear Independence

We now show $\mathcal{B}_{n,\lambda,s}$ is linearly independent in $R_{n,\lambda,s}$. Denote by $\widehat{\lambda}' = (\lambda'_2, \ldots, \lambda'_m)$ the partition obtained by removing the first column of $\lambda$ and taking the conjugate.

**Proposition 5.1.** *Let $\lambda \vdash k \leq n$, $s \geq \ell(\lambda)$, and $\widehat{\lambda}$ be as above. Given any subset $A \subset [n]$ of size $|A| = \lambda'_1$, the composite map*

$$\widetilde{\varphi}_A : \mathbb{Q}[\mathbf{x}_n] \to \mathbb{Q}[\mathbf{x}_A] \otimes \mathbb{Q}[\mathbf{x}_{[n]\setminus A}] \to R_{\lambda'_1} \otimes R_{n-|A|,\widehat{\lambda},s}$$

*given by the canonical projection in both tensor components descends to a well-defined map $\varphi_A : R_n \to R_{\lambda'_1} \otimes R_{n,\widehat{\lambda},s}$.*

This follows from showing that each of the generators of $I_{n,\lambda,s}$, where $R_{n,\lambda,s} = \mathbb{Q}[\mathbf{x}]/I_{n,\lambda,s}$, vanish in the image under $\tilde{\psi}_A$. Using the maps from Proposition 5.1, we can prove the following:

**Proposition 5.2.** *Let* $\mathbf{a} \in \mathcal{D}_{n,\lambda,s}$*, and suppose* $\sum_{\mathbf{b} \leq_{\mathrm{des}} \mathbf{a}} c_{\mathbf{b}} x^{\mathbf{b}} = 0$ *in* $R_{n,\lambda,s}$*. Then, we have* $c_{\mathbf{a}} = 0$*. In particular, the monomials* $\mathcal{B}_{n,\lambda,s}$ *are linearly independent in* $R_{n,\lambda,s}$*.*

This proof involves the shuffle construction of $\mathcal{D}_{n,\lambda,s}$ given in Definition 3.1, along with Lemma 2.7.

Our first main result (Theorem 3.3) follows from Proposition 5.2 and Corollary 4.11.

**Remark 5.3.** Our result unifies the seemingly disjoint results on descent bases of $R_\lambda$ and $R_{n,k}$. Since we know $\Sigma_{k,\lambda,\ell(\lambda)} = \Sigma_\lambda$, it is immediate that Theorem 3.3 reduces to Theorem 2.11. If we set $(n, \lambda, s) = (n, 1^k, k)$, we get the generalized descent basis of $R_{n,k}$, defined by Haglund–Rhoades–Shimozono, though their presentation differs from ours.

# 6 Frobenius character of $R_{n,\lambda,s}$

In this section, we use Theorem 3.3 to give a combinatorial formula for the graded Frobenius character $\mathrm{Frob}_q(R_{n,\lambda,s})$ in terms of $\Sigma_{n,\lambda,s}$.

Given a Young subgroup $S_\gamma = S_{\gamma_1} \times \cdots \times S_{\gamma_l} \subset S_n$ indexed by $\gamma \models n$, we define $N_\gamma = \sum_{\sigma \in S_\gamma} \mathrm{sgn}(\sigma)\sigma$ to be the antisymmetrizer with respect to $S_\gamma$. For any $\mathbb{C}S_n$-module $V$, the vector space $N_\gamma V$ is the subspace of elements of $V$ that are antisymmetric with respect to $S_\gamma$.

**Proposition 6.1.** *For any graded* $\mathbb{C}S_n$*-module* $V = \oplus_{d \geq 0} V_d$ *and Young subgroup* $S_\gamma \subset S_n$*, we have* $\mathrm{Hilb}_q(N_\gamma V) = \langle e_\gamma, \mathrm{Frob}_q(V) \rangle$*, where* $\langle -, - \rangle$ *denotes the Hall inner product on symmetric functions.*

In particular, $\mathrm{Frob}_q(V)$ is uniquely determined by $\mathrm{Hilb}_q(N_\gamma V)$ for all partitions $\gamma \vdash n$. From this, it suffices to determine $\mathrm{Hilb}_q(N_\gamma R_{n,\lambda,s})$ for $\gamma \vdash n$. To do this, we will construct a basis of $N_\gamma R_{n,\lambda,s}$ using Theorem 3.3.

For any composition $\gamma \models n$ and word $\mathbf{z}$ of length $n$, we say $\mathbf{z}$ is **strictly increasing on** $\gamma$ if when we divide $\mathbf{z}$ into blocks of sizes $\gamma_1, \gamma_2, \ldots, \gamma_l$, we have that the entries within the blocks are strictly increasing. This condition picks out a subset of $\mathcal{D}_{n,\lambda,s}$ that gives a spanning set of $N_\gamma R_{n,\lambda,s}$.

**Example 6.2.** Consider $\mathbf{z} = 2423679$ and $\gamma = (2, 2, 3)$. Then when we divide $\mathbf{z}$ into three blocks 24|23|679, the entries are strictly increasing. Thus $\mathbf{z}$ is strictly increasing on $\gamma$.

**Lemma 6.3.** *Let* $\mathcal{D}^\gamma_{n,\lambda,s} := \{\mathbf{a} \in \mathcal{D}_{n,\lambda,s} : \mathbf{a}$ *is strictly increasing on* $\gamma\}$*. Then* $\{N_\gamma \mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathcal{D}^\gamma_{n,\lambda,s}\}$ *spans* $N_\gamma R_{n,\lambda,s}$*.*

Since $\{N_\gamma \mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathcal{D}_{n,\lambda,s}\}$ clearly spans $N_\gamma R_{n,\lambda,s}$, we remove immediate linear dependencies in the set. In particular, if $\mathbf{a} \in \mathcal{D}_{n,\lambda,s} \setminus \mathcal{D}^\gamma_{n,\lambda,s}$, there exists a $j$ such that $a_j \geq a_{j+1}$ within the same block. In the case of equality we have $N_\gamma \mathbf{x}^{\mathbf{a}} = 0$. Otherwise, we can find $\mathbf{b} \in \mathcal{D}^\gamma_{n,\lambda,s}$ such that $N_\gamma \mathbf{x}^{\mathbf{a}} = \pm N_\gamma \mathbf{x}^{\mathbf{b}}$ via the following lemma. Note that this is analogous to [2, Lemma 3.4] and the same proof holds.

**Lemma 6.4.** *Consider $\mathbf{a} \in \mathcal{D}_{n,\lambda,s}$ with $a_j > a_{j+1}$ Let $\tilde{\mathbf{a}}$ be the word we get by swapping $a_j$ and $a_{j+1}$. Then $\tilde{\mathbf{a}} \in \mathcal{D}_{n,\lambda,s}$.*

Thus we have a spanning set of $N_\gamma R_{n,\lambda,s}$. We will show that $|\mathcal{D}_{n,\lambda,s}| = \dim(N_\gamma R_{n,\lambda,s})$ using the ungraded $S_n$ structure of $R_{n,\lambda,s}$. We can rewrite formula for $\mathrm{Frob}(R_{n,\lambda,s})$ in Theorem 2.12 to be in terms of the set $\Sigma_{n,\lambda,s}$ by composing the map $(D, B) \mapsto D$ with the bijection $\Psi$ in Proposition 4.9. Using this formula, we can derive the following combinatorial formula for $\dim(N_\gamma R_{n,\lambda,s})$.

**Corollary 6.5.** *Let $\mathrm{Asc}(w)$ denote the* ***ascent set*** *of $w$, defined to be $\mathrm{Asc}(w) = \{i : w_i < w_{i+1}\}$. We have the following:*

$$\dim(N_\gamma R_{n,\lambda,s}) = \sum |\{w \in S_n, P(w) = S, Asc(w) \subset \{\gamma_l, \gamma_l + \gamma_{l-1}, \ldots, \gamma_l + \cdots + \gamma_2\}|,$$

*where the sum ranges over $(S, \mathbf{i}) \in \Sigma_{n,\lambda,s}$.*

Using Proposition 4.3, we get the following relation.

**Lemma 6.6.** *For $\gamma \models n$, we have $\mathrm{rev}(\mathrm{cc}(w, \mu)) \in \mathcal{D}^\gamma_{n,\lambda,s}$ if and only if $\mathrm{Asc}(w) \subset \{\gamma_l, \gamma_l + \gamma_{l-1}, \ldots, \gamma_l + \gamma_{l-1} + \cdots + \gamma_2\}$. In particular, we have $|\mathcal{D}^\gamma_{n,\lambda,s}| = \dim(N_\gamma R_{n,\lambda,s})$.*

Combining Lemmas 6.3, 6.6, we obtain a basis of $N_\gamma R_{n,\lambda,s}$.

**Proposition 6.7.** *$\{N_\gamma \mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathcal{D}^\gamma_{n,\lambda,s}\}$ is a basis of $N_\gamma R_{n,\lambda,s}$.*

Using the expression of $\mathrm{Hilb}_q(N_\gamma R_{n,\lambda,s})$ that we get from Proposition 6.7, we obtain an explicit Schur expansion of $\mathrm{Frob}_q(R_{n,\lambda,s})$ using $\Sigma_{n,\lambda,s}$.

**Theorem 6.8.** $\mathrm{Frob}_q(R_{n,\lambda,s}) = \sum_{(S,\mu) \in \Sigma_{n,\lambda,s}} q^{\mathrm{cocharge}(S) + |\mu|} s_{\mathrm{sh}(S)}$

**Remark 6.9.** Theorem 6.8 unifies the known formulas for $\mathrm{Frob}_q(R_\lambda)$ and $\mathrm{Frob}_q(R_{n,k})$.

- When $(n, \lambda, s) = (k, \lambda, \ell(\lambda))$, we immediately recover (1.1) up to $\mathrm{std} : \mathrm{SSYT}(\lambda) \to \Sigma_\lambda$ and Theorem 2.9.
- When $(n, \lambda, s) = (n, 1^k, k)$, we recover (1.2) using the equidistribution of maj and cocharge with respect to descents.

Furthermore, applying $\Psi : \Sigma_{n,\lambda,s} \to \mathcal{T}^+_{n,\lambda,s}$ from Proposition 4.9 to Theorem 6.8 recovers the battery-powered tableaux formula (1.3). From this, our proof of Theorem 6.8, along with Proposition 4.9 provides an algebraic and combinatorial proof for (1.3).

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