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Chow classes of matroids and standard Young tableaux

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Abstract. We study the Chow classes of arbitrary matroids in the Grassmannian. Our main contribution identifies the Poincaré dual of the Chow class of a snake matroid with a specific ribbon Schur function, providing an explicit formula for its coefficients in the Schubert basis as the number of standard Young tableaux of a given shape with a prescribed descent set. This extends to a standard Young tableaux count for the Chow class of any lattice path matroid, and by valuativity to a new expression of this class for any matroid. As consequences, we recover an enumeration by Gessel and Viennot of permutations with fixed descent sets, and provide a formula for the volume of lattice path matroids. Furthermore, we demonstrate the power of our findings by proving that certain Schubert coefficients are positive for all connected paving matroids.

Keywords: matroid theory, Schubert calculus, symmetric functions, ribbon Schur functions, valuative matroid invariants, standard Young tableaux

1 Introduction

The algebraic torus $T = (\mathbb{C}^*)^n$ acts naturally on the complex Grassmannian $G(k,n)$ of k-subspaces of $\mathbb{C}^n$. To each point x in $G(k,n)$ one may associate the *torus orbit closure* $\overline{Tx}$, i.e., the Zariski closure of the T-orbit of x. It is a classical result that $\overline{Tx}$ is isomorphic to the projective toric variety of the matroid (base) polytope of the linear matroid M_x, which is represented by the columns of any matrix whose rowspace is x [10]. In light of this result, it is natural to expect that properties of $\overline{Tx}$ solely depend on the combinatorics of the matroid M_x. One example of such a property is the class of $\overline{Tx}$ in the Chow ring $A^\bullet(G(k,n))$, as studied in [9]. This notion can be extended uniquely to a valuative invariant defined on arbitrary (not necessarily $\mathbb{C}$-representable) matroids. For a matroid

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M of rank k on a ground set of size n, we call its class $\mathsf{Sc}(\mathsf{M}) \in A^\bullet(G(k,n))$ the *Chow class* of M and write

$$\mathsf{Sc}(\mathsf{M}) \;=\; \sum_\eta d_\eta(\mathsf{M})\, s_\eta\,,$$

where we use that the Chow ring $A^\bullet(G(k,n))$ is isomorphic to the quotient of the ring of symmetric functions by the ideal $\langle s_\eta \mid \eta \not\subseteq k \times (n-k)\rangle$, and s_η is the Schur function associated to a partition $\eta \subseteq k \times (n-k)$. The unique integers $d_\eta(\mathsf{M})$ appearing in this expansion are called the *Schubert coefficients* of the matroid M.

The Chow classes of matroids can be computed using equivariant K-theory [9, 3] or tautological bundles [2]. These methods are, however, relatively sophisticated and computationally expensive. This is one of the difficulties in the attempt to understand these Chow classes, and it is therefore of interest to develop new techniques that are more direct and conceptually more transparent.

The central contribution of this paper is a purely combinatorial description of the Chow classes of matroids, which sheds new light on their properties and allows for drastically more efficient computations. Our first main result is the following.

Theorem. *Let* S *be the snake matroid associated to the ribbon* ρ*. Then the Poincaré dual of its Chow class is equal to the ribbon Schur function* s_ρ*.*

Ribbon Schur functions are well studied objects, see for example [15, 17, 21]. Together with a theorem by Gessel [11], our result provides the following combinatorial interpretation. If $\mathbf{b}$ is a composition, let $\rho(\mathbf{b})$ be the ribbon with row lengths given by $\mathbf{b}$. Then, the Schubert coefficient d_{η^c} of the snake matroid with shape $\rho(\mathbf{b})$ is the number of standard Young tableaux of shape η with descent set $\{\sum_{i=1}^s b_i \mid s \leq k-1\}$. This extends to all lattice path matroids. Let $\mathsf{M} = \mathsf{M}[\mathbf{L},\mathbf{U}]$ be a connected lattice path matroid of rank k, and let $\mathcal{A} = \{\,[u_{i+1}-1, \ell_i - 1] \mid 1 \leq i \leq k-1\}$ where u_i (resp. ℓ_i) denote the index of the i-th north step of $\mathbf{U}$ (resp. $\mathbf{L}$).

Theorem. *Let* M *be a connected lattice path matroid of rank* k *on* $[n]$*, and let* η *be a partition of* $n-1$ *in* $k \times (n-k)$*. The Schubert coefficient of* M *associated to* η^c *is given by*

$$d_{\eta^c}(\mathsf{M}) \;=\; \left|\left\{\, T \in \mathrm{SYT}_\eta \;\middle|\; \mathrm{Des}(T) \textit{ is a transversal of } \mathcal{A} \right\}\right|\,.$$

By results of [7] it is known that any valuative invariant is completely determined by its values on lattice path matroids. Together with results in [13] the above theorem provides a concrete computational algorithm for evaluating the Chow class of any matroid. We demonstrate this algorithm in Section 4. In section 5 we apply our results. We give a formula for the volume of the matroid polytope of a lattice path matroid (Corollary 5.2). And we reprove a determinantal formula by Gessel and Viennot for the number of permutations with fixed descent set (Theorem 5.3). We also give a sufficient condition for when the Schubert coefficients of a snake matroid is zero (Proposition 5.4).

Furthermore, we make some progress on a conjecture by Berget and Fink [3], according to which all Schubert coefficients are nonnegative. The conjecture is true for matroids representable over $\mathbb{C}$, since their Schubert coefficients count intersection points of $\overline{Tx}$ with certain Schubert varieties. In addition, the conjecture has been verified for sparse paving matroids [14], when η^c is a hook [23], and when η^c has two rows or two columns [2, Section 9]. We use our combinatorial interpretation to prove positivity for certain shapes η when the matroid is paving (Theorem 5.6).

2 Preliminaries and notation

2.1 Matroids

Definition 2.1. *A matroid* M *of rank* k *on* $[n]$ *is given by a nonempty set of bases* $\mathcal{B} \subseteq \binom{[n]}{k}$ *satisfying the exchange axiom*

$$\forall B_1, B_2 \in \mathcal{B}\ \forall x \in B_1\ \exists y \in B_2 \ s.t.\ B_1 \setminus \{x\} \cup \{y\} \in \mathcal{B}\,.$$

The class of all matroids of rank k on $[n]$ will be denoted by $\mathcal{M}_{n,k}$. To any matroid M we associate the *matroid (base) polytope*,

$$\mathscr{P}(\mathsf{M}) \ = \ \operatorname{conv}\{\, e_B \in \mathbb{R}^n \mid\ B \in \mathcal{B}(\mathsf{M})\} \ ,$$

where $e_B = \sum_{i \in B} e_i$ is a sum of standard basis vectors. The most important examples of matroids in this article fall into the class of *lattice path matroids*, which were first introduced in [5]. Let $k \leq n$ and choose two lattice paths $\mathbf{U}$ and $\mathbf{L}$ from $(0,0)$ to $(k, n-k)$ using steps $\mathbf{N} = (0,1)$ and $\mathbf{E} = (1,0)$, so that $\mathbf{U}$ always stays above $\mathbf{L}$. The collection

$$\mathcal{B}(\mathbf{L},\mathbf{U}) \ = \ \{\, B(T) \mid\ T \text{ is always between } \mathbf{L} \text{ and } \mathbf{U}\} \ ,$$

where $B(T)$ is the set of indices of north steps of T, forms the set of bases of a matroid of rank k on $[n]$ that we denote by $\mathsf{M}[\mathbf{L},\mathbf{U}]$. A matroid that arises in this way is called a *lattice path matroid*. All lattice path matroids are representable over infinite fields.

A connected lattice path matroid of rank k on $[n]$ can also be encoded by a skew shape λ/μ, where λ has k parts and $\lambda_1 = n-k$, and we sometimes use the alternative notation $\mathsf{M}(\lambda/\mu)$. A *snake matroid* is a connected lattice path matroid whose associated skew shape is a ribbon, i.e., a skew shape not containing a 2×2 square. If $\mathbf{b}$ is a composition, we write $\rho(\mathbf{b})$ for the ribbon whose i-th row from the bottom is of length b_i. Let $\mathsf{S}(\mathbf{b})$ be the snake matroid $\mathsf{M}(\rho(\mathbf{b}))$.

Example 2.2. *The skew shape defining the snake matroid* $\mathsf{S}(2,1,2,3)$ *is depicted in Figure 1. It is the lattice path matroid* $\mathsf{M}(\lambda/\mu)$ *of the ribbon* λ/μ *where* $\lambda = [5,3,2^2]$ *and* $\mu = [2,1^2]$.

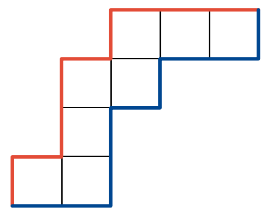


Figure 1: The snake matroid $\mathsf{S}(2,1,2,3)$ from Example 2.2. We highlighted the upper path $\mathbf{U} = \mathbf{NENNENEEE}$ (red) and lower path $\mathbf{L} = \mathbf{EENNENEEN}$ (blue).

2.2 The Chow class of a matroid

For nonnegative integers $k \leq n$ we denote the Grassmannian of k-dimensional linear subspaces of $\mathbb{C}^n$ by $G(k,n)$. For any point $x \in G(k,n)$ we denote by M_x the matroid of rank k on $[n]$ represented by the columns of a matrix whose rowspace is x. That is, the bases of M_x are the sets $B \in \binom{[n]}{k}$ such that the Plücker coordinate $p_B(x)$ is nonzero. The algebraic torus $T = (\mathbb{C}^*)^n$ acts on $G(k,n)$ by scaling the standard coordinates of $\mathbb{C}^n$. We are interested in the *torus orbit closure* of x, i.e., the Zariski closure of the T-orbit of x, denoted by $\overline{Tx}$. The study of the relationship between $\overline{Tx}$ and the matroid M_x was initiated by Gelfand, Goresky, MacPherson and Serganova in [10]. The authors showed that $\overline{Tx}$ is isomorphic to the projective toric variety given by the matroid polytope $\mathscr{P}(\mathsf{M}_x)$. In light of this result it is natural to expect that properties of $\overline{Tx}$ depend solely on the matroid M_x. An example is the class $[\overline{Tx}]$ in the Chow ring $A^\bullet(G(k,n))$, which is the main object of interest in the remainder of this article. Throughout, we are going to identify the Chow ring $A^\bullet(G(k,n))$ with the quotient of the ring of symmetric functions by the ideal $\langle s_\eta \mid \eta \not\subseteq k \times (n-k)\rangle$, where s_η is the Schur function associated to the partition η. We also observe that the set $\{s_\eta \mid \eta \subseteq k \times (n-k)\}$ is an additive basis for $A^\bullet(G(k,n))$. In particular, if Y is a subvariety of $G(k,n)$ of codimension m, then we are able to write the Chow class $[Y] \in A^m(G(k,n))$ as a linear combination $[Y] = \sum_\eta d_\eta s_\eta$, where the sum is taken over all partitions $\eta \vdash m$ with $\eta \subseteq k \times (n-k)$. The torus orbit closure $\overline{Tx}$ is of dimension $n - \kappa(\mathsf{M}_x)$, where $\kappa(\mathsf{M})$ denotes the number of *connected components* of M, hence of codimension $m := k(n-k) - (n - \kappa(\mathsf{M}_x))$. Thus we may write

$$[\overline{Tx}] \;=\; \sum_{\substack{\eta \vdash m \\ \eta \subseteq k \times (n-k)}} d_\eta s_\eta \;\in\; A^m(G(k,n)) \,.$$

It follows from [23, Proposition A.5] that the class $[\overline{Tx}]$, and each coefficient d_η only depends on the isomorphism class of the matroid M_x. The coefficients d_η are called the *Schubert coefficients* of the representable matroid M_x. A function on matroids is called *valuative* if it satisfies an inclusion-exclusion property on matroid base polytope subdivisions. If such a function is defined only for representable matroids, one can uniquely extend it to a valuative function on all matroids, see [7]. This is the case for the map $\mathsf{M}_x \mapsto [\overline{Tx}]$.

Theorem 2.3 ([9]). *Fix integers $0 \leq k \leq n$ and $(k-1)(n-k-1) \leq m \leq k(n-k)$. There exists a unique valuative matroid invariant*

$$\mathrm{Sc}\colon \mathcal{M}_{n,k} \longrightarrow A^m(G(k,n))\,,$$

which is zero when $m \neq k(n-k) - (n - \kappa(\mathsf{M}))$ and such that $\mathrm{Sc}(\mathsf{M}) = [\overline{Tx}]$ whenever $\mathsf{M} = \mathsf{M}_x$.

Definition 2.4. *Let M be a matroid of rank k on $[n]$. The* Schubert coefficients *of the matroid M are the integers $d_\eta(\mathsf{M})$ such that*

$$\mathrm{Sc}(\mathsf{M}) \;=\; \sum_\eta d_\eta(\mathsf{M}) s_\eta\,.$$

We also consider the Poincaré dual of the class $\mathrm{Sc}(\mathsf{M})$ and use the notation

$$\mathrm{Sc}^c(\mathsf{M}) \;=\; \sum_\eta d_\eta(\mathsf{M}) s_{\eta^c}\,.$$

Example 2.5. *Corollary 3.7 below provides a formula for the Schubert coefficients of uniform matroids. This gives that the Chow class of $\mathsf{U}_{3,6}$ is*

$$\mathrm{Sc}(\mathsf{U}_{3,6}) \;=\; 3s_{\ydiagram{3,1}} + 6s_{\ydiagram{2,2}} + 3s_{\ydiagram{2,1,1}}\;.$$

The dual class is

$$\mathrm{Sc}^c(\mathsf{U}_{3,6}) \;=\; 3s_{\ydiagram{3,2}} + 6s_{\ydiagram{3,1,1}} + 3s_{\ydiagram{2,2,1}}\;.$$

It follows from Theorem 2.3 that for a fixed partition η, the Schubert coefficient

$$d_\eta\colon \mathcal{M}_{n,k} \longrightarrow \mathbb{Z}$$

is a valuative matroid invariant. When $\mathsf{M} = \mathsf{M}_x$ for some $x \in G(k,n)$ the Schubert coefficients of M are nonnegative integers, as they count the number of points in an intersection of varieties. A main open problem stated in [3, Conjecture 9.13] is whether this holds for arbitrary matroids.

Conjecture 2.6. *For any matroid M and partition η, $d_\eta(\mathsf{M})$ is nonnegative.*

In [23, Theorem 5.1] Speyer recognizes one of the Schubert coefficients as the well known β-invariant. Specifically, $d_{h^c}(\mathsf{M}) = \beta(\mathsf{M})$ where h is the hook partition $h = [n-k, 1^{k-1}]$. It is well known that the β-invariant is nonnegative for all matroids, see [6]. In [14, Theorem 4.1] the first author showed how to compute the Schubert coefficients of a matroid in terms of its direct summands, reducing Conjecture 2.6 to the case of connected matroids. Moreover [14, Theorem 1.1] shows that Schubert coefficients of sparse paving matroids are nonnegative. This class of matroids is conjectured to be predominant among all matroids, see [18].

3 Chow classes of snake matroids

In this section, our aim is to prove that the dual Chow class of a snake matroid is a ribbon Schur function (Theorem 3.3). The key tool is the following identity.

Proposition 3.1. *Let* $\mathbf{b} = (b_1, \dots, b_k, b_{k+1})$*. For the snake matroid* $\mathsf{S}(\mathbf{b})$

$$\mathrm{Sc}^c(\mathsf{S}(\mathbf{b})) \;=\; s_{[b_k]}\,\mathrm{Sc}^c(\mathsf{S}(\mathbf{b}')) - \mathrm{Sc}^c(\mathsf{S}(\mathbf{b}''))\,.$$

where $\mathbf{b}' = (b_1, \dots, b_k)$ *and* $\mathbf{b}'' = (b_1, \dots, b_{k-1}, b_k + b_{k+1})$*.*

Remark 3.2. *We prove that a more general identity in terms of series and parallel extensions holds for any matroid. The formula above follows since the class of snake matroids is closed under such operations.*

Since $\mathsf{S}(\mathbf{b}')$ and $\mathsf{S}(\mathbf{b}'')$ have strictly lower rank than $\mathsf{S}(\mathbf{b})$, this result can be used as a recursion. Unraveling it and expressing the resulting formula as a determinant of Schur functions leads to the following expression for the dual Chow class of snake matroids.

Theorem 3.3. *Let* S *be a snake matroid given by the ribbon* $\rho(\mathbf{b})$*. Then*

$$\mathrm{Sc}^c(\mathsf{S}) \;=\; s_{\rho(\mathbf{b})}\,,$$

where $s_{\rho(\mathbf{b})}$ *is the ribbon Schur function of shape* $\rho(\mathbf{b})$ *.*

3.1 A descent statistic interpretation

Theorem 3.3 tells us that the Schubert coefficients of snake matroids are the well-studied Littlewood–Richardson coefficients appearing in the Schur expansion of a ribbon Schur function. These coefficients were first studied by MacMahon in [17]. We follow Gessel who described them in the following combinatorial fashion, see [11]. A *descent* in a standard Young tableau is an entry i such that the entry $i+1$ is in a row below i. For a standard Young tableau T we denote the set of descents of T by $\mathrm{Des}(T)$, and the number of descents by $\mathrm{des}(T) = |\,\mathrm{Des}(T)|$. For a given composition $\mathbf{b}$ of length k and a partition η of $\sum_i b_i$, let $\mathrm{SYT}_\eta(\mathbf{b})$ denote the set of standard Young tableaux T of shape η with

$$\mathrm{Des}(T) \;=\; \mathrm{Des}(\mathbf{b}) := \left\{ \sum_{i=1}^{s} b_i \;\middle|\; 1 \le s \le k-1 \right\}. \tag{3.1}$$

Gessel describes the Littlewood–Richardson coefficients $c^{\lambda}_{\mu,\eta}$ when λ/μ is a ribbon in terms of standard Young tableaux with a predetermined descent set. The following theorem is a consequence of Theorem 3.3 and [11, Theorem 7].

Theorem 3.4. *Let η be a partition of $n-1$ and $\mathsf{S} = \mathsf{S}(\mathbf{b})$ a snake matroid of rank k on the ground set $[n]$. The Schubert coefficient of S associated to η^c is given by*

$$d_{\eta^c}(\mathsf{S}(\mathbf{b})) \;=\; |\mathrm{SYT}_\eta(\mathbf{b})|\,.$$

Example 3.5. *Consider the snake matroid $\mathsf{S} = \mathsf{S}(2,1,2,3)$. The standard Young tableaux with descent set given by (3.1) are illustrated in Figure 2. By Theorem 3.4 it follows that*

$$\mathrm{Sc}^c(\mathsf{S}) \;=\; s_{(5,2,1)} + s_{(5,1,1,1)} + s_{(4,3,1)} + s_{(4,2,2)} + 2s_{(4,2,1,1)} + s_{(3,3,1,1)} + s_{(3,2,2,1)}\,.$$

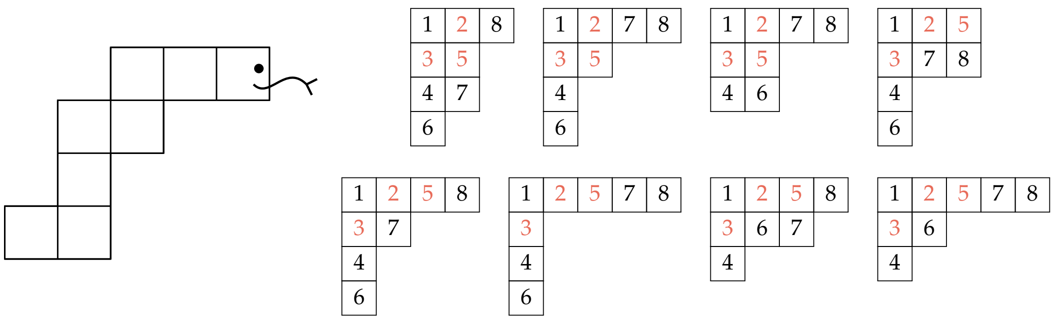


Figure 2: The snake matroid $\mathsf{S}(2,1,2,3)$ and the standard Young tableaux with their descent sets highlighted in red.

3.2 Schubert coefficients of lattice path matroids

We now extend Theorem 3.4 to give a combinatorial formula for the Schubert coefficients of connected lattice path matroids, in particular uniform matroids. Let $\mathsf{M} = \mathsf{M}[\mathbf{L},\mathbf{U}]$ be a connected lattice path matroid of rank k, and consider the set system

$$\mathcal{A} \;=\; \{\,[u_{i+1}-1, \ell_i - 1] \mid 1 \le i \le k-1\}$$

where u_i (resp. ℓ_i) denote the index of the i-th north step of $\mathbf{U}$ (resp. $\mathbf{L}$).

Theorem 3.6. *Let M be a connected lattice path matroid of rank k on $[n]$ and let η be a partition of $n-1$ in $k \times (n-k)$. Then with notation as above,*

$$d_{\eta^c}(\mathsf{M}) \;=\big|\big\{\,T \in \mathrm{SYT}_\eta \;\big|\; \mathrm{Des}(T) \textit{ is a transversal of } \mathcal{A}\big\}\big|\;.$$

In the case of the uniform matroid $\mathsf{U}_{k,n}$ this recovers a theorem proved independently by Klyachko in [16] and Nadeau and Tewari in [22].

Corollary 3.7. *Let η be a partition of $n-1$ in $k \times (n-k)$. The Schubert coefficient of the uniform matroid $\mathsf{U}_{k,n}$ associated to η^c is given by*

$$d_{\eta^c}(\mathsf{U}_{k,n}) \;=\; |\mathrm{SYT}_\eta^{k-1}|\,.$$

4 Computing the Chow class of an arbitrary matroid

It is a result by Derksen and Fink [7] that any valuative matroid invariant is completely determined by its values on *nested* matroids. These matroids are, up to isomorphism, lattice path matroids with lower path $\mathbf{L} = \mathbf{E}^{n-k}\mathbf{N}^k$. To be more precise, for any matroid M, there are unique integers $c(\mathsf{N},\mathsf{M})$ associated to any nested matroid N, of the same rank and size as M, such that for any valuative function on matroids f we have

$$f(\mathsf{M}) \;=\; \sum_{\mathsf{N}} c(\mathsf{N},\mathsf{M}) f(\mathsf{N})\,.$$

In [13] Hampe provides a method for computing these coefficient in terms of the *lattice of cyclic flats* of the matroid. The aim of this section is to demonstrate how to use Hampe's method together with Theorem 3.6 to compute the Chow class of an arbitrary matroid. For this purpose consider the matroid M of rank 3 on $[1,7] = \{1,\ldots,7\}$ whose lattice of cyclic flats $\mathcal{Z}(\mathsf{M})$ is depicted on the left in Figure 3.

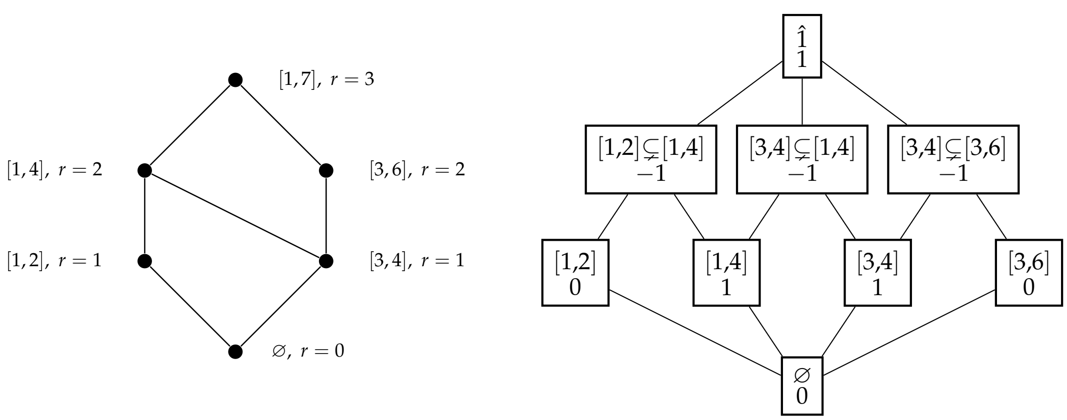
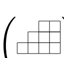
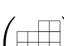
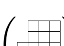


Figure 3: The ranked lattice of cyclic flats and the Möbius values on the poset of chains.

Each chain in $\mathcal{Z}(\mathsf{M})$ corresponds to a nested matroid N and by [13, Theorem 3.12] the coefficient $c(\mathsf{N},\mathsf{M})$ is equal to $-\mu(\mathsf{N},\hat{1})$ where μ is the Möbius function on the chain lattice of $\mathcal{Z}(\mathsf{M})$. This chain lattice and the values of its Möbius function are depicted on the right in Figure 3. In our example, for the valuative invariant Sc, we get five summands leading us to the equation

$\mathrm{Sc}(\mathsf{M}) \;=\; 3\,\mathrm{Sc}\big($  $\big) - \mathrm{Sc}\big($  $\big) - \mathrm{Sc}\big($  $\big)\,.$ (4.1)

Using Theorem 3.6 we obtain

$$
\begin{aligned}
\mathrm{Sc}\left(\ydiagram{2+2,1+3,4}\right) &= 3s_{\ydiagram{4,2}} + 5s_{\ydiagram{3,3}} + 2s_{\ydiagram{4,1,1}} + 5s_{\ydiagram{3,2,1}} + s_{\ydiagram{2,2,2}} \\
\mathrm{Sc}\left(\ydiagram{2+2,4,4}\right) &= 5s_{\ydiagram{4,2}} + 7s_{\ydiagram{3,3}} + 3s_{\ydiagram{4,1,1}} + 6s_{\ydiagram{3,2,1}} + s_{\ydiagram{2,2,2}} \\
\mathrm{Sc}\left(\ydiagram{1+3,1+3,4}\right) &= 3s_{\ydiagram{4,2}} + 6s_{\ydiagram{3,3}} + 2s_{\ydiagram{4,1,1}} + 6s_{\ydiagram{3,2,1}} + s_{\ydiagram{2,2,2}}\,.
\end{aligned}
$$

Now (4.1) gives

$$
\mathrm{Sc}(\mathsf{M}) \;=\; s_{\ydiagram{4,2}} + 2s_{\ydiagram{3,3}} + s_{\ydiagram{4,1,1}} + 3s_{\ydiagram{3,2,1}} + s_{\ydiagram{2,2,2}}\,.
$$

Alternatively, one can further decompose each of the matroids in (4.1) into the snake matroids contained in their skew shape, see [4]. By Theorem 3.3 this yields the following expression for the Poincaré dual of $\mathrm{Sc}(\mathsf{M})$,

$$
\mathrm{Sc}^c(\mathsf{M}) \;=\; s_{\ydiagram{2+2,1+2,2}} + s_{\ydiagram{3+1,1+3,2}} + s_{\ydiagram{2+2,2+1,3}} + s_{\ydiagram{3+1,2+2,3}} + s_{\ydiagram{3+1,3+1,4}} - s_{\ydiagram{2+2,3,1}} - s_{\ydiagram{3+1,4,1}} - s_{\ydiagram{1+3,2,1}}\,.
$$

5 Applications

5.1 Volume

In [1, Theorem 5.4] the authors give a recursive formula for the volume of the matroid base polytope of lattice path matroids of rank 2. With our results we are able to provide a concrete formula for the volume of any lattice path matroid. A helpful observation is the following.

Lemma 5.1 ([14, Proposition 5.1])**.** *Let* M *be a matroid of rank* k *on* $[n]$*. Then*

$$
\sum_{\eta} |\mathrm{SYT}_{\eta^c}|\; d_\eta(\mathsf{M}) \;=\; \mathrm{V}(\mathsf{M})\,.
$$

Combining this with the Robinson–Schensted–Knuth correspondence and Theorem 3.6 gives a proof of the following formula.

Corollary 5.2. *Let* M *be a connected lattice path matroid, and let* $\mathcal{A}$ *be the corresponding set system as in Theorem 3.6. Then the normalized volume of* $\mathscr{P}(\mathsf{M})$ *equals the number of permutations* $\pi \in \mathfrak{S}_{n-1}$ *such that* $\mathrm{Des}(\pi)$ *is a transversal of* $\mathcal{A}$*.*

5.2 Permutations with fixed descent set

We use our results to provide a simple proof for a determinantal formula by Gessel and Viennot for the number of permutations with fixed descent set. Let $A = (a_{ij})_{i,j\geq 0}$ be the infinite matrix where $a_{ij} = \binom{i}{j}$ for $j \leq i$ and 0 for $j > i$. A *binomial determinant* is any minor of A. The minor corresponding to rows $0 \leq a_1 < a_2 < \ldots < a_k$ and columns $0 \leq b_1 < b_2 < \ldots < b_k$ is denoted by $\binom{a_1,\ldots,a_k}{b_1,\ldots,b_k}$.

Theorem 5.3 ([12, Corollary 6])**.** *If $\mathcal{D} = \{d_1, \ldots, d_k\} \subseteq [n-2]$, the number of permutations in $\mathfrak{S}_{n-1}$ whose descent set is $\mathcal{D}$ is*

$$\binom{\mathcal{D} \cup \{n-1\}}{\{0\} \cup \mathcal{D}}.$$

Sketch of proof. Corollary 5.2 applied to the snake matroid $\mathsf{S}(\mathbf{b})$ with $\mathrm{Des}(\mathbf{b}) = \mathcal{D}$ shows that the volume of this snake matroid is exactly the number of permutations in $\mathfrak{S}_{n-1}$ with descent set $\mathcal{D}$. Expanding the minor $\binom{\mathcal{D}\cup\{n-1\}}{\{0\}\cup\mathcal{D}}$ along the last column gives a recursion. Using Lemma 5.1 and Proposition 3.1 one can see that the same recursion is satisfied by the volume of the snake matroids. Now Theorem 5.3 follows inductively. □

5.3 Support

We define the support of a matroid M of rank k on $[n]$ to be the set

$$\mathrm{supp}(\mathsf{M}) \ = \ \left\{\eta \vdash n - \kappa(\mathsf{M}) \mid \ d_{\eta^c}(\mathsf{M}) \neq 0\right\} .$$

When restricting to snake matroids, by Theorem 3.3 this is the same as the support of ribbon Schur functions. This has been the subject of much work, see for example [19, 15, 20] and references therein. By considering how we may build the standard Young tableaux in Theorem 3.4 recursively we prove the following necessary condition for a partition to be in the support of a snake matroid.

Proposition 5.4. *Let S be a snake matroid given by the ribbon $\rho = \lambda/\mu$, and η a partition. If $\eta \in \mathrm{supp}(\mathsf{S})$ then $\eta \subseteq \lambda$.*

5.4 Positivity

In this section we provide a new nonnegativity result in support of Conjecture 2.6. Consider the partitions of the form

$$\eta(m) = [n-k, m+1, 1^{k-m-2}]$$

and the lattice path matroids of the form

$$\Lambda_{k,h,n} := \mathsf{M}(\lambda/\mu)$$

with $\lambda = [n-k, (h-k+1)^{k-1}]$ and $\mu = \emptyset$. These matroids are also known as *panhandle matroids* and fall inside the larger class of cuspidal matroids. Using Theorem 3.6 we get the following explicit formulas.

Lemma 5.5. *Let $\eta(m)$ be the partition defined above. Then*

$$\begin{aligned} d_{\eta(m)^c}(\mathsf{U}_{k,n}) &= \frac{n-k}{k-1}\binom{n-m-2}{n-k}\binom{n-k-1}{m} \text{ and} \\ d_{\eta(m)^c}(\Lambda_{k,h,n}) &= \frac{h-k+1}{k-1}\binom{h-m-1}{h-k+1}\binom{h-k}{m}. \end{aligned}$$

Panhandle matroids appear in the subdivisions of the matroid polytope of uniform matroids involving paving matroids. Combining our explicit formulas with results from [8], we prove the following result.

Theorem 5.6. *Let* M *be a connected paving matroid of rank* k *on* n *elements and* $\eta(m)$ *the partition we defined above. Then* $d_{\eta(m)^c}(\mathsf{M}) > 0$.

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