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Odd Shifted Parking Functions

Zachary Hamaker, Jesse Kim

[1]*Department of Mathematics, University of Florida*

Abstract. Stanley recently introduced the shifted parking function symmetric function SH_n as the shiftification of Haiman's parking function symmetric function PF_n. The function SH_n lives in the subalgebra of symmetric functions generated by odd power sums. Stanley showed how to expand SH_n into the V–basis of this algebra, which is indexed by partitions with all parts odd and is analogous to the complete homogeneous (or elementary) basis of symmetric functions. We introduce *odd shifted parking functions* to give combinatorial and representation-theoretic realizations of the V–expansion of SH_n, resolving the main open problem in his paper. We conclude with further directions, including a relationship between SH_n and Haglund's (q,t)–Schröder theorem.

Keywords: parking functions, spin characters, symmetric functions

1 Introduction

A *parking function* of size n is a tuple $(a_1, \ldots, a_n)$ of positive integers so that the ith smallest entry is at most i. The symmetric group $\mathfrak{S}_n$ acts on parking functions of size n by permuting their entries. The *parking function symmetric function* PF_n is the Frobenius character of this $\mathfrak{S}_n$–action on the set $\mathrm{Pf}(n)$ of parking functions of size n. Parking functions have been studied extensively in the combinatorics community (see [14] for a recent survey), in large part due to Haiman's observation that a natural bi-graded version of PF_n is the Frobenius character for the space of diagonal harmonics [7, 8].

The *shiftification map* $\mathtt{sh}$, sometimes denoted Θ, is an operator on symmetric functions Sym defined on power sums by $\mathtt{sh}(p_{2k-1}) = 2p_{2k-1}$ and $\mathtt{sh}(p_{2k}) = 0$. The image of shiftification is the subalgebra SymP generated by odd power sums. The algebra SymP has several bases, for instance $\{p_\lambda : \lambda \text{ odd}\}$ where λ is *odd* if each λ_i is. The *Schur–P functions* P_λ, indexed by partitions with all parts distinct, form a basis analogous to Schur functions. For k a positive integer, single part Schur–P functions satisfy the relation

$$\sum_{i=1}^{k} P_{2i} \cdot P_{2(k-i)} = \sum_{i=1}^{k} P_{2i-1} \cdot P_{2(k-i)+1}. \tag{1.1}$$

Therefore, with $V_\lambda = \prod_i P_{\lambda_i}$ we see $\{V_\lambda : \lambda \text{ odd}\}$ is a basis of SymP as well.

Let $\bigwedge(\mathbb{C}^n)$ be the exterior algebra and $\mathfrak{S}_n$ act on it by permuting standard basis elements. It is well-known that the Frobenius character $\mathtt{ch}(\bigwedge(\mathbb{C}^n)) = 2P_n$ [10, III.8 Ex. 6(b)]. Meanwhile, shiftification has an alternate characterization as $f \mapsto f * 2P_n$ where $*$ is the Kronecker product [13, Prop. 9.1]. Therefore:

Proposition 1.1. *Let M be an $\mathfrak{S}_n$–module with Frobenius character f. Then the Frobenius character of $M \otimes \bigwedge(\mathbb{C}^n)$ is $\mathtt{sh}(f)$.*

In [12], Stanley applied shiftification to the parking function symmetric function PF_n, constructing the *shifted parking function symmetric function SH_n*. Using symmetric function theory methods, Stanley showed how to expand SH_n into the power sum, Schur–P and V–bases of SymP. Additionally, he introduced the *naive shifted parking functions* $\mathrm{NShPf}(n)$ of size n, which are pairs (a, σ) with $a \in \mathrm{Pf}(n)$ and $\sigma \in \{-1, 1\}^n$. By identifying σ with a basis element of the exterior algebra $\bigwedge^n(\mathbb{C})$, Proposition 1.1 shows the $\mathfrak{S}_n$–action on $\mathbb{C}[\mathrm{NShPf}(n)] \cong \mathbb{C}[\mathrm{Pf}(n)] \otimes \bigwedge^n(\mathbb{C})$ has Frobenius character SH_n (see Corollary 3.2). This gives a combinatorial/representation-theoretic explanation for Stanley's expansion [12, Thm. 3.4] of SH_n into the spanning set $\{V_\lambda : \lambda \vdash n\}$.

Next, we introduce *odd shifted parking functions*, which are triples (a, σ, τ). Here a is a parking function whose shape is odd, σ is a sign function satisfying certain parity constraints and τ is a non-crossing partial matching of the integers occurring in a. Then $\mathfrak{S}_n$ acts on the set of odd shifted parking functions $\mathrm{OShPf}(n)$ by permuting a and σ. A *sorted odd shifted parking function* is a triple $(a, \overline{\sigma}, \tau)$ where a is a weakly increasing parking function whose shape $\lambda(a)$ is odd, τ is a non-crossing matching as above and $\overline{\sigma}$ assigns a sign to each integer occurring in a satisfying certain constraints determined by τ. These objects are defined in detail in Sections 2 and 3. Our main result is:

Theorem 1.2. *The Frobenius character of the $\mathfrak{S}_n$–action on* $\mathrm{OShPf}(n)$ *is SH_n. Moreover,*

$$SH_n = \sum_{(a, \overline{\sigma}, \tau) \text{ sorted}} V_{\lambda(a)}. \tag{1.2}$$

Theorem 1.2 gives a combinatorial description of the V–basis expansion of SH_n, resolving Stanley's main open problem [12]. Stanley also asks for a projective representation on shifted parking functions whose spin character is SH_n, concluding 'probably this is too much to hope for.' Adapting work of Stembridge [13, §9], a construction similar to Proposition 1.1 using the Clifford algebra in place of the exterior algebra allows us to extend our $\mathfrak{S}_n$ action on $\mathrm{OShPf}(n)$ to a projective representation (see our arXiv preprint [9] for details). Additionally, we show that the naive V–expansion and Theorem 1.2 both extend to a t–graded version (see Proposition 4.6).

Let R_{2k} equal either side of (1.1), $R_{2k-1} = P_{2k-1}$ and $R_\lambda = \prod_i R_{\lambda_i}$. Our proof of Theorem 1.2 proceeds by grouping terms from the naive shifted parking function description of SH_n using the left-hand-side of (1.1) so that even parts now contribute to R_{2k} terms. This gives a combinatorial expansion of SH_n into the spanning set $\{R_\lambda : \lambda \vdash n\}$. We then show our sorted odd shifted parking functions can be obtained by splitting R_{2k}'s using the right-hand-side of (1.1) to obtain (1.2), and in turn the Frobenius character result.

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2 Background

Let $[n] = \{1, 2, \ldots, n\}$ and $\mathfrak{S}_n$ be the permutations of $[n]$. Note that $\mathfrak{S}_n$ acts on tuples $(a_1, \ldots, a_n)$ by $w \cdot (a_1, \ldots, a_n) = (a_{w(1)}, \ldots, a_{w(n)})$. Recall an integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a weakly descending tuple of non-negative integers with *size* $\sum \lambda_i$ and length $\ell(\lambda) = \max\{i : \lambda_i > 0\}$. Let $m_i(\lambda) = |\{j : \lambda_j = i\}|$. We say λ is *odd* if λ_i is odd when positive and *strict* if $\lambda_1 > \cdots > \lambda_{\ell(\lambda)}$.

Parking functions: For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, let $\mathtt{sort}(a) = \pi \cdot a$ where $\pi \in \mathfrak{S}_n$ satisfies $a_{\pi(1)} \leq \cdots \leq a_{\pi(n)}$. As stated in the introduction, a *parking function* of *size* n is a tuple a of n positive integers so that for $i \in [n]$ the ith entry in $\mathtt{sort}(a)$ is at most i. We say a is *sorted* if $a = \mathtt{sort}(a)$. Let $\mathrm{Pf}(n)$ be the set of all parking functions of size n and $\overline{\mathrm{Pf}}(n)$ be the set of sorted parking functions of size n. It is a classical result that $|\mathrm{Pf}(n)| = (n+1)^{n-1}$ and $|\overline{\mathrm{Pf}}(n)|$ is the nth Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$.

The *content* of $a \in \mathrm{Pf}(n)$ is $\alpha(a) = (\alpha_1(a), \ldots, \alpha_n(a))$ where $\alpha_k(a) = |\{j \in [n] : a_j = k\}|$, and the *shape* of a is the partition $\lambda(a)$ obtained by reversing $\mathtt{sort}(\alpha(a))$. For example, the parking function $a = (4, 1, 1, 3, 6, 3)$ has $\mathtt{sort}(a) = (1, 1, 3, 3, 4, 6)$, $\alpha(a) = (2, 0, 2, 1, 0, 1)$ and $\lambda(a) = (2, 2, 1, 1)$.

Symmetric Functions: Let Sym denote the algebra of symmetric functions. We refer the reader to [10] or [11] for descriptions of the power sum, elementary, homogeneous, Schur symmetric functions and the Hall inner product.

Our results concern $\mathrm{SymP} = \langle p_{2k+1} : k \in \mathbb{N} \rangle$. Recall the *shiftification operator* $\mathtt{sh} : \mathrm{Sym} \to \mathrm{SymP}$ is defined by $p_{2k-1} \mapsto 2p_{2k+1}$ and $p_{2k} \mapsto 0$ for k a positive integer. Equivalently, $\mathtt{sh}(h_k) = 2P_k$ and $\mathtt{sh}(e_k) = (-1)^k \cdot 2P_k$ where

$$P_k = \frac{1}{2}\sum_{i=0}^{k} h_i e_{k-i} = \sum_{\substack{i=0\\ i \text{ odd}}}^{k} h_i e_{k-i} = \sum_{\substack{i=0\\ i \text{ even}}}^{k} h_i e_{k-i}. \tag{2.1}$$

Recall for $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition that $V_\lambda = \prod_{i=1}^{k} P_{\lambda_i}$. From (1.1) we see the V_λ's are not linearly independent. However, $\{V_\lambda : \lambda \vdash n \text{ odd}\}$ is a basis for SymP. There are other bases for SymP, most notably the Schur-P and Schur-Q functions indexed by strict partitions, which figure less prominently in our work.

Shiftification is self-adjoint with respect to the Hall inner product.

Proposition 2.1. *For $f, g \in \mathrm{Sym}$, $\langle \mathtt{sh} f, g\rangle = \langle f, \mathtt{sh} g\rangle$.*

The *parking function symmetric function* is

$$PF_n = \sum_{a \in \overline{\mathrm{Pf}}(n)} h_{\lambda(a)} = \sum_{\lambda \vdash n} \frac{1}{(n-\ell(\lambda)+1)} \binom{n}{m_1(\lambda), \ldots, m_n(\lambda), n-\ell(\lambda)} h_\lambda. \tag{2.2}$$

The coefficient of h_λ in the second sum is the *Kreweras number* $\mathrm{Krew}(\lambda)$. We are interested in understanding the *shifted parking function symmetric function* $SH_n = \mathtt{sh}(PF_n)$. By applying $\mathtt{sh}$ to the right-hand-side of (2.2), we have

$$SH_n = \sum_{a \in \overline{\mathrm{Pf}}(n)} 2^{\ell(\lambda(a))} V_{\lambda(a)} = \sum_{\lambda \vdash n} \frac{2^{\ell(\lambda)}}{(n-\ell(\lambda)+1)} \binom{n}{m_1(\lambda), \ldots, m_n(\lambda), n-\ell(\lambda)} V_\lambda. \tag{2.3}$$

As we are summing over V_λ of every shape, the expansion in (2.3) is not unique. In [12], Stanley used generating function arguments to give several expansions of SH_n into bases of SymP. Most notably for our purposes, he showed:

Theorem 2.2 ([12, Thm. 3.4])**.** *For every n, $SH_n = \sum_{\lambda \vdash n \text{ odd}} \mathrm{OKrew}(\lambda) V_\lambda$ with*

$$\mathrm{OKrew}(\lambda) = \frac{2^{\ell}(\lambda)}{\ell(\lambda)!} \binom{\ell(\lambda)}{m_1(\lambda), m_3(\lambda), \ldots} (n+\ell(\lambda)-1)(n+\ell(\lambda)-3)\ldots(n-\ell(\lambda)+3).$$

For λ odd, we call $\mathrm{OKrew}(\lambda)$ the *odd Kreweras number* of λ.

Let Z_n denote the algebra of class functions of $\mathfrak{S}_n$. The *Frobenius characteristic map* is the linear isomorphism $\mathtt{ch} : Z_n \to \mathrm{Sym}$ defined by

$$\mathtt{ch}(\chi) = \sum_{\lambda \vdash n} z_\lambda^{-1} \chi(\pi_\lambda) p_\lambda. \tag{2.4}$$

Here π_λ is a permutation with cycle type λ and z_λ is the size of the stabilizer of π_λ under conjugation. The *Frobenius character* of a $\mathbb{C}[\mathfrak{S}_n]$-module V is the image of the character of V under $\mathtt{ch}$.

3 Shifted Parking Functions and Garages

In this section, we introduce naive and odd shifted parking functions. First, we realize SH_n combinatorially via naive shifted parking functions. To prove our main theorem, we introduce garages, which are used to partition both naive and odd shifted parking functions so that the associated S_n–submodules are isomorphic. This allows us to show the odd shifted parking functions provide a combinatorial realization of Theorem 2.2.

3.1 Naive shifted parking functions

A *naive shifted parking function* of *size* n is a pair of n–tuples (a, σ) where p is a parking function and $\sigma \in \{-1, 1\}^n$. Define $\texttt{sort}(a, \sigma) = (\texttt{sort}(a), \overline{\sigma})$ where

$$\overline{\sigma}_k = \begin{cases} \prod_{a_i = k} \sigma_i & \alpha_k(a) > 0 \\ 0 & \alpha_k(a) = 0 \end{cases} \tag{3.1}$$

is the parity of σ_i's where $a_i = k$ when this quantity is non-zero. A *sorted naive shifted parking function* of size n is a pair $(a, \overline{\sigma})$ where a is a sorted parking function and $\overline{\sigma} \in \{-1, 0, 1\}^n$ where $\overline{\sigma}_k = 0$ if and only if $\alpha_k(a) = 0$. The naive shifted parking function (a, σ) inherits its content and shape from a. Let $\mathrm{NShPf}(n)$ be the set of naive shifted parking functions of size n and $\overline{\mathrm{NShPf}}(n)$ be the set of sorted naive shifted parking functions of size n. There are $2^{\ell(\lambda(a))}$ choices for $\overline{\sigma}$ associated to each sorted a in (2.3), so:

Proposition 3.1. *For $n > 0$, $SH_n = \sum_{(a,\overline{\sigma}) \in \overline{\mathrm{NShPf}}(n)} V_{\lambda(a)}$.*

As mentioned in Section 1, we identify $\mathbb{C}[\mathrm{NShPf}(n)]$ with $\mathbb{C}[\mathrm{Pf}(n)] \otimes \bigwedge(\mathbb{C}^n)$ where $(a, \sigma) \in \mathrm{NShPf}(n)$ corresponds to $a \otimes \vee_{\{i : \sigma_i = -1\}} e_i$. Applying Proposition 1.1, we then have:

Corollary 3.2. *The Frobenius character of $\mathbb{C}[\mathrm{NShPf}(n)] \cong \mathbb{C}[\mathrm{Pf}(n)] \otimes \bigwedge(\mathbb{C}^n)$ is SH_n.*

The resulting $\mathfrak{S}_n$–action on NShPf preserves the sort of a naive shifted parking function and thus $\mathbb{C}[\mathrm{NShPf}]$ decomposes into submodules, giving a representation theoretic interpretation of Proposition 3.1. For $(p, \overline{\sigma}) \in \overline{\mathrm{NShPf}}$, let $N_{(p,\overline{\sigma})}$ be the set of naive shifted parking functions whose sort is $(p, \overline{\sigma})$.

Lemma 3.3. *The $\mathfrak{S}_n$–module $\mathbb{C}[N_{(p,\overline{\sigma})}]$ is an $\mathfrak{S}_n$–submodule of $\mathbb{C}[\mathrm{NShPf}]$ with character $V_{\lambda(p)}$.*

3.2 Garages

The key to our argument is to relate naive and (forthcoming) odd shifted parking functions by grouping them via garages. To define them, we first introduce a lattice path associated to each sorted naive shifted parking function.

Definition 3.4. For $(a, \overline{\sigma})$ a sorted naive shifted parking function, let $v \in \{0, 1, 2\}^n$ with

$$v_i = \begin{cases} 0 & \alpha_i(a) = 0; \\ 1 & \alpha_i(a) \text{ is odd}; \\ 2 & \alpha_i(a) > 0 \text{ is even}. \end{cases} \tag{3.2}$$

the *matching path* L associated to $(a, \overline{\sigma})$ by, for each i, adding a step according to the following rules:

- $L_i = (0,1)$ if $v_i = 2$ and $\overline{\sigma}_i = +$ or $v_i = 0$ and the path is currently below the diagonal;
- $L_i = (1,0)$ if $v_i = 2$ and $\overline{\sigma}_i = -$ or $v_i = 0$ and the path is currently above the diagonal;
- $L_i = (1,1)$ if $v_i = 1$ or $v_i = 0$ and the path is currently on the diagonal.

See Figure 1 and Example 3.14 for examples of this construction.

To a lattice path L we associate the matching $\tau(L)$ where i and k are matched, i.e., $(i,k) \in \tau(L)$, if $i < k$, L_i and L_k intersect the same diagonal $x - y = d + 1/2$ and no intermediate step L_j does. Restricted to Dyck paths, this is the standard map to non-crossing matchings. Note for $(i,k) \in \tau(L)$ that L_i must be moving away from the diagonal, so $v_i = 2$. Also, note diagonal steps can never be matched.

Definition 3.5. A sorted naive shifted parking function $(a,\overline{\sigma})$ is a *garage* if and only if when the ith step in its matching path is towards the diagonal, $v_i = 0$. Equivalently, $(a,\overline{\sigma})$ with matching path L is a garage if for each $(i,k) \in \tau(L)$ we see $v_k = 0$.

Garages can be modified to construct other sorted naive shifted parking functions.

Lemma 3.6. *Let $(a,\overline{\sigma}) \in \overline{\mathrm{NShPf}}(n)$ with matching path L and $(i,k) \in \tau(L)$ so that we have $\alpha_i(a) + \alpha_k(a) = 2\ell$. For $j \in [\ell]$, construct $(a',\overline{\sigma}')$ from $(a,\overline{\sigma})$ by replacing all occurrences of i and k in a with $2j$ occurrences of i and $2\ell - 2j$ occurrences of k, with $\overline{\sigma}'_k = -\overline{\sigma}_i$ if $j \neq \ell$ and 0 if $j = \ell$. Then $(a',\overline{\sigma})$ is a naive shifted parking function with matching path L.*

We call $(a,\overline{\sigma}),(a',\overline{\sigma}') \in \overline{\mathrm{NShPf}}(n)$ *garage equivalent* if they have the same matching path L with matching $\tau(L)$, $\alpha_j(a) = \alpha_j(a')$ for j unmatched and for each $(i,k) \in \tau(L)$

$$\alpha_i(a) + \alpha_k(a) = \alpha_i(a') + \alpha_k(a').$$

Lemma 3.7. *Every $(a,\overline{\sigma}) \in \overline{\mathrm{NShPf}}(n)$ is garage equivalent to exactly one garage.*

Proof. This follows immediately from Lemma 3.6 and the definition of garage. □

Define $\varphi : \mathrm{NShPf}(n) \to \mathrm{Gar}(n)$ by mapping each naive shifted parking function to the garage that its sort is garage equivalent to. Lemma 3.7 shows φ is well-defined for all n. From the introduction, recall for $\lambda = (\lambda_1, \ldots, \lambda_k)$ that $R_\lambda = R_{\lambda_1} \cdot \ldots \cdot R_{\lambda_k}$ where

$$R_{2k+1} = P_{2k+1} \quad \text{and} R_{2k} = \sum_{i=1}^{k} P_{2i}P_{2k-2i} = \sum_{i=1}^{k} P_{2i-1}P_{2k-2i+1}. \tag{3.3}$$

Proposition 3.8. *For $(a,\overline{\sigma}) \in \mathrm{Gar}(n)$ with $\lambda(a) = \lambda$, $\mathbb{C}[\varphi^{-1}(a,\overline{\sigma})]$ is an $\mathfrak{S}_n$-submodule of $\mathbb{C}[\mathrm{NShPf}(n)]$ whose Frobenius character is*

$$\mathtt{ch}(\mathbb{C}[\varphi^{-1}(a,\overline{\sigma})]) = R_\lambda$$

Proof sketch. Using Lemma 3.6, we can modify $(a,\overline{\sigma})$ to produce a sorted naive shifted parking function $(a',\overline{\sigma}')$ with $\alpha_i(a') = 2\ell$ and $\alpha_k(a') = 2j - 2\ell$ for any $\ell \in [j]$ and $\overline{\sigma}' = \overline{\sigma}$ unless $\ell = j$, in which case $\overline{\sigma}'_k = 0$. Each choice of ℓ corresponds to one term in $R_{2j} = \sum_{i=1}^{j} P_{2i}P_{2j-2i}$. Doing so concurrently for each matched pair, the result follows. □

See Example 3.14 for an example. Combining Lemma 3.7 and Proposition 3.8:

Theorem 3.9. *For each n, we have*

$$SH_n = \sum_{(a,\overline{\sigma})\in \mathrm{Gar}(n)} R_{\lambda(a)}.$$

3.3 Odd shifted parking functions

We now introduce odd shifted parking functions, the key objects in Theorem 1.2.

Definition 3.10. An *odd shifted parking function* is a triple (a,σ,τ), where a is a parking function whose shape is odd, $\sigma \in \{-1,1\}^n$, and τ is a noncrossing matching of the integers appearing in a satisfying:

1. If $(i,j) \in \tau$, then $\overline{\sigma}_i = -\overline{\sigma}_j$
2. If $(i,k) \in \tau$, and $i < j < k$, then j appears in a.
3. If $(i,\ell) \in \tau$, $(j,k) \in \tau$ and $i < j < k < \ell$, then $\overline{\sigma}_i = \overline{\sigma}_j$ (here $\overline{\sigma}_k$ is defined as in (3.1)).

For (a,σ,τ) an odd parking function, $\texttt{sort}(a,\sigma,\tau)$ is the triple $(\texttt{sort}(a),\overline{\sigma},\tau)$, A *sorted odd parking function* is a triple $\texttt{sort}(a,\sigma,\tau)$ of some odd parking function. Let $\mathrm{OShPf}(n)$ be the set of odd shifted parking functions of size n and $\overline{\mathrm{OShPf}}(n)$ be the set of sorted odd shifted parking functions of size n.

Let $\mathfrak{S}_n$ act on $(a,\sigma,\tau) \in \mathrm{OShPf}(n)$ by $\pi \cdot (a,\sigma,\tau) = (\pi \cdot (a,\sigma),\tau)$, i.e. $\mathfrak{S}_n$ acts as in the naive case on a and σ and leaves τ fixed. Since the action of $\mathfrak{S}_n$ does not change $\texttt{sort}(a,\sigma,\tau)$, $\mathbb{C}[\mathrm{OShPf}(n)]$ decomposes into submodules indexed by sorted odd shifted parking functions. For $(a,\overline{\sigma},\tau)$ a sorted odd shifted parking function, define

$$O_{(a,\overline{\sigma},\tau)} = \{(a',\sigma',\tau') \in \mathrm{OShPf}(n) \mid \texttt{sort}(a',\sigma',\tau') = (a,\overline{\sigma},\tau)\}. \tag{3.4}$$

Lemma 3.11. *Let $(a,\overline{\sigma},\tau) \in \overline{\mathrm{OShPf}}(n)$. Then $\mathbb{C}[O_{(a,\overline{\sigma},\tau)}]$ is an $\mathfrak{S}_n$-module with character $V_{\lambda(a)}$.*

To show the Frobenius character of $\mathbb{C}[\mathrm{OShPf}(n)]$ is SH_n, we construct an analogue of φ mapping odd shifted parking functions to garages. To each odd shifted parking function we associate a garage in the following way.

Definition 3.12. Let $(a,\sigma,\tau) \in \mathrm{OShPf}(n)$. Construct a' where:

- if $a_i = j$ and j is unmatched or a left endpoint of an arc (j,k) in τ, then $a'_i = j$;
- if $a_i = k$ and k is the right endpoint of an arc (j,k) in τ, then $a'_i = j$.

Let $\overline{\sigma}'$ be the restriction of $\overline{\sigma}$ to the set of integers appearing in a'. Define

$$\varphi_o : \mathrm{OShPf}(n) \to \mathrm{Gar}(n) \quad \text{by} \quad \varphi_o(a,\sigma,\tau) = (\mathtt{sort}(a'),\overline{\sigma}').$$

Proposition 3.13. *The map φ_o is well defined. Furthermore, for $(a,\overline{\sigma}) \in \mathrm{Gar}(n)$ with shape λ we see $\mathbb{C}[\varphi_o^{-1}(a,\overline{\sigma})]$ is an $\mathfrak{S}_n$-submodule of $\mathbb{C}[\mathrm{OShPf}(n)]$ with character*

$$\mathtt{ch}(\mathbb{C}[\varphi_o^{-1}(a,\overline{\sigma})]) = R_\lambda$$

Proof Sketch. By Lemma 3.11 it suffices to show that $\overline{\varphi}_o^{-1}((a,\overline{\sigma}))$ is in bijection with terms in the $\{V_\lambda : \lambda \text{ odd}\}$–basis expansion of R_λ. But this is clear, since for $(a',\overline{\sigma}',\tau) \in \overline{\varphi}_o^{-1}(a,\overline{\sigma})$ the following are determined: $\overline{\sigma}'$, τ, $\alpha_c(a')$ for c unmatched in τ and the quantity $2k = \alpha_a(a') + \alpha_b(a')$ for each pair $(a,b) \in \tau$. The only choice comes from how many a's appear, and this corresponds to a choice of a term in the sum

$$R_{2k} = \sum_{i=1}^{k} P_{2i-1} \cdot P_{2(k-i)+1}$$

Each choice corresponds to a valid sorted odd shifted parking function so the result follows by making all possible choices for each matched pair. $\square$

Proof of Theorem 1.2. By Theorem 3.9 and Proposition 3.13, the result follows. $\square$

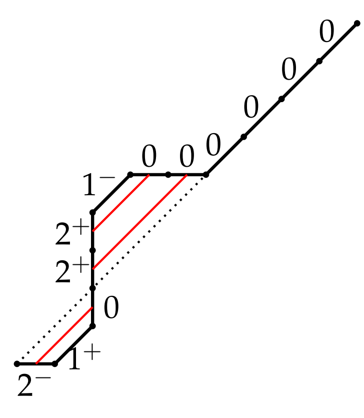

Figure 1: The lattice path L associated to the garage in Example 3.14. Steps are labelled by v with $\overline{\sigma}$ in the superscripts. The matching $\tau(L)$ is shown in red.

Example 3.14. Consider the garage

$$(a,\overline{\sigma}) = ((1,1,1,1,2,4,4,4,4,5,5,6),(-1,1,0,1,1,-1,0,0,0,0,0,0)),$$

and note $\lambda(a) = (4,4,2,1,1)$. Its associated lattice path L appears in Figure 1, and it has the matching $\tau(L) = \{(1,3),(4,8),(5,7)\}$. The four sorted naive shifted parking functions with the same lattice path are determined by the choices 1111 or 1133 and 4444 or 4488:

$$\varphi^{-1}(a,\overline{\sigma}) = \begin{Bmatrix} ((1,1,1,1,2,4,4,4,4,5,5,6),(-1,1,0,1,1,-1,0,0,0,0,0,0)) \\ ((1,1,1,1,2,4,4,5,5,6,8,8),(-1,1,0,1,1,-1,0,-1,0,0,0,0)) \\ ((1,1,2,3,3,4,4,4,4,5,5,6),(-1,1,1,1,1,-1,0,0,0,0,0,0)) \\ ((1,1,2,3,3,4,4,5,5,6,8,8),(-1,1,1,1,1,-1,0,-1,0,0,0,0)) \end{Bmatrix}.$$

The shapes of these four preimages are $(4,4,2,1,1),(4,2,2,2,1,1),(4,2,2,2,1,1)$ and $(2,2,2,2,2,1,1)$ respectively, so by Lemma 3.3 $\mathtt{ch}(\mathbb{C}[\varphi^{-1}(a,\overline{\sigma})])$ is

$$V_{4,4,2,1} + 2V_{4,2,2,2,1} + V_{2,2,2,2,2,1} = (P_4 + P_2P_2)(P_4 + P_2P_2)P_2P_1P_1 = R_{4,4,2,1,1}.$$

There are four sorted odd shifted parking functions mapped to $(a,\overline{\sigma})$ by φ_o, obtained by the splits $1111 \mapsto 1113$ or 1333, $4444 \mapsto 4448$ or 4888 and $55 \mapsto 57$, i.e., $\varphi_o^{-1}(p,\overline{\sigma})$ is

$$\begin{Bmatrix} ((1,1,1,2,3,4,4,4,5,6,7,8),(-1,1,1,1,1,-1,-1,-1,0,0,0,0),\{(1,3),(4,8),(5,7)\}) \\ ((1,1,1,2,3,4,5,6,7,8,8,8),(-1,1,1,1,1,-1,-1,-1,0,0,0,0),\{(1,3),(4,8),(5,7)\}) \\ ((1,2,3,3,3,4,4,4,5,6,7,8),(-1,1,1,1,1,-1,-1,-1,0,0,0,0),\{(1,3),(4,8),(5,7)\}) \\ ((1,2,3,3,3,4,5,6,7,8,8,8),(-1,1,1,1,1,-1,-1,-1,0,0,0,0),\{(1,3),(4,8),(5,7)\}) \end{Bmatrix}.$$

The shapes of these four preimages are all $(3,3,1,1,1,1,1,1)$, so by Lemma 3.11

$$\mathtt{ch}(\mathbb{C}[\varphi^{-1}(a,\overline{\sigma})]) = 4V_{3,3,1,1,1,1,1,1} = (2P_3P_1)(2P_3P_1)(P_1P_1)P_1P_1 = R_{4,4,2,1,1}$$

4 Problems and Extensions

Enumerative aspects: From the definition, it is easy to see

$$\#\{(a,\overline{\sigma}) \in \mathrm{NShPf}(n) : \lambda(a) = \mu\} = 2^{\ell}(\mu)\mathrm{Krew}(\mu).$$

Since there are direct combinatorial arguments showing $\mathrm{Krew}(\mu)$ counts sorted parking functions with shape μ, this identity has a combinatorial proof. By contrast:

Problem 4.1. For $\mu \vdash n$ odd, give a direct combinatorial argument showing

$$\mathrm{OKrew}(\mu) = \#\{(a,\overline{\sigma}) \in \overline{\mathrm{OShPf}}(n) : \lambda(a) = \mu\}.$$

As mentioned previously, Stanley observed $\#\overline{\mathrm{NShPf}}(n)$ is the nth *large Schröder number* s_n, which enumerate many combinatorial objects such as Schröder paths and are defined by the initial conditions and recurrence

$$s_0 = 1, \quad s_1 = 2, \quad s_n = 3s_{n-1} + \sum_{k=1}^{n-2} s_k s_{n-k-1} \ (n \geq 2).$$

There is an straightforward extension of the map from $\overline{\mathrm{Pf}}(n)$ to Dyck paths mapping $\overline{\mathrm{NShPf}}(n)$ to *Schröder paths* of length n, which are lattice paths from $(0,0)$ to (n,n) with steps $(1,0)$, $(0,1)$ and $(1,1)$ staying weakly above the line $y = x$.

Note $\#\overline{\mathrm{OShPf}}(n) = s_n$ as well. Our proof of Theorem 1.2 can be used to give a bijection between odd and naive sorted shifted parking functions. However, we were unable to find any objects enumerated by large Schröder numbers placing a similar emphasis on odd parity in its constituent components, suggesting $\overline{\mathrm{OShPf}}(n)$ could be the first instance in a new and combinatorially distinct family of large Schröder objects.

Algebraic aspects: As discussed in the introduction, a major motivation for studying the parking function symmetric function is its bigraded form

$$PF_n(q,t) = \sum_{a \in \mathrm{Pf}(n)} t^{\mathrm{area}(a)} q^{\mathrm{dinv}(a)} F_{n,\mathrm{ides}(a)}. \tag{4.1}$$

Here dinv and area are $\mathbb{N}$–valued statistics on a and ides is a set-valued statistic on a. The Shuffle conjecture, now a theorem [3], says $PF_n(q,t) = \nabla e_n$ where ∇ is Bergeron and Garsia's celebrated *nabla operator* [1]. Define

$$C_n(q,t) = \sum_{a \in \overline{\mathrm{Pf}}(n)} t^{\mathrm{area}(a)} q^{\mathrm{dinv}(a)}$$

As Haglund and Garsia showed [5], $C_n(q,t)$ is the coefficient of e_n in ∇e_n. Haiman showed ∇e_n is the bi-graded Frobenius character for the $\mathfrak{S}_n$–module of *diagonal harmonics*

$$\mathcal{D}_n = (\mathbb{C}[x_1, \ldots, x_n] \otimes \mathbb{C}[y_1, \ldots, y_n]) / \mathcal{I}_n$$

where $\mathcal{I}_n$ is the degree ≥ 1 invariants under the diagonal $\mathfrak{S}_n$–action [7].

Applying shiftification, we obtain a (q,t)–analogue $SH_n(q,t) = \mathtt{sh} PF_n(q,t)$ of the shifted parking function symmetric function. By Proposition 1.1, we have:

Proposition 4.2. *For $n \geq 1$, $SH_n(q,t)$ is the bigraded Frobenius character of $(D_n \otimes \Lambda_n)$.*

Note, in the natural presentation of $D_n \otimes \Lambda_n$, we now have two sets of commuting variables and one set of anti-commuting variables modulo the positive degree $\mathfrak{S}_n$–invariants from the diagonal action on the commuting variables. This is somewhat reminiscent of the *super diagonal coinvariant algebra* introduced in [15], and more generally the (r,s)–*bosonoic-fermionic coinvariant algebra* $R_n^{(r,s)}$ from [2].

Let $\mathrm{S}(n)$ be the set of Schröder paths of size n and $\mathrm{S}_k(n)$ be the subset with k horizontal steps. In [4] statistics area and bounce on Schröder paths are introduced so:

Theorem 4.3 ([6]). *For $k \in [0, n]$,*

$$\langle \nabla e_n, h_k e_{n-k} \rangle = \sum_{P \in \mathrm{S}_k(n)} q^{\mathrm{area}(P)} t^{\mathrm{bounce}(P)}. \tag{4.2}$$

Summing (4.2) over all values k, then applying Proposition 2.1 and (2.1), we obtain

$$\langle \nabla e_n, 2P_n \rangle = \langle \mathtt{sh} \nabla e_n, h_n \rangle = \langle SH_n(q,t), h_n \rangle = \sum_{s \in \mathrm{S}(n)} q^{\mathrm{area}(s)} t^{\mathrm{bounce}(s)}. \tag{4.3}$$

Since Schröder paths can be identified with sorted naive shifted parking functions, we can interpret (4.3) as a statement about sorted naive shifted parking functions with appropriate statistics.

Problem 4.4. Define statistics stat_1 and stat_2 on $\overline{\mathrm{OShPf}}(n)$ so that

$$\langle \nabla e_n, 2P_n \rangle = \langle SH_n(q,t), h_n \rangle = \sum_{(a,\overline{\sigma},\tau) \in \overline{\mathrm{OShPf}}(n)} q^{\mathrm{stat}_1(a,\overline{\sigma},\tau)} t^{\mathrm{stat}_2(a,\overline{\sigma},\tau)}.$$

Alternatively, it is plausible that SH_n has a different q,t–analogue more naturally associated to odd shifted parking functions. Perhaps this would be based on a bi-grading of the projective representation associated to odd shifted parking functions.

Problem 4.5. Is there another $\mathfrak{S}_n$–module (or $\overline{\mathfrak{S}}_n$–module) analogous to the diagonal harmonics whose bi-graded Frobenius character specializes to SH_n when $q = t = 1$?

We remark that Haglund's area statistic for Schröder paths (hence for $\overline{\mathrm{NShPf}}(n)$) has a simple interpretation for $\overline{\mathrm{OShPf}}(n)$ via the bijection implicit in our proof. For $(a, \overline{\sigma}) \mapsto (a', \overline{\sigma}', \tau)$ we have the relation

$$\mathrm{area}(a) = \mathrm{area}(a') + \sum_{(i,j) \in \tau} j - i. \tag{4.4}$$

Call the right-hand side of (4.4) area_o. Then:

Proposition 4.6.

$$SH_n(1,t) = \sum_{(a,\overline{\sigma},\tau) \in \overline{\mathrm{OShPf}}(n)} t^{\mathrm{area}_o(a,\overline{\sigma},\tau)} V_{\lambda(a)}.$$

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