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A Macdonald expansion of the q-chromatic symmetric functions and the Stanley–Stembridge Conjecture

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Abstract. The Stanley–Stembridge Conjecture asserts that the chromatic symmetric function of the incomparability graph of a $(3+1)$-free poset is e-positive. Recently, Hikita proved this conjecture by giving an explicit e-expansion of the Shareshian–Wachs q-chromatic refinement for unit interval graphs. Using the $\mathbb{A}_{q,t}$ algebra, we give an expansion of these q-chromatic symmetric functions into Macdonald polynomials. Since q-chromatic symmetric functions do not depend on t, we obtain expansions into different bases of symmetric functions by specializing t. Upon setting $t = 1$, we rederive Hikita's formula and obtain another proof of the Stanley–Stembridge Conjecture. Upon setting $t = 0$, we obtain an expansion into Hall–Littlewood symmetric functions.

Keywords: Chromatic symmetric function, e-positivity, Macdonald polynomials

1 Introduction

In 1995, Stanley [14] introduced the *chromatic symmetric functions*, which generalize the chromatic polynomial of a graph. The Stanley–Stembridge Conjecture [15], a long-standing open problem in algebraic combinatorics, states that the chromatic symmetric function of the incomparability graph of a finite $(3+1)$-free poset is *e-positive*, meaning that it expands positively in the elementary symmetric function basis. By [7], the conjecture can be reduced to proving the e-positivity of unit interval graphs. Shareshian and Wachs [13] subsequently generalized the conjecture by introducing a certain refinement, the *chromatic quasisymmetric functions* (or *q-chromatic symmetric functions*) $\chi_{\mathbf{e}}[X;q]$, which they conjecture are also e-positive.

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Recently, Hikita [10] proved the Stanley–Stembridge Conjecture by giving a formula for the coefficients $c_{\mathbf{e},\lambda}(q)$ in the e expansion of $\chi_{\mathbf{e}}[X;q]$ as a sum of rational functions in q, which then specialize to nonnegative numbers upon taking $q=1$.

On the other hand, Carlsson and Mellit [2], in their proof of the long-standing Shuffle Conjecture, proved a plethystic formula relating $\chi_{\mathbf{e}}[X;q]$ to LLT polynomials, and they related the corresponding LLT polynomials to the action of a certain algebra, denoted by $\mathbb{A}_{q,t}$, on symmetric functions. In [1] the action of $\mathbb{A}_{q,t}$ was explicitly computed in a certain basis, generalizing the Macdonald basis.

In this article, we show that combining these results produces an explicit formula for the expansion of the LLT polynomials in the basis of modified Macdonald polynomials $\widetilde{\mathrm{H}}_{\lambda}[X;q,t]$. Applying the plethysm leads to an explicit expansion of $\chi_{\mathbf{e}}[X;q]$ in the basis of integral form Macdonald polynomials (2.1). For all details not written here, see the full article [6].

Our main result can be stated as follows:

Theorem 1.1 (Theorem 3.3)**.** *We have explicit tableau formulas for the coefficients $C_{\mathbf{e},\mu}(q,t)$ appearing in the expansion*

$$\chi_{\mathbf{e}}[X;q]=\sum_{\mu}C_{\mathbf{e},\mu}(q,t)(q-1)^{-n}\widetilde{\mathrm{H}}_{\mu}[(q-1)X;q,t], \tag{1.1}$$

expanding $\chi_{\mathbf{e}}[X;q]$ in terms of plethystically evaluated modified Macdonald polynomials.

Observe that since $\chi_{\mathbf{e}}[X;q]$ does not depend on t, the sum on the right-hand side of (1.1) is independent of t.

Corollary 1.2 (Corollary 3.7)**.** *Setting $t=1$ in* (1.1)*, we get*

$$\chi_{\mathbf{e}}[X;q]=\sum_{\mu}C_{\mathbf{e},\mu}(q,1)\left(\prod_{i}[\mu_i]_q!\right)e_{\mu}[X].$$

Upon setting $q=1$, we find that the coefficient of e_μ is nonnegative, so the Stanley–Stembridge Conjecture holds. Furthermore, the coefficient of e_μ is the same as the one found in [10].

The coefficient of e_μ in $\chi_{\mathbf{e}}[X;q]$ can be counted by certain admissible proper colorings, see Corollary 3.7 and Remark 3.8.

Corollary 1.3. *Upon setting $t=0$, we expand $\chi_{\mathbf{e}}[X;q]$ into Hall–Littlewood Q polynomials, see Corollary 4.1. Setting $t=q^{-1}$ gives an expansion of $\chi_{\mathbf{e}}[X;q]$ into Schur functions, see Remark 4.2.*

It would be interesting to see if there is any connection between this formula and the (q,t)-chromatic symmetric functions recently defined by Hikita [11].

2 Background

2.1 The $\mathbb{A}_{q,t}$ algebra

Our main method is to use the $\mathbb{A}_{q,t}$ algebra, which was introduced by Carlsson and Mellit [2] to prove the Shuffle Theorem [8] and its compositional refinement [9].

Let $\Lambda[X]$ be the space of symmetric functions with coefficients in $\mathbb{C}(q,t)$. Let $V_k = \Lambda[X] \otimes \mathbb{C}[y_1,\dots,y_k]$. We will only need the following operators appearing in the $\mathbb{A}_{q,t}$ algebra.

Definition 2.1. For any expression $F(y_i, y_{i+1})$, let

$$T_i F = \frac{(q-1)y_i F(y_i,y_{i+1}) + (y_{i+1} - q y_i) F(y_{i+1}, y_i)}{y_{i+1} - y_i}.$$

Define $d_+ : V_k \to V_{k+1}$ by setting

$$d_+ F[X] = T_1 \cdots T_k F[X + (q-1) y_{k+1}],$$

and define $d_- : V_k \to V_{k-1}$ by setting

$$d_- F[X] = -F[X - (q-1)y_k] \operatorname{Exp}\left[-y_k^{-1} X\right] \Big|_{y_k^{-1}},$$

where $\operatorname{Exp}[X] = \sum_{n\geq 0} h_n[X]$ is the plethystic exponential, and $f\big|_{y_k^{-1}}$ means take the coefficient of y_k^{-1} in f.

2.2 q-chromatic symmetric functions

Given a finite graph G with edges E and vertices $V = \{v_1,\dots,v_m\}$, a **proper coloring of** G is a function $\kappa : V \to \mathbb{Z}_{>0}$ such that $\kappa(v) \neq \kappa(w)$ whenever $vw \in E$. Given a proper coloring κ (and the ordering on V), the **ascents** of κ are

$$\operatorname{asc}(\kappa) := \#\{(i<j) : v_i v_j \in E,\ \kappa(v_i) < \kappa(v_j)\}.$$

The **chromatic quasisymmetric function** introduced by Shareshian and Wachs [13] is given by

$$\chi_G[X;q] := \sum_{\kappa} q^{\operatorname{asc}(\kappa)} x^{\kappa},$$

where the sum is over all proper colorings of G and $x^\kappa = \prod_i x_{\kappa(v_i)}$.

A **unit interval graph** is a type of graph whose chromatic quasisymmetric function has particularly nice properties. One equivalent way of defining such a graph is as follows: Let $[n] = \{1,\dots,n\}$ and $\mathbf{e} : [n] \to \{0,1,\dots,n-1\}$ be a **reverse Hessenberg**

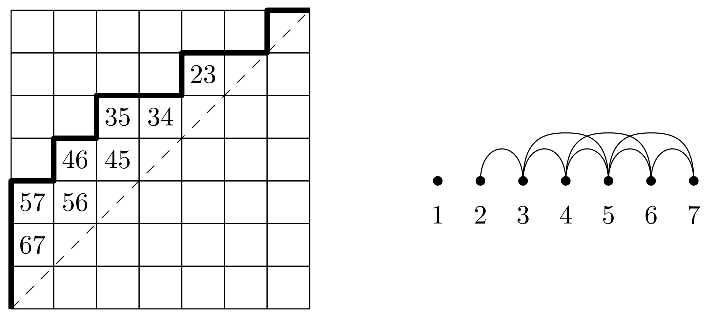

Figure 1: The Dyck path D and unit interval graph $\Gamma_{\mathbf{e}}$ associated to the reverse Hessenberg function $\mathbf{e} = (0,1,1,2,2,3,4)$. Cells are labeled by the edges, ij, present in the graph.

function, which is a function that is weakly increasing such that $\mathbf{e}(i) < i$ for all i. Then the unit interval graph $\Gamma = \Gamma_{\mathbf{e}}$ is the graph with vertex set $V = [n]$ and edges $\{ij : \mathbf{e}(j) < i < j\}$. We will write $\chi_{\mathbf{e}}[X;q] = \chi_{\Gamma_{\mathbf{e}}}[X;q]$.

Before Hikita's proof of e-positivity, it was known that $\chi_{\mathbf{e}}[X;q]$ is a Schur positive symmetric function in the case of unit interval graphs by Shareshian and Wachs [13] who refined a formula for $\chi_{\mathbf{e}}[X;1]$ of Gasharov [5]. In this article, we will restrict our attention to unit interval graphs and refer to the chromatic quasisymmetric function as a **q-chromatic symmetric function**.

A Dyck path $D = W^{a_1}S^{b_1}\cdots W^{a_\ell}S^{b_\ell}$ is a sequence of West and South unit lattice steps, so that the lattice path starting at (n,n) and taking West $(-1,0)$ and South $(0,-1)$ steps according to the word D stays weakly above the diagonal of the $n \times n$ grid and ends at $(0,0)$.

Let $\mathbf{e}_D(i)$ be the number of South steps preceding the i-th West step of D. In this way, we associate to every $D \in D_n$ a reverse Hessenberg function $\mathbf{e}_D$ of length n. Any reverse Hessenberg function arises from a unique Dyck path in this way. So we will identify Dyck paths with reverse Hessenberg functions and write $\mathbf{e} \in D_n$. See Figure 1 for D and the unit interval graph corresponding to $\mathbf{e} = (0,1,1,2,2,3,4)$.

Carlsson and Mellit showed that the q-chromatic symmetric function can be obtained from d_+ and d_- by first writing

$$F_D[X;q] := d_-^{b_\ell} d_+^{a_\ell} \cdots d_-^{b_1} d_+^{a_1}(1),$$

then setting

$$\chi_D[X;q] := (q-1)^{-n} F_D[(q-1)X].$$

Proposition 2.2 ([2, Theorem 4.4+Proposition 3.5]). *For any reverse Hessenberg function* $\mathbf{e}$ *with the corresponding Dyck path* $D \in D_n$, *we have*

$$\chi_D[X;q] = \chi_{\mathbf{e}}[X;q].$$

We will write $F_{\mathbf{e}}$ for the corresponding function F_D where D is such that $\mathbf{e} = \mathbf{e}_D$.

2.3 Macdonald symmetric functions

For the reader's convenience, we mention the following conversion formulas between the modified Macdonald polynomials from [4] and the integral form Macdonald polynomials from [12]:

$$\widetilde{\mathrm{H}}_\mu[(q-1)X;q,t] = q^{n(\mu')+|\mu|} J_{\mu'}[X;t,q^{-1}] = (-1)^{|\mu|} t^{n(\mu)} J_{\mu'}[X;t^{-1},q] \tag{2.1}$$

where $n(\mu) = \sum_i (i-1)\mu_i$ and $|\mu| = \sum_i \mu_i$.

The modified Macdonald polynomials $\widetilde{\mathrm{H}}_\mu[X;q,t] = \widetilde{\mathrm{H}}_\mu[X]$ specialize to modified homogeneous symmetric functions:

$$\widetilde{\mathrm{H}}_\mu[X;q,1] = (q;q)_\mu h_\mu \left[\frac{X}{1-q}\right],$$

where $(q;q)_\mu = h_\mu[1/(1-q)]^{-1}$ is the Pochhammer symbol. Therefore,

$$\widetilde{\mathrm{H}}_\mu[(q-1)X;q,1] = (-1)^{|\mu|}(q;q)_\mu e_\mu[X],$$

where we have used that $h_\mu[-X] = (-1)^{|\mu|}\omega(h_\mu)[X] = (-1)^{|\mu|} e_\mu$. As a consequence, the elementary basis expansion of $\chi_{\mathbf{e}}[X;q]$ can be found by the expansion of $F_{\mathbf{e}}$ in the basis $\{h_\mu[X/(1-q)]\}_\mu$.

Our goal will be to first write $F_{\mathbf{e}}$ in terms of the modified Macdonald basis:

$$F_{\mathbf{e}}[X;q] = \sum_\mu C_{\mathbf{e},\mu}(q,t)\ \widetilde{\mathrm{H}}_\mu[X;q,t].$$

Since $\chi_{\mathbf{e}}[X;q]$ has no t variable, we must then also have

$$\begin{aligned} F_{\mathbf{e}}[X;q] &= \sum_\mu C_{\mathbf{e},\mu}(q,1)\ \widetilde{\mathrm{H}}_\mu[X;q,1] \\ &= \sum_\mu C_{\mathbf{e},\mu}(q,1)(q;q)_\mu h_\mu \left[\frac{X}{1-q}\right]. \end{aligned}$$

From here, it follows that

$$\chi_{\mathbf{e}}[X;q] = \sum_\mu C_{\mathbf{e},\mu}(q,1)(1-q)^{-|\mu|}(q;q)_\mu e_\mu[X], \tag{2.2}$$

and Theorem 1.1 will follow.

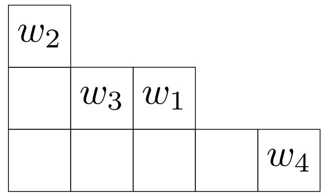

Figure 2: An indexing object for the basis of V_k where $\lambda = (5,3,1)$ and $k = 4$

2.4 The action of $d_\pm$ on $I_{\lambda,w}$

We now think of partitions as being given by a sum of cells. We say $(i,j) \in \lambda$ if $0 \leq j < \ell(\lambda)$ and $0 \leq i < \lambda_{j+1}$. To simplify notation, we will then write

$$\lambda = \sum_{(i,j)\in\lambda} q^i t^j.$$

For a given λ, we say $x = q^r t^s$ is an **addable cell** if $\lambda + x = \mu$ is a partition, and we will then write $x \in \mathrm{Add}(\lambda)$. The cells in λ that are in the same row as x are denoted by $\mathcal{R}_{\lambda,x}$; and similarly, the cells in λ that are in the same column as x are denoted by $\mathcal{C}_{\lambda,x}$. For a given cell $c \in \lambda$, let $a_\lambda(c)$ denote the arm length of c in λ (that is, the number of cells to its right), and let $l_\lambda(c)$ be its leg length (that is, the number of cells above c in the French convention).

If $x \in \mathrm{Add}(\lambda)$, we let

$$d_{\lambda,x} = \prod_{c\in\mathcal{R}_{\lambda,x}} \frac{q^{a_\lambda(c)} - t^{l_\lambda(c)+1}}{q^{a_\lambda(c)+1} - t^{l_\lambda(c)+1}} \prod_{c\in\mathcal{C}_{\lambda,x}} \frac{q^{a_\lambda(c)+1} - t^{l_\lambda(c)}}{q^{a_\lambda(c)+1} - t^{l_\lambda(c)+1}}.$$

These factors appear in the Pieri formula for $e_1\widetilde{\mathrm{H}}_\lambda[X]$, see [3].

A basis for V_k is given by certain elements of the form $I_{\lambda,w}$, where $w = (w_1, \ldots, w_k)$ is a sequence of cells in λ which form a horizontal strip, meaning no two cells are in the same column and each row is connected and right justified in λ. The list is written such that $\lambda - w_1 - w_2 - \cdots - w_s$ forms a partition for each $s \leq k$, meaning that if we place i in the cell w_{k-i+1}, we get a standard tableau of the horizontal strip $w_1 + \cdots + w_k$.

In [1], Carlsson, Gorsky, and Mellit construct an action of $\mathbb{A}_{q,t}$ on the direct sum of the (localized) equivariant K-theory of parabolic flag Hilbert schemes, $K_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{PFH}_{n,n-k})$; moreover, they show that

$$\bigoplus_k K_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{PFH}_{n,n-k}) \cong \bigoplus_k V_k$$

as $\mathbb{A}_{q,t}$-modules. The basis $\{I_{\lambda,w}\}$ corresponds to the distinguished basis of the left-hand side given by the classes of torus fixed points.

It will not be important to know explicitly what the elements $I_{\lambda,w}$ are. Instead, we rely on the following three facts.

Proposition 2.3 ([1])**.** *Given a partition λ and sequence $w = (w_1, \ldots, w_k)$ forming a horizontal strip (as described above), we have*

$$I_{\lambda,\varnothing} = \widetilde{\mathrm{H}}_\lambda[X]/\widetilde{\mathrm{H}}_\lambda[-1] \in V_0, \tag{2.3}$$

$$d_- I_{\lambda,wx} = I_{\lambda,w}, \text{ and} \tag{2.4}$$

$$d_+ I_{\lambda,w} = -q^k \sum_{x \in \mathrm{Add}(\lambda)} x d_{\lambda,x} \left(\prod_{i=1}^{k} \frac{x - tw_i}{x - qtw_i} \right) I_{\lambda+x,xw}. \tag{2.5}$$

We will also need the following identity:

$$\widetilde{\mathrm{H}}_\mu[-1] = (-1)^{|\mu|} \prod_{(i,j)\in\mu} q^i t^j. \tag{2.6}$$

3 Main proof

In this section, we prove Theorem 1.1 using the $\mathbb{A}_{q,t}$ algebra.

Notation 3.1. Let $\mathbf{e}$ be a reverse Hessenberg function. If $\mathbf{e}(j) < i < j$, then we will write $i \prec j$. Note that $\prec$ is not a partial order since it is not necessarily transitive.

For example, in the case $\mathbf{e} = (0, 1, 1, 2, 2, 3, 4)$ considered in Figure 1, the pairs $i \prec j$ are given by the edges of $\Gamma_{\mathbf{e}}$. Namely, $2 \prec 3$, $3 \prec 4$, $3 \prec 5$ and so on. This function corresponds to $\mathbf{e} = \mathbf{e}_D$ for the Dyck path $D = WSWWSWWSWSWSSS$, and the pairs $i \prec j$ are in bijection with the cells in the 7×7 grid between the diagonal and D.

Let SYT_μ be the set of standard Young tableaux of shape μ. For a given j, let $T_{<j}$ be the standard tableau given by the entries smaller than j. We will also let $T_{\prec j}$ be the skew-shaped tableau with entries i satisfying $i \prec j$. Similarly, define $T_{\leq j}$ and $T_{\preceq j}$ to include the entry j. We will say that a skew tableau is *strict* if its entries are increasing along rows and columns. A strict skew tableau T gives a sequence $w(T) = (w_1, \ldots, w_k)$ of cells, where w_1 is the cell with the largest entry, w_2 is the cell with the second largest entry, and so on until we reach the cell w_k with the smallest entry. For a strict skew tableau, we will also write $\mathrm{sh}(T)$ to denote the skew diagram with no fillings.

First, we write

$$F_{\mathbf{e}}[X; q] = d_-^{r_n} d_+ d_-^{r_{n-1}} \cdots d_+ d_-^{r_1} d_+ (I_{\varnothing,\varnothing}), \tag{3.1}$$

where $r_1 + \cdots + r_j = \mathbf{e}(j+1)$, $r_1 + \cdots + r_n = n$. Note that $\mathbf{e}(1) = 0$. Suppose that after we apply $d_+ d_-^{r_{j-1}} \cdots d_-^{r_1} d_+$ we arrive at a term with $I_{\lambda,w}$. Applying $d_-^{r_j}$ eliminates r_j elements from w giving w', leaving only the last $j - \mathbf{e}(j+1)$ cells which were added to λ; and $j - \mathbf{e}(j+1)$ is the number of $i \prec j+1$. Applying d_+ then adds a cell x to λ so that the sequence xw forms a strict horizontal strip. Keeping track of the order in which cells were added throughout this process gives us a tableau T, see Figure 3.

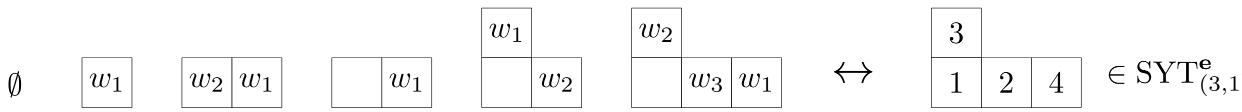

Figure 3: A sequence of indexing objects obtained in the expansion of $d_-^3 d_+^2 d_- d_+^2 (I_{\emptyset,\emptyset})$ and its corresponding tableau. Here, $\mathbf{e} = (0,0,1,1)$.

Let $\mathrm{SYT}^{\mathbf{e}}_{\lambda}$, be the set of standard tableaux T such that for each i, $T_{\preceq i}$ is a horizontal strip. It can then easily be checked that the tableaux obtained in the expansion of (3.1) are precisely $\mathrm{SYT}^{\mathbf{e}}_{\lambda}$.

Remark 3.2. We may alternatively define $\mathrm{SYT}^{\mathbf{e}}_{\lambda}$ as follows. Define a partial order on $[n]$, called the *unit interval order*, by $i \ll j$ if $i \leq \mathbf{e}(j)$. Then $\mathrm{SYT}^{\mathbf{e}}_{\lambda}$ is the set of standard tableaux whose rows increase with respect to $<$ and whose columns increase with respect to $\ll$.

Let

$$A_x^{\lambda,w} := \prod_{c \in \mathcal{R}_{\lambda,x}} \frac{q^{a_\lambda(c)} - t^{l_\lambda(c)+1}}{q^{a_\lambda(c)+1} - t^{l_\lambda(c)+1}} \times \prod_{c \in \mathcal{C}_{\lambda,x}} \frac{q^{a_\lambda(c)+1} - t^{l_\lambda(c)}}{q^{a_\lambda(c)+1} - t^{l_\lambda(c)+1}} \times \prod_{i=1}^{k} \frac{x - tw_i}{x - qtw_i}. \tag{3.2}$$

Let $T^{(i)} := \mathrm{sh}(T_{\leq i}) - \mathrm{sh}(T_{<i})$ be the cell containing i, and let

$$A_T^{\mathbf{e}}(q,t) := \prod_{i=1}^{n} A_{T^{(i)}}^{\mathrm{sh}(T_{<i}), w(T_{\prec i})}. \tag{3.3}$$

Lastly, note that $\prod_{i=1}^{n} -q^{|T_{\prec i}|} T^{(i)} = q^{\binom{n}{2} - |\mathbf{e}|} \widetilde{\mathrm{H}}_\lambda[-1]$, where $|\mathbf{e}| = \sum_i \mathbf{e}(i)$. Proposition 2.3 then tells us that

$$F_{\mathbf{e}}[X;q] = \sum_{\lambda} \widetilde{\mathrm{H}}_\lambda[X] / \widetilde{\mathrm{H}}_\lambda[-1] \left(\sum_{T \in \mathrm{SYT}^{\mathbf{e}}_{\lambda}} \prod_{i=1}^{n} -q^{|T_{\prec i}|} T^{(i)} A_{T^{(i)}}^{\mathrm{sh}(T_{<i}), w(T_{\prec i})} \right).$$

This gives us the next theorem.

Theorem 3.3. *We have the following expansions of $F_{\mathbf{e}}$ and $\chi_{\mathbf{e}}$:*

$$F_{\mathbf{e}}[X;q] = q^{\binom{n}{2} - |\mathbf{e}|} \sum_{\lambda} \left(\sum_{T \in \mathrm{SYT}^{\mathbf{e}}_{\lambda}} A_T^{\mathbf{e}}(q,t) \right) \widetilde{\mathrm{H}}_\lambda[X;q,t], \tag{3.4}$$

$$\chi_{\mathbf{e}}[X;q] = (q-1)^{-n} q^{\binom{n}{2} - |\mathbf{e}|} \sum_{\lambda} \left(\sum_{T \in \mathrm{SYT}^{\mathbf{e}}_{\lambda}} A_T^{\mathbf{e}}(q,t) \right) \widetilde{\mathrm{H}}_\lambda[(q-1)X;q,t]. \tag{3.5}$$

Let $\overline{\mathrm{SYT}}^{\mathbf{e}}_{\mu}$ denote all standard tableaux in $\mathrm{SYT}^{\mathbf{e}}_{\mu}$ with the following extra condition:

$$\text{For every } i \text{ not in the first column, there is a } j \text{ in the column immediately left of } i \text{ such that } j \prec i. \quad (*)$$

Remark 3.4. Having condition $(*)$ in place, together with the property that the columns increase with respect to $\ll$ (see Remark 3.2), it can be checked that the standard tableau conditions hold automatically.

Example 3.5. When $\mathbf{e} = (0,1,1,2,2,3,4)$,

$$T = \begin{array}{|c|c|c|} \hline 5 \\ \cline{1-1} 2 & 6 & 7 \\ \hline 1 & 3 & 4 \\ \hline \end{array}$$

is an element of $\mathrm{SYT}^{\mathbf{e}}_{3,3,1}$ since $T_{\preceq i}$ is a horizontal strip for each i. In order to satisfy $(*)$, 2 is forced to be in the first column since $1 \not\prec 2$. Furthermore, since $2 \prec 3$, $3 \prec 4$, $5 \prec 6$, and $6 \prec 7$, all entries i satisfy condition $(*)$. Hence, T is in $\overline{\mathrm{SYT}}^{\mathbf{e}}_{(3,3,1)}$.

Lemma 3.6. *For $T \in \mathrm{SYT}^{\mathbf{e}}_{\mu}$, $A^{\mathbf{e}}_{T}(q,1) \neq 0$ if and only if $T \in \overline{\mathrm{SYT}}^{\mathbf{e}}_{\mu}$.*

Thus, $\overline{SYT}^{\mathbf{e}}_{\mu}$ are exactly the tableaux that remain after setting $t = 1$. In fact, we get an explicit expansion for $A^{\lambda,w}_{x}(q,1)$, and therefore, an explicit expansion of $\chi_{\mathbf{e}}[X;q]$.

Corollary 3.7. *For a given standard tableau T, let $c_i(T)$ be the column of i, so that $T^{(i)}|_{t=1} = q^{c_i(T)}$. Then $\chi_{\mathbf{e}}[X;q]$ expands as*

$$q^{\binom{n}{2}-n-|\mathbf{e}|}\sum_{\lambda}\left(q^{\ell(\lambda)}[\lambda_1]_q\cdots[\lambda_{\ell(\lambda)}]_q \sum_{T\in\overline{\mathrm{SYT}}^{\mathbf{e}}_{\lambda}} \prod_{\substack{i\prec j\\ c_j(T)\neq c_i(T)+1}} \frac{q^{c_j(T)}-q^{c_i(T)}}{q^{c_j(T)}-q^{c_i(T)+1}}\right)e_{\lambda}[X]. \qquad (3.6)$$

Moreover, setting $q = 1$ in this identity yields a positive expansion of $\chi_{\mathbf{e}}[X;q]$ into the elementary basis. This proves the Stanley–Stembridge Conjecture. Explicitly:

$$\chi_{\mathbf{e}}[X;1] = \sum_{\lambda}\left(\lambda_1\cdots\lambda_{\ell(\lambda)} \sum_{T\in\overline{\mathrm{SYT}}^{\mathbf{e}}_{\lambda}} \prod_{\substack{i\prec j\\ c_j(T)\neq c_i(T)+1}} \frac{|c_j(T)-c_i(T)|}{|c_j(T)-c_i(T)-1|}\right)e_{\lambda}[X]. \qquad (3.7)$$

Remark 3.8. One can also interpret this formula in terms of proper colorings. We call a proper coloring κ on the graph $\Gamma_{\mathbf{e}}$ *admissible* if for every i with $\kappa(i) > 1$, there is an edge $v_j v_i$ (with $j < i$) for which $\kappa(j) = \kappa(i) - 1$.

Let $K^{\mathbf{e}}_{\lambda'}$ be the set of all admissible colorings κ on the graph $\Gamma_{\mathbf{e}}$ such that λ'_i is the multiplicity of the color i. To such a coloring we associate a weight

$$\mathrm{weight}(\kappa) = \prod_{\substack{\text{edges } v_iv_j, i<j \\ \kappa(j)\neq\kappa(i)+1}} \frac{|\kappa(j)-\kappa(i)|}{|\kappa(j)-\kappa(i)-1|}.$$

By interpreting $T \in \overline{\mathrm{SYT}}^{\mathbf{e}}_{\lambda}$ as a coloring κ_T where i is assigned the color $\kappa_T(i) = c_i(T)+1$, then (3.7) translates to

$$\chi_{\mathbf{e}}[X;1] = \sum_{\lambda} e_\lambda \sum_{\kappa\in K^{\mathbf{e}}_{\lambda'}} \lambda_1\cdots\lambda_{\ell(\lambda)}\mathrm{weight}(\kappa).$$

Example 3.9. For $\mathbf{e}$ in Figure 1, the $e_{3,3,1}$ coefficient of the q-chromatic symmetric function $\chi_{(0,1,1,2,2,3,4)}$ is $q^5+q^4+q^3$. From Corollary 3.7, it comes from a single tableau, see Example 3.5. Equation (3.6) produces

$$q^{21-7-13+3}[3]_q[3]_q[1]_q \times \frac{1-q}{1-q^2}\times\frac{1-q^2}{1-q^3}\times\frac{q-q^2}{q-q^3}\times\frac{q^2-1}{q^2-q},$$

the fractions coming from the pairs $(i,j) = (3,5),(4,5),(4,6),(5,7)$. Performing cancellations results in $q^3[3]_q$. On the other hand, the coefficient of $e_{4,2,1}$ comes from 3 tableaux and we observe non-trivial denominators:

$$q^2[4]_q[2]_q + q^3\frac{[4]_q[2]_q}{[3]_q} + q^3\frac{[2]_q^4}{[3]_q}.$$

For instance, setting $q=1$ produces a decomposition of 16 as $16 = 8+\frac{8}{3}+\frac{16}{3}$.

3.1 The relation to Hikita's formula

We will now find an equivalent expression that lets us re-derive Hikita's formula. Let, once again, λ, w, and x be given as in Proposition 2.3. Let $c_1<\cdots<c_k$ be the columns for $w=(w_1,\ldots,w_k)$ in increasing order. Let $w_1^L<\cdots<w_m^L$ be the subsequence of columns such that $w_j^L-1\neq c_i$ for some i, and let $w_1^R<\cdots<w_m^R$ be the subsequence of columns for which $w_j^R+1\neq c_i$ for some i. In other words, the sequence of columns $c_1<\cdots<c_k$ consists of a union of contiguous segments, and the left-most elements of these contiguous segments are denoted by w_i^L and the right-most elements of these contiguous segments are denoted by w_i^R.

Lemma 3.10. *Let $x=q^rt^s$ as in Theorem 3.3. When $r=0$,*

$$A_x^{\lambda,w}(q,1) = \prod_{i=1}^{m}\frac{[w_i^L]_q}{[w_i^R+1]_q}. \tag{3.8}$$

When $r > 0$, letting j be maximal such that w_j^R is on the left of x (in other words, $w_j^R = r-1$), then

$$A_x^{\lambda,w}(q,1) = q^{b(w,x)-r} \prod_{i=1}^{j} \frac{[r-w_i^L]_q}{[r-w_{i-1}^R-1]_q} \prod_{i=j+1}^{m} \frac{[w_i^L-r]_q}{[w_i^R+1-r]_q}, \tag{3.9}$$

where $w_0^R := -1$, and $b(w,x) = \sum_{i=1}^{j}(w_i^L - w_{i-1}^R - 1)$ counts the number of columns left of x with no w_j.

The proof is to specialize $t = 1$ in (3.2), divide numerators and denominators by appropriate powers of $(q-1)$ and cancel telescoping factors, and carefully keep track of q powers.

Using Lemma 3.10 on each factor of (3.3) (at $t = 1$), one can then relate $A_T^{\mathbf{e}}(q,1)$ to the product over transition probabilities in [10, Equation 5]. In particular, it can be checked that the tableaux T that appear with positive probability in [10, Theorem 3] are exactly the tableaux $\overline{\mathrm{SYT}}_\lambda^{\mathbf{e}}$ appearing in our formula (3.6).

4 A Hall–Littlewood expansion

In this section, we use Theorem 1.1 to give an expansion of $\chi_{\mathbf{e}}$ in terms of Hall–Littlewood symmetric functions. Combining Equation (2.1) with [12, p. VI.8.4.ii], we find that

$$\widetilde{\mathrm{H}}_\lambda[(q-1)X;q,0] = q^{|\lambda'|+n(\lambda')} J_{\lambda'}[X;0,q^{-1}] = q^{|\lambda'|+n(\lambda')} Q_{\lambda'}[X;q^{-1}].$$

Specializing $t = 0$ in Theorem 3.3 and reducing the expression gives us the following expansion in $Q_{\lambda'}[X;q^{-1}]$ functions.

Given $T \in \mathrm{SYT}_\lambda^{\mathbf{e}}$ and a label i, let $\mathrm{m}(T,i)$ be the number of entries of $T_{\prec i}$ that are at least 2 rows below $T^{(i)}$ in French notation. Furthermore, if $T^{(i)}$ is in row $r_i + 1$ (starting from 1), then let $\mathrm{d}(T,i) := \mathrm{sh}(T_{<i})_{r_i} - \mathrm{sh}(T_{<i})_{r_i+1}$. If $T_{\prec i}$ contains elements in row r_i, let $\mathrm{L}(T,i)$ be the column of its left-most element in row r_i of $T_{\prec i}$. Finally, we write T_1 for the first row of T.

Corollary 4.1. *We have the following expansion of $\chi_{\mathbf{e}}$ in terms of reversed Q_λ functions:*

$$\chi_{\mathbf{e}}[X;q] = \sum_\lambda Q_{\lambda'}[X;q^{-1}] \frac{q^{\binom{n+1}{2}-|\mathbf{e}|}}{(q-1)^{\lambda_1}} \times \left(\sum_{T\in\mathrm{SYT}_\lambda^{\mathbf{e}}} \prod_{i\notin T_1} q^{-\mathrm{m}(T,i)-\mathrm{d}(T,i)} \prod_{\substack{i\notin T_1,\\ (T_{\prec i})_{r_i}=\emptyset}} [\mathrm{d}(T,i)]_q \prod_{i:(T_{\prec i})_{r_i}\neq\emptyset} [\mathrm{L}(T,i) - \mathrm{sh}(T_{<i})_{r_i+1}]_q \right).$$

Remark 4.2. We note that a Schur expansion of $\chi_{\mathbf{e}}[X;q]$ can also be obtained by setting $t = q^{-1}$ in Theorem 3.3, which we do not write out explicitly here. It is not clear how this formula matches Shareshian and Wachs' Schur expansion in terms of P-tableaux [13].

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