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Matroids arising from algebraic shifting

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Abstract. We characterize the shifted simple graphs and the shifted 3-uniform hypergraphs whose inverse image under exterior shifting is the set of bases of a matroid: those are exactly the hypergraphs whose hyperedges form an initial lex-segment. There are several examples of known matroids arising in this way: the simplicial matroid, the hyperconnectivity matroid and the area-rigidity matroid. For $k \geq 4$, we provide a similar characterization for shifted k-uniform hypergraphs satisfying an additional combinatorial condition.

Keywords: hypergraphs, algebraic shifting, matroids

1 Introduction

A collection H of k-element subsets of $[n]$ is *shifted* if for every $i < j$, $j \in S \in H$, $i \notin S$, also $(S \setminus \{j\}) \cup \{i\} \in H$. After fixing a field, a canonical operation of *algebraic shifting*, which associates with every simplicial complex K a shifted simplicial complex $\Delta(K)$, was introduced by Kalai [8, 10]. This operation preserves some important properties of K, such as its f-vector and β-vector, while loses others, such as the homotopy type of K. Various properties of K are described in terms of its shifting, such as its homology and Cohen-Macaulyness over the fixed field.

Two different variations of algebraic shifting were introduced by Kalai: *exterior shifting* and *symmetric shifting*. Given a simplicial complex K we denote by $\Delta^e(K)$ and $\Delta^s(K)$ its exterior and symmetric shifting respectively, and sometimes omit the superscript for statements that hold for both types of algebraic shifting. Throughout the paper all results on exterior shifting are satisfied over any field, and all results on symmetric shifting are satisfied over any field of characteristic zero.

We start our exposition with a simple known example. Let n be a positive integer. The *graphic matroid* of the complete graph on n vertices is a collection of all spanning trees on the vertex set $[n]$. Let G be such a spanning tree. Note that since algebraic shifting preserves Betti numbers and by the shiftedness of $\Delta(G)$, for every edge

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$e \in \Delta(G)$, we have $1 \in e$. Note also that since algebraic shifting preserves f-vector, it then follows that $\Delta(G) = \{12, 13, \ldots, 1n\}$. Conversely, one can similarly verify that every $G \in \Delta^{-1}(\{12, 13, \ldots, 1n\})$ is a spanning tree on $[n]$. Thus, we conclude that $\Delta^{-1}(\{12, 13, \ldots, 1n\})$ coincides with the graphic matroid of the complete graph on n vertices. The main goal of this paper is to study this phenomenon in a general setting.

Let H be a shifted k-uniform hypergraph on $[n]$ (equivalently, the collection of facets of a pure simplicial complex). Denote by

$$M(H) = \left\{ G \subseteq \binom{[n]}{k} \mid \Delta^e(G) = H \right\}$$

the set of all k-uniform hypergraphs whose exterior shifting is H. We say that H is *matroidal* if $M(H)$ is the set of bases of a matroid; we abuse notation and also denote that matroid by $M(H)$.

For example, if H is the collection of k-subsets of $[n]$ containing 1, then $M(H)$ is the Crapo-Rota simplicial matroid [6], whose independent sets are the $(k-1)$-acyclic collections of k-subsets of $[n]$, namely those that do not support any $(k-1)$-homology cycle over a fixed field; see e.g. [9, Section 4, Remark (2)] for further background. If $H = \{ij : 1 \leq i < j \leq n, i \leq k\}$ then $M(H)$ is Kalai's k-hyperconnectivity matroid [7]; see e.g. [9, Section 4, Remark (2)] and [10, Section 2.7] for further background. If $H = \{1ij : 2 \leq i < j \leq n, i \leq 3\}$ then $M(H)$ is the area-rigidity matroid, as recently proved in [5, Theorem 1.2].

We conjecture a combinatorial characterization of the algebraic property of being *matroidal*:

Conjecture 1.1. *For every natural numbers $1 \leq k \leq n$, a shifted k-uniform hypergraph $H \subseteq \binom{[n]}{k}$ is matroidal if and only if the hyperedges of H form an initial segment with respect to the lexicogrpaphic order (initial lex-segment for short).*

The "if" part follows easily from a result of Kalai [9, Section 4]. We prove the "only if" part for $k = 2$ (i.e. for graphs) and for $k = 3$:

Theorem 1.2. *For $k = 2, 3$, a shifted k-uniform hypergraph $H \subseteq \binom{[n]}{k}$ is matroidal if and only if the (hyper)edges of H form an initial lex-segment.*

For $k \geq 4$ we prove the "only if" part for k-uniform hypergraphs satisfying some technical condition on their structure.

Theorem 1.3. *Let $k \geq 4$ and let $H \subseteq \binom{[n]}{k}$ be a shifted k-uniform hypergraph such that $s_H \notin H$, where s_H is defined in terms of the lex-maximum element of H by (3.2)-(3.4). Then H is matroidal if and only if the hyperedges of H form an initial lex-segment.*

For the "only if" part of Theorem 1.2 we first make a reduction of the general problem to a certain restricted subset of hypergraphs; and for each of them we find a suitable permutation $\pi : [n] \to [n]$ such that the pair $(H, \pi(H))$, where $\pi(H) = \{\pi(T) :\ T \in H\}$, violates the exchange axiom among the elements of $M(H)$. For the "only if" part of Theorem 1.3 we find a suitable such permutation $\pi = \pi(s_M)$ for each H.

Let H be a shifted k-uniform hypergraph on $[n]$. For any $S \in \binom{[n]}{k}$ denote by

$$H(S) = \{T \in \binom{[n]}{k} \mid T \in H \text{ and } T \leq_{\text{lex}} S\}$$

the sub-hypergraph of H consisting of all hyperedges in H that are lexicographically not greater than S. H is *lex-matroidal* if $H(S)$ is matroidal for every S.

We prove the analog of Conjecture 1.1 for this stronger notion:

Proposition 1.4. *A shifted k-uniform hypergraph $H \subseteq \binom{[n]}{k}$ is lex-matroidal if and only if the hyperedges of H form an initial lex-segment.*

This paper is organized as follows. In Section 2 we give a background on algebraic shifting and matroids. In Section 3.1 we prove lemmas needed for proofs of Theorem 1.2, Theorem 1.3 and Proposition 1.4, and in Section 3.2 we prove these results. In Section 4 we consider matroids arising from symmetric shifting, and in Proposition 4.1 classify all matroidal graphs with respect to symmetric shifting. For $H = \{ij :\ 1 \leq i \leq j \leq n, i \leq k\}$ the corresponding matroid is the (generic) k-rigidity matroid, see e.g. [12] and [10, Section 2.7], [14, Section 3.2]. In Section 5 we conclude with related open problems.

2 Preliminaries

2.1 Hypergraphs and simplicial complexes

Let $[n] := \{1, 2, \ldots, n\}$. Denote by $\binom{[n]}{k}$ the set of subsets of $[n]$ of size k. A *simplicial complex* K is a collection of subsets of $[n]$ which is closed under inclusion, i.e. $S \subseteq T \in K$ implies $S \in K$. The elements of K are called *faces*. The *i-th skeleton* of K is $K_i = \{S \in K : |S| = i+1\}$. Faces which belong to K_i are *i-dimensional* faces. The 0-dimensional faces are called *vertices* and the maximum faces with respect to inclusion are called *facets*. If all facets have the same dimension, K is *pure*.

The *f-vector* (face vector) of K is $f(K) = (f_{-1}, f_0, f_1, \ldots)$ where $f_i = |K_i|$. The *dimension* of K is $\dim(K) := \max\{i : f_i(K) \neq 0\}$. In particular, a 1-dimensional simplicial complex is a *simple graph*.

The join of two simplicial complexes K and L with disjoint sets of vertices is the simplicial complex $K * L = \{S \cup T : S \in K, T \in L\}$. The *cone* over a simplicial complex K is the join of K with a single vertex, denoted by $\mathrm{Cone}(K)$.

Denote by $\beta_i(K, \mathbb{L})$ the i-th Betti number of K with coefficients in $\mathbb{L}$, or simply $\beta_i(K)$ if $\mathbb{L}$ was specified earlier. Denote by $\beta(K) = (\beta_0(K), \beta_1(K), \dots)$ the *Betti vector* of K.

A simplicial complex K is *shifted* if for every $i < j$, $j \in S \in K$, also $(S \setminus \{j\}) \cup \{i\} \in K$. Let $<_p$ be the product partial order on equal sized ordered subsets of $\mathbb{N}$. That is, for $S = \{s_1 < \dots < s_k\}$ and $T = \{t_1 < \dots < t_k\}$ $S \leq_p T$ if and only if $s_j \leq t_j$ for every $1 \leq j \leq k$. Then K is shifted if and only if $S <_p T \in K$ implies $S \in K$.

A *k-uniform hypergraph* on n vertices is a collection of subsets of $\binom{[n]}{k}$. In particular, k-uniform hypergraphs are in bijection with pure $(k-1)$-dimensional simplicial complexes, by closing down to subsets $H \mapsto K(H)$.

Let $<_{\text{lex}}$ be the lexicographic order on equal sized subsets of $\mathbb{N}$, i.e. $S <_{\text{lex}} T$ if and only if $\min(S \triangle T) \in S$. Throughout the paper, we will use a convention that writing $\{i_1, i_2, \dots, i_k\}$ means an ordered set, namely $i_1 < i_2 < \dots < i_k$, unless specified otherwise.

A *matroid* M is an ordered pair $(E, \mathcal{B})$ consisting of a finite set E and a non-empty collection $\mathcal{B}$ of subsets of E having the following property: if B_1 and B_2 are members of $\mathcal{B}$ and $e_1 \in B_1 \setminus B_2$, then there is an element $e_2 \in B_2 \setminus B_1$ such that $(B_1 \setminus e_1) \cup e_2 \in \mathcal{B}$. This property is called the *exchange axiom*. The elements of $\mathcal{B}$ are called the *bases*. The subsets of bases are called the *independent sets* of M.

2.2 Algebraic shifting

The presentation in this section is based mainly on [10, 14].

Exterior shifting. Let $\mathbb{F}$ be a field and let $\mathbb{L}$ be a field extension of $\mathbb{F}$ of transcendental degree $\geq n^2$. (e.g. $\mathbb{F} = \mathbb{Q}$ and $\mathbb{L} = \mathbb{R}$). Let $X = (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ be an n by n matrix over $\mathbb{L}$ which is generic over $\mathbb{F}$, i.e. a matrix whose entries are algebraically independent over $\mathbb{F}$. Let $X\left(\binom{[n]}{k}; \binom{[n]}{k}\right)$ be the *k-th compound matrix* of X, i.e., the $\binom{n}{k}$ by $\binom{n}{k}$ matrix of k by k minors of X. Assume that its rows and columns are ordered lexicographically.

Given a collection H of k-subsets of $[n]$ with $|H| = m$, let $X\left(H; \binom{[n]}{k}\right)$ be the $m \times \binom{n}{k}$ submatrix of $X\left(\binom{[n]}{k}; \binom{[n]}{k}\right)$ whose rows correspond to the k-sets in H. Choose a basis of columns for the column-space of $X\left(H; \binom{[n]}{k}\right)$ in the greedy way with respect to the lexicographic order: by simply taking those columns which are not spanned by previous columns in the lexicographic order. Starting from a generic over $\mathbb{F}$ matrix X define $\Delta^e(H)$ to be the family of sets which are the indices of the chosen columns. For a simplicial complex K define $\Delta(K) = \cup_i \Delta(K_i)$, where K_i is an i-dimensional skeleton of K. One can reformulate this definition in terms of exterior algebra, see [10, Section 2.1].

Symmetric shifting. Let K be a simplicial complex and let $R(K)$ be its Stanley-Reisner ring (face ring): $R(K) = \mathbb{L}[x_1, \dots, x_n]/I_K$, where I_K is the homogeneous ideal generated by the monomials $x_{i_1} \cdot x_{i_2} \cdot \dots \cdot x_{i_r}$, where $\{i_1, i_2, \dots, i_r\} \notin K$. Let $y_1, \dots, y_n$ be generic over $\mathbb{F}$ linear combinations of $x_1, \dots, x_n$. We now construct the basis $\mathrm{GIN}(K)$ of monomials in the variables y_i in the greedy way with respect to the lexicographic order:

$$\mathrm{GIN}(K) = \{m : \tilde{m} \notin \mathrm{span}_k\{\tilde{m'} : \deg(m') = deg(m), m' <_{\mathrm{lex}} m\}\}.$$

$\mathrm{GIN}(K)$ can be reconstructed from its monomials of the form $m = y_{i_1} \cdot y_{i_2} \cdot \ldots \cdot y_{i_r}$ where $r \leq i_1 \leq i_2 \leq \ldots \leq i_r$, $r \leq \dim(K) + 1$. Denote this set by $gin(K)$, and associate the set $S_{un}(m) = \{i_1 - r + 1, i_2 - r + 2, \ldots, i_r\}$ with such a monomial m. Denote $\Delta^s(K) = \cup\{S_{un}(m) : m \in \mathrm{gin}(K)\}$ to be the *symmetric shifting* of K.

Given a simplicial complex K, we will use the notation $\Delta(K)$ in the cases where a statement is satisfied for both $\Delta^e(K)$ and $\Delta^s(K)$.

Both exterior and symmetric shiftings are canonical in the sense that they do not depend on the choice of a generic matrix and for a permutation $\pi : [n] \to [n]$ a simplicial complex K satisfies $\Delta(\pi(K)) = \Delta(K)$. The following Theorem summarizes properties of algebraic shifting (from now on we will consider Betti numbers with coefficients in $\mathbb{L}$):

Theorem 2.1. [10, Theorems 2.1, 2.2 and 3.2] *Let K and L be simplicial complexes. Then:*

1. *$\Delta(K)$ is a shifted simplicial complex depending only on the characteristic of $\mathbb{F}$.*
2. *$\Delta(K) = \Delta(L)$ for L combinatorially isomorphic to K, see* [4, Claim 1].
3. *$f(K) = f(\Delta(K))$ and $\beta(K) = \beta(\Delta(K))$, see* [4, Theorem 3.1].
4. *If K is shifted then $\Delta(K) = K$, see* [4, Claim 1], [9, Remark (4)].
5. *If $L \subseteq K$, then $\Delta(L) \subseteq \Delta(K)$.*
6. *$\Delta(Cone(K)) = Cone(\Delta(K))$, see* [13, Corollary 5.4], [2, Lemma 3.3].

3 Matroids arising from exterior shifting

Let H be a shifted k-uniform hypergraph on $[n]$. Recall the definition of the set $M(H)$ and when H is matroidal from the Introduction.

Kalai's construction: Let $X = (x_{ij})_{1\leq i\leq n, 1\leq j\leq n}$ be an $n \times n$-matrix over $\mathbb{L}$ which is generic over $\mathbb{F}$. For $S, T \in \binom{[n]}{k}$ let X_{ST} be the minor which corresponds to the k rows of X indexed by the elements of S and the k columns of X indexed by the elements of T.

Let B and H be subsets of $\binom{[n]}{k}$. Denote by $X(B, H)$ the matrix $(X_{ST})_{S\in B, T\in H}$. Here, the rows and the columns of $X(B, H)$ are ordered by the lexicographic order on $\binom{[n]}{k}$.

The following definition appeared in [9, Section 4]. For a k-uniform hypergraph $H \subseteq \binom{[n]}{k}$ define a matroid $M'(H)$ on $\binom{[n]}{k}$ as follows: $B \subseteq \binom{[n]}{k}$ is independent in $M'(H)$ if and only if the rows of $X(B, H)$ are linearly independent.

Remark 3.1. *[9, Section 4, Remark (2)] For $H = \{S \subseteq \binom{[n]}{2} : S \cap [k] \neq \emptyset\}$, $M'(H)$ is the k-hyperconnectivity matroid.*

In fact, a more general statement is straightforward from the construction above. We call $H \subseteq \binom{[n]}{k}$ an *initial lex-segment* if $H = \left\{T' \in \binom{[n]}{k} | \; T' \leq_{\text{lex}} T\right\}$ for some $T \in \binom{[n]}{k}$.

Corollary 3.2. *Let $H \subseteq \binom{[n]}{k}$ be a k-uniform hypergraph whose hyperedges form an initial lex-segment. Then H is matroidal.*

Proof. We will verify that $M(H)$ is the set of bases of the matroid $M(H')$. Let $B \in M(H)$, then H is a column basis chosen in the greedy way for the matrix $X(B, \binom{[n]}{k})$. Thus, the columns of the matrix $X(B, H)$ are linearly independent, hence $B \in M'(H)$.

For a basis $B \in M'(H)$ let us compute the exterior shifting of B: as the columns in $X(B, H)$ are linearly independent, and as for exterior shifting the basis of columns in the compound matrix is chosen in the greedy way, $\Delta^e(B)$ is given by some set G corresponding to $|H|$ columns lexicographically smaller than or equal to H. However, since H is an initial lex-segment, we conclude that this set G coincides with H, namely $\Delta^e(B) = H$. Thus, $M(H)$ and $M'(H)$ coincide as desired. $\square$

3.1 Preparatory lemmas

In this section we present a series of lemmas and reductions which we will use in proofs of Theorems 1.2 and 1.3. In order to show that a shifted but not initial lex-segment k-uniform hypergraph H, satisfying some *extra condition*, is not matroidal, we will choose a permutation π on $[n]$ and an element $e_1 \in H$ for which we then show that for every $e_2 \in \pi(H) \setminus H$ holds $\Delta((H \setminus e_1) \cup e_2) \neq H$. Thus, using Theorem 2.1(2), we will conclude that $M(H)$ violates the exchange axiom. Here is the first such *extra condition*, giving the first reduction.

Lemma 3.3 (Isolated vertex)**.** *Let $H \subseteq \binom{[n]}{k}$ be a shifted k-uniform hypergraph such that n is an isolated vertex and $\{2, 3, \ldots, k+1\} \in H$. Then H is not matroidal.*

Sketch of proof. For the transposition $\pi = (1, n)$, the pair $(H, \pi(H))$ violates the exchange axiom on $M(H)$. To see this take $e_1 = [k]$ and argue by homology (Theorem 2.1(3)). $\square$

The lemma below serves in the inductive step in the proof of Theorem 1.2. While the base case is proven differently in the 2-uniform case and the 3-uniform case of Theorem 1.2, the induction step is proved uniformly in both cases, so it is separated in the lemma below. Denote $\mathrm{st}(1) := \{S \in \binom{[n]}{k} | \; 1 \in S\}$.

Lemma 3.4 (Reduction)**.** *Assume that for every shifted k-uniform hypergraph H whose hyperedges do not form an initial lex-segment and additionally satisfies the two conditions below, it holds that H is not matroidal.*

(i) $\{2,3,\ldots,k,k+1\}$ *is a hyperedge of* H.
(ii) There exists a subset $\{1,i_2,\ldots,i_k\}\notin H$ *with* $i_2>2$.
Then, for every shifted k-uniform hypergraph H whose hyperedges do not form an initial lex-segment it holds that H is not matroidal.

Sketch of proof. Let H be a shifted k-uniform hypergraph H which is not an initial lex-segment. If H fails (1) then H is the cone (with apex 1) over a shifted $(k-1)$-uniform hypergraph H' which is not an initial lex-segment. The assertion follows by induction on k from the observation that the cone of a not matroidal hypergraph is not matroidal.

Assume H fails (ii). Then $\mathrm{st}(1)\subseteq H$. We now prove by induction on n that H is not matroidal. By assumption $H'=H\setminus \mathrm{st}(1)$ is not an initial lex-segment (on $[n]\setminus\{1\}$), and by the induction hypothesis H' is not matroidal. Now show that if $G_1,G_2\in M(H')$ form a violating pair for the exchange axiom on $M(H')$ then $\mathrm{st}(1)\cup G_1,\mathrm{st}(1)\cup G_2\in M(H)$ form a violating pair for the exchange axiom on $M(H)$, using Theorem 2.1(6). $\square$

Lemma 3.5 (Homology)**.** *Let* $H\subseteq\binom{[n]}{k}$ *be a shifted hypergraph such that* $\{1,i_2,\ldots,i_k\}\notin H$ *and all facets of a k-simplex boundary complex* ∂T *are hyperedges of* H*. Then, for every facet* e_1 *of* ∂T *and every* $e_2\in\binom{[n]}{k}$ *such that* $\{i_2,\ldots,i_k\}\subset e_2$*, we have* $\Delta((H\setminus e_1)\cup e_2)\neq H$.

Sketch of proof. Show that $\beta_{k-1}((H\setminus e_1)\cup e_2)=\beta_{k-1}(H\setminus e_1)<\beta_{k-1}(H)$, and conclude by Theorem 2.1(3) on homology preserving. $\square$

Lemma 3.6 (Small e_2)**.** *Let* $H\subseteq\binom{[n]}{k}$ *be a shifted k-uniform hypergraph and let* $e_1\in H$*. Then for every* $e_2\in\binom{[n]}{k}$ *such that* $e_2\notin H$ *and* $e_2<_{lex}e_1$, $\Delta^e((H\setminus e_1)\cup e_2)\neq H$.

Sketch of proof. Denote $H':=(H\setminus e_1)\cup e_2$. Deduce from the definition of exterior shifting that $\Delta^e(H')\leq_{\mathrm{lex}}H'$. On the other hand, $H'<_{\mathrm{lex}}H$ since $e_2<_{\mathrm{lex}}e_1$. $\square$

Lemma 3.7 (Induced subgraph)**.** *Let* $H\subseteq\binom{[n]}{k}$ *be a shifted k-uniform hypergraph, let* $e_1\in H$ *and let* $e_2\in\binom{[n]}{k}$ *such that* $\max e_2<\max e_1$*. Then,* $\Delta((H\setminus e_1)\cup e_2)\neq H$.

Sketch of proof. Denote $t=\max e_2$ and $H':=\left\{I\in H|\ I\in\binom{[t]}{k}\right\}$. Then H' is shifted. By Theorem 2.1(4,5) conclude that $\Delta(H'\cup e_2)=H'\cup\{J\}$ for some $J\in\binom{[t]}{k}$. But $J\notin H$. $\square$

Lemma 3.8 (Case e_2 is almost small.)**.** *Let* $H\in\binom{[n]}{k}$ *be a shifted k-uniform hypergraph. Let* $e_1\in H$ *and let* $e_2\in\binom{[n]}{k}$ *such that* $e_2\notin H$*. Let* $S\in\binom{[n]}{k}$ *such that* $S<_{lex}e_1$, $S\notin H$ *and* $|e_2\setminus S|=|S\setminus e_2|=1$*. Then* $\Delta^e((H\setminus e_1)\cup e_2)\neq H$.

Sketch of proof. We prove that the compound matrix $X((H\setminus e_1)\cup e_2;(H\setminus e_1)\cup S)$ is invertible. In order to do that we first reorder its rows and columns in a certain convenient way, see Fig. 1. Then we prove that in $\det X((H\setminus e_1)\cup e_2;(H\setminus e_1)\cup S)$ the coefficient of the monomial obtained by choosing the dashed elements in Fig. 1 is equal to 1. The key observation here is that this monomial is a unique monomial of the form $\prod_{j=1}^{k(|H|-1)+(k-1)} x_{i_j i_j} \cdot x_{pq}$, where $p \neq q$.

Since the matrix $X((H \setminus e_1) \cup e_2; (H \setminus e_1) \cup S)$ is invertible, the basis for the column space of $X((H \setminus e_1) \cup e_2; \binom{[n]}{k}))$ chosen in the greedy way is not lexicographically greater than $(H \setminus e_1) \cup S$. Thus, $\Delta^e((H \setminus e_1) \cup e_2) \leq_{\text{lex}} (H \setminus e_1) \cup S$. However, since $S <_{\text{lex}} e_1$, we have that $(H \setminus e_1) \cup S <_{\text{lex}} H = \Delta^e(H)$. Taking these together we conclude that $\Delta^e((H \setminus e_1) \cup e_2) <_{\text{lex}} H$. □

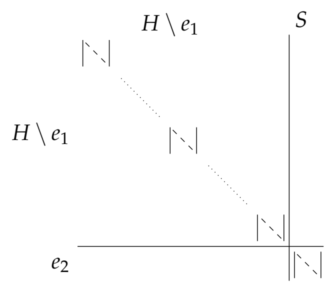

Figure 1: Compound matrix $X((H \setminus e_1) \cup e_2; (H \setminus e_1) \cup S)$ and diagonal monomial.

3.2 Proofs of main results

In this section we prove Theorems 1.2 and 1.3 and Proposition 1.4.

Proof of Theorem 1.2, 2-uniform case. By Corollary 3.2 if the edges of H form an initial lex-segment, then H is matroidal. For the opposite direction, by Lemma 3.4 and the shiftedness of H we can assume that $\{1, n\} \notin H$ and $\{2, 3\} \in H$. Thus, by Lemma 3.3 we get that H is not matroidal. □

Recall that m_H denotes the maximum element of H with respect to the lexicographic order, and the convention of writing in order the elements of a set.

Sketch of proof of Theorem 1.2, 3-uniform case. By Corollary 3.2 if H is an initial lex-segment, then H is matroidal. For the opposite direction, By Lemma 3.4 we can assume that $\{2, 3, 4\} \in H$, and that there is $I \in \binom{[n]}{3}$ such that $1 \in I \notin H$. Denote by $\{1, j, i\}$ the triple not in H such that for every other triple $\{1, j', i'\} \notin H$, either $i < i'$, or $i = i'$ and $j < j'$.

We now define a suitable permutation π and prove that the pair $(H, \pi(H))$ violates the exchange axiom among the elements of $M(H)$. In order to do that we have to provide an element $e_1 \in H \setminus \pi(H)$ such that for every $e_2 \in \pi(H) \setminus H$ we have that $(H \setminus e_1) \cup e_2 \notin M(H)$. We split the argument according to the following two cases.

Case 1: $\{2, 3, n\} \in H$. Consider the transposition $\pi = (1, i-1)$ on $[n]$ written in one-row notation. We have $\{1, 2, n\} \in H$ and so we set $e_1 = \{1, 2, n\}$.

By the definition of π we can also get that $i-1 \in e_2$, and we argue depending on whether $i-1 = \max e_2$ either by Lemma 3.5 or by Lemma 3.7.

Case 2: $\{2,3,n\} \notin H$. Denote $m_H =: \{a,b,c\}$, in order. We split the argument according to the following two subcases: either $a=2, b=3$ or not, and find a suitable permutation π for each. We set $e_1 = m_H$ for both cases.

1. $\{a,b\} \neq \{2,3\}$. Define a permutation $\pi : [n] \to [n]$ by the following conditions: the restrictions $\pi : [b] \to [b-1] \cup \{n\}$ and $\pi : [n] \setminus [b] \to [n] \setminus ([b-1] \cup \{n\})$ are the order-preserving bijections.

If $e_2 <_{\text{lex}} m_H$, then by Lemma 3.6 we get $(H \setminus m_H) \cup e_2 \notin M(H)$. If $e_2 >_{\text{lex}} m_H$, then by the definition of π and m_H we get that e_2 has the form $e_2 = \{a, \pi(i), n\}$ for some $b < i \leq c$. We argue by Lemma 3.8 with $e_1 = m_H$, and with $S = \{2, a, n\}$ if $a \neq 2$ and $S = \{a = 2, b-1, n\}$ otherwise (here $b - 1 \neq 2$ as we assumed $\{a,b\} \neq \{2,3\}$).

2. $\{a,b\} = \{2,3\}$. Define a permutation $\pi : [n] \to [n]$ by the following conditions: the restrictions $\pi : [3] \to \{1, n-1, n\}$ and $\pi : [n] \setminus [3] \to [n] \setminus \{1, n-1, n\}$ are the order-preserving bijections. Set $e_1 = m_H$.

If $e_2 <_{\text{lex}} m_H$, then by Lemma 3.6 we get $(H \setminus m_H) \cup e_2 \notin M(H)$. If $e_2 >_{\text{lex}} m_H$, then e_2 has the form $e_2 = \{\pi(i), n-1, n\}$ for some $b < i \leq c$. We argue by Lemma 3.5, with $T = \{1,2,3,c\}$ and $i_2 = n-1$, $i_3 = n$. $\square$

We now turn to prove Theorem 1.3. Recall that this theorem applies to shifted k-uniform hypergraphs which do not contain a certain k-subset s_H defined in terms of m_H. We define s_H by (3.2)-(3.4) based on the following two cases:

Case 1: The maximum element of m_H is not equal to n. Then m_H has the form

$$m_H = \{a, \ldots, b, c, c+1, \ldots, c+l, d\} \tag{3.1}$$

with $d \neq n$ and $b < c-1$, where $l \in \mathbb{Z}_{\geq 0}$. Note that possibly c is the smallest element of m_H in which case $m_H = \{c, c+1, c+2, \ldots, c+l, d\}$. By the shiftedness of H we can assume $c \neq 1$. Then define:

$$s_H = \{a, \ldots, b, c-1, n-(l+1), n-l, \ldots, n-1\}. \tag{3.2}$$

Case 2: The maximum element of m_H is equal to n. Then m_H has the form

$$m_H = \{a, \ldots, b, c, c+1, c+2, \ldots, c+l, n-m, n-m+1, \ldots, n-1, n\}, \tag{3.3}$$

where $b < c-1$, $l, m \in \mathbb{Z}_{\geq 0}$. By the shiftedness of H we can assume $c + l < n - m - 1$ and $c \neq 1$ (still c might be the smallest one, and then $m_H = \{c, c+1, c+2, \ldots, c+l, n-m, n-m+1, \ldots, n-1, n\}$). Then define:

$$s_H = \{a, \ldots, b, c-1, n-m-l, n-m-l+1, \ldots, n-m-1, n-m, \ldots, n\}. \tag{3.4}$$

Note that in both cases s_H is a k-subset of $[n]$ since $b \neq c-1$ and $c > 1$.

Sketch of proof of Theorem 1.3. By Corollary 3.2 if the hyperedges of H form an initial lex-segment, then H is matroidal. We now prove the opposite direction.

It suffices to find for each of the cases 1 and 2 separately a permutation such that there exists a unique s_H obtaining the following properties: $S <_{\text{lex}} m_H$, $S \notin H$ and for every $e_2 \in \pi(H) \setminus H$ such that $e_2 \geq_{\text{lex}} m_H$ holds $|e_2 \setminus S| = |S \setminus e_2| = 1$. Then we can argue by Lemma 3.8 in this case; and by Lemma 3.6 for $e_2 \leq_{\text{lex}} m_H$.

Figures 2(a) and 2(b) pictorially explain the definition of the appropriate π according to Cases 1 and 2 respectively. The element $m_H \in H$ is drawn in gray above the horizontal line and its image $\pi(m_H)$ is drawn below the horizontal line. On the set of elements $[n] \setminus m_H$ the permutation π is defined to be the order-preserving bijection. □

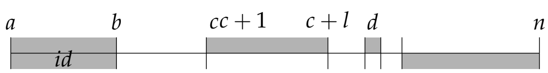

Figure 2(a): Image of m_H under the permutation π, Case 1.

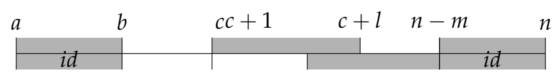

Figure 2(b): Image of m_H under the permutation π, Case 2.

Remark 3.9. *Let us now restrict our attention to a class of shifted k-uniform hypergraphs which are given by picking a single k-subset and closing down with respect to the partial order $<_p$. Shifted pure simplicial complexes of this type were called shifted matroids in [1, Section 4], and principal order ideals in [11, Section 5.4]. For a shifted hypergraph which belongs to this class and which is not an initial lex-segment, we have $s_H \notin H$. It follows that Theorem 1.3 covers all shifted hypergraphs from this class.*

Sketch of proof of Proposition 1.4. By Corollary 3.2 if H is an initial lex-segment, then H is matroidal, and hence lex-matroidal. For the opposite direction, by Lemma 3.4 we can assume that $S := \{2, 3, \ldots, k+1\} \in H$, and that there is $I \in \binom{[n]}{k}$ such that $1 \in I$, $2 \notin I$ and $I \notin H$. Consider the permutation $\pi : [n] \to [n]$ defined by the following conditions: the restrictions $\pi : \{3, 4, \ldots, k+1\} \to I \setminus \{1\}$ and $\pi : [n] \setminus \{3, 4, \ldots, k+1\} \to [n] \setminus (I \setminus \{1\})$ are the order-preserving bijections. Apply Lemma 3.5 to the pair $(H(S), \pi(H(S)))$, with $T = \{1, 2, 3, \ldots, k+1\}$ and $\{i_2, \ldots, i_k\} = I \setminus \{1\}$. □

4 Matroids arising from symmetric shifting

Let H be a shifted k-uniform hypergraph on $[n]$. Denote by

$$M^s(H) = \{G \subseteq \binom{[n]}{k} \mid \Delta^s(G) = H\}$$

the set of all k-uniform hypergraphs whose symmetric shifting is H. We say that H is *s-matroidal* if $M^s(H)$ is the set of bases of a matroid on $\binom{[n]}{k}$.

Proposition 4.1. *A simple graph $H \subseteq \binom{[n]}{2}$ is s-matroidal if and only if the edges of H form an initial lex-segment.*

Sketch of proof. The proof of the "if" direction is similar in nature to its exterior analog but is more technical; we omit the details. For the proof of the "only if" direction the same proof as of Theorem 1.2($k = 2$) works, as the only properties of exterior shifting it used are Theorem 2.1(3,6) which hold for symmetric shifting as well. □

5 Concluding remarks

We state here several directions for further research related to our results.

We characterized s-matroidal simple graphs in Proposition 4.1. The next natural question is to study s-matroidal hypergraphs.

Problem 5.1. *Characterize which shifted k-uniform hypergraphs are s-matroidal.*

Note that algebraic shifting can be defined not only with respect to the lexicographic order, but also with respect to any linear extension of the partial order $<_p$, see [10, Section 2.1, Remark]. Thus, one can extend the definition of matroidal hypergraph with respect to any linear extension of $<_p$ and try to characterize them. Knowing Corollary 3.2 it is easy to see that if $H \subseteq \binom{[n]}{k}$ is an initial segment with respect to some linear extension of $<_p$, then H is matroidal in this extended sense if we shift with respect to the same linear extension. The other direction is not clear.

Problem 5.2. *Characterize combinatorially which shifted k-uniform hypergraphs are matroidal when we shift with respect to an arbitrary linear extension of the partial order $<_p$.*

Any initial segment H with respect to the lexicographic order gives rise to a matroid $M(H)$ which we can describe in two different ways. The first one is via exterior shifting. The second one is via a concrete representation given by the matroid $M'(H)$, as described in Section 3. For the Crapo-Rota simplicial matroid we have a third description which is a combinatorial characterization, see Section 1. However, for other initial segments H such a combinatorial characterization of the matroid $M(H)$ is not known. For area-rigidity, and more generally volume-rigidity, this was addressed in [5, Problem 5.3].

Problem 5.3. *Find a combinatorial characterization of the matroid $M(H)$ for every initial lex-segment H.*

One can also hope to find a rigidity-style description for other matroids given by an initial lex-segment, as it exists for $H = \{\{i,j\} | \ \{i,j\} \leq_{\text{lex}} \{k,n\}\}$ via hyperconnectivity and for $H = \{\{1,i,j\} | \ \{1,i,j\} \leq_{\text{lex}} \{1,3,n\}\}$ via area-rigidity. A computational direction is whether there exists a deterministic polynomial-time algorithm that given a uniform hypergraph checks its membership in $M(H)$ for a fixed initial lex-segment hypergraph H. The computation of algebraic shifting involves the computation of symbolic determinants; see [10, Problem 8]. Even the case of 2-hyperconnectivity is open [3].

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