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Bijectivity of a generalized Pak-Stanley labeling

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Abstract. The Pak-Stanley labeling is a bijection between the regions of the m-Shi arrangement and the m-parking functions. Mazin generalized this labeling to every deformation of the braid arrangement and proved that this labeling is always surjective onto a set of directed multigraph parking functions. We provide a right inverse to the generalized Pak-Stanley labeling, and identify a class $\mathcal{C}$ of arrangements for which this labeling is bijective. The class $\mathcal{C}$ includes the multi-Shi arrangements and the multi-Catalan arrangements. We also show that the arrangements in $\mathcal{C}$ are the only transitive arrangements for which the generalized Pak-Stanley labeling is bijective.

1 Introduction

A (real) hyperplane arrangement is a finite collection of affine hyperplanes in $\mathbb{R}^n$ for some integer $n \geq 1$. The hyperplanes partition the space into convex *regions*. It is a central question in the study of hyperplane arrangements to count the number of regions of a given hyperplane arrangement, and to provide a bijective encoding of these regions by some combinatorial objects.

For the Shi arrangement, Pak suggested a labeling of the regions using parking functions [6]. Stanley extended this labeling to m-Shi arrangements [7], and proved that it is a bijection between the regions of the n dimensional m-Shi arrangement and the m-parking functions of length n.

In [3], Mazin generalized the Pak-Stanley labeling to every *deformation of the braid arrangement*. These are the arrangements in which every hyperplane has the form

$$H_{i,j,s} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i - x_j = s\},$$

for some indices $i, j \in [n]$ and real number s. To any such arrangement $\mathcal{A}$ one can associate a directed multigraph $D_{\mathcal{A}}$, and Mazin showed that the generalized Pak-Stanley labeling is always surjective onto the set of $D_{\mathcal{A}}$-parking functions [3] (see Section 2.1).

However, the generalized Pak-Stanley labeling is not bijective in general, and it is an open question to characterize the arrangements for which it is bijective. In [1], Baker

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proved a necessary (but not sufficient) condition for injectivity (see also the slight improvement in [5, Theorem 1.30]). Mazin and Miller then showed that for central graphical arrangements, this condition is sufficient in [4].

In this paper, we define a right inverse to the generalized Pak-Stanley labeling. We then show that the generalized Pak-Stanley labeling is bijective for a family of arrangements that generalize the m-Shi and m-Catalan arrangements. Let us introduce some notation to describe this family.

Notation 1.1. Let $\mathbf{S} = (S_{i,j})_{1\leq i<j\leq n}$ be a collection of finite sets of integers. We define the *$\mathbf{S}$-braid arrangement* $\mathcal{A}_{\mathbf{S}}$ as the collection of the following hyperplanes:

$$\mathcal{A}_{\mathbf{S}} = \{H_{i,j,s} \mid 1 \leq i < j \leq n,\, s \in S_{i,j}\}.$$

For $i < j$, we denote $S^+_{i,j} = \{s > 0 \mid H_{i,j,s} \in \mathcal{A}_{\mathbf{S}}\}$ and $S^+_{j,i} = \{s \geq 0 \mid H_{j,i,s} \in \mathcal{A}_{\mathbf{S}}\}$.

The *m-Shi arrangement* is the arrangement $\mathcal{A}_{\mathbf{S}}$ for $S_{i,j} = [-m+1; m]$ for all $i < j$. The *m-Catalan arrangement* is the arrangement $\mathcal{A}_{\mathbf{S}}$ for $S_{i,j} = [-m; m]$ for all $i < j$.

We call *m-Catalan region* (resp. *$\mathbf{S}$-braid region*) a region of the m-Catalan (resp. $\mathbf{S}$-braid) arrangement.

Remark 1.2. Any deformation of the braid arrangement is combinatorially equivalent to an arrangement where the hyperplanes are of the form $H_{i,j,s}$ with $s \in \mathbb{Z}$, and any such arrangement is of the form $\mathcal{A}_{\mathbf{S}}$ for some tuple $\mathbf{S}$ of sets of integers.

Definition 1.3. An *$(\boldsymbol{m}, \varepsilon)$-arrangement* is an $\mathbf{S}$-braid arrangement for a tuple $\mathbf{S} = (S_{i,j})_{i<j}$ such that

$$S^+_{i,j} = \begin{cases} [1; m_j - \varepsilon_{i,j}] & \text{if } i < j \\ [0; m_j - \varepsilon_{i,j}] & \text{if } i > j \end{cases},$$

where $\mathbf{m} = (m_1, \ldots, m_n)$ is a tuple of non-negative integers, and $\varepsilon = (\varepsilon_{i,j})_{i\neq j\in[n]}$ is a tuple of numbers in $\{0, 1\}$ such that for all $i, j, k \in [n]$ with $i < j$ one has $\varepsilon_{i,k} \leq \varepsilon_{j,k}$.

We establish the bijectivity of the generalized Pak-Stanley labeling for every $(\mathbf{m}, \varepsilon)$-arrangement. Note that if $m_j = m$ for all j and $\varepsilon_{i,j} = 0$ if and only if $i < j$, the $(\mathbf{m}, \varepsilon)$-arrangement is the m-Shi arrangement. Hence we obtain a new proof of the bijectivity of the Pak-Stanley labeling. If $m_j = m$ for all j and $\varepsilon_{i,j} = 0$ for all i, j, the $(\mathbf{m}, \varepsilon)$-arrangement is the m-Catalan arrangement. From our bijectivity result, one can express the number of regions in any $(\mathbf{m}, \varepsilon)$-arrangement as a number of multigraph parking functions, which admits a determinantal formula via the matrix-tree theorem. The $(\mathbf{m}, \varepsilon)$-arrangements satisfy a condition called *transitivity* [2] (see Section 2.2) and we show that they are the only transitive arrangements for which the generalized Pak-Stanley labeling is bijective.

This paper is structured as follows. In Section 2, we set our notation and recall some known results. In Section 3, we define a map that is a right inverse for the generalized

Pak-Stanley labeling, thus recovering Mazin's surjectivity result. In Section 4 we establish the bijectivity of the labeling for $(\mathbf{m}, \varepsilon)$-arrangements. In Section 5, we show that $(\mathbf{m}, \varepsilon)$-arrangements are the only transitive arrangements such that the labeling is bijective, and recover Mazin and Miller's result for transitive central graphical arrangements.

2 Background and notation

In this section we state some definitions and recall some known results. For the rest of this paper, for a positive integer n, we denote $[n] := \{1, \ldots, n\}$ and for integers a and b, we denote $[a; b] := \{a, \ldots, b\}$. Further, $\mathbf{S} = (S_{i,j})_{1 \leq i < j \leq n}$ denotes a collection of finite sets of integers, and $R(\mathcal{A}_\mathbf{S})$ denotes the set of regions of the arrangement $\mathcal{A}_\mathbf{S}$.

2.1 Parking functions and the generalized Pak-Stanley labeling

Definition 2.1. Let $\mathbf{S} = (S_{i,j})_{1 \leq i < j \leq n}$ be a collection of finite sets of integers. We define $D_\mathbf{S}$ to be the directed multigraph on $[n+1]$ with the arc set containing precisely $|S^+_{i,j}|$ arcs (i, j) and one arc $(i, n+1)$ for all $i, j \in [n]$.

Definition 2.2. Let $D = ([n+1], A)$ be a directed multigraph. A sequence $p = (p_1, \ldots, p_n)$ is a *D-parking function of length n* if and only if for every non-empty set $U \subseteq [n]$,

$$\exists u \in U \text{ such that } p_u < |\{(u, v) \in A \mid v \notin U\}|.$$

We denote by $\mathrm{Park}_\mathbf{S}$ the set of $D_\mathbf{S}$-parking functions. Then, the *generalized Pak-Stanley labeling*, or *GPS labeling* for short, is the map $\lambda : R(\mathcal{A}_\mathbf{S}) \to \mathrm{Park}_\mathbf{S}$ defined as follows.

We denote by R_0 the region of $\mathcal{A}_\mathbf{S}$ containing the fundamental alcove $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > x_2 > \ldots > x_n > x_1 - 1\}$. Then, for a region R of $\mathcal{A}_\mathbf{S}$, we define $\lambda(R) = (p_1, \ldots, p_n)$ with

$$\begin{aligned} p_i &= \#H_{i,j,s} \in \mathcal{A}_\mathbf{S} \text{ separating } R \text{ from } R_0 \text{ with } s > 0 \text{ or } (s = 0 \text{ and } i > j) \\ &= |\{j \in [n], s \in S^+_{i,j} \mid H_{i,j,s} \text{ separates } R \text{ from } R_0\}|. \end{aligned}$$

Mazin proved the following result in [3].

Theorem 2.3. *[3] For any tuple $\mathbf{S} = (S_{i,j})$, the GPS labeling $\lambda : R(\mathcal{A}_\mathbf{S}) \to \mathrm{Park}_\mathbf{S}$ is surjective.*

2.2 Sketches and Bernardi labeling

In [2], Bernardi defined a bijection between m-Catalan regions and (m, n)-sketches.

Definition 2.4. An (m, n)-*sketch* is a word $w = w_1 w_2 \ldots w_{(m+1)n}$ such that

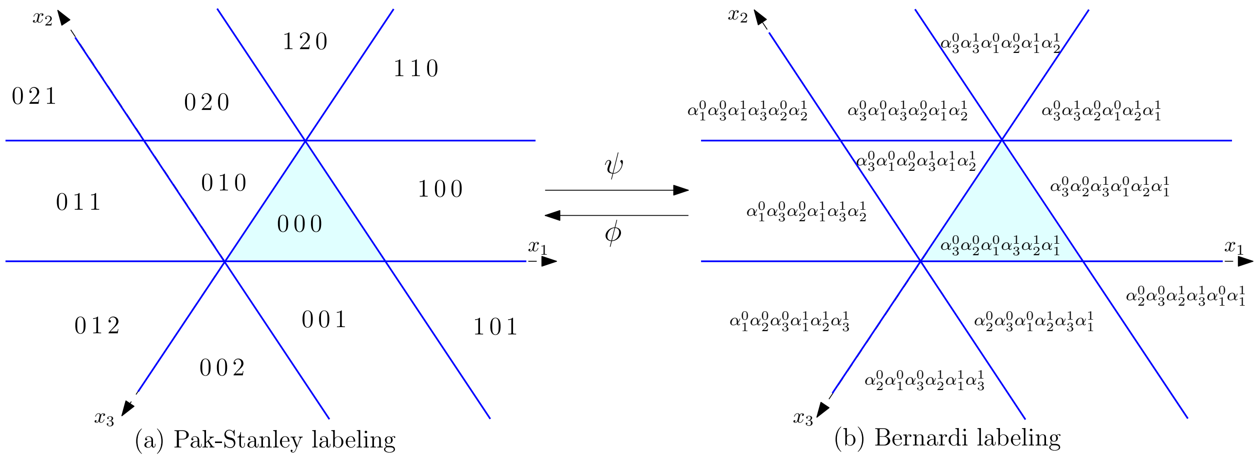

Figure 1: The generalized Pak-Stanley labeling, and Bernardi labeling for $\mathcal{A}_{\mathbf{S}}$ with $S_{1,2} = \{0\}$, $S_{1,3} = S_{2,3} = \{0,1\}$. The region R_0 is shaded.

- $\{w_1, \dots, w_{(m+1)n}\} = \{\alpha_i^s \mid i \in [n],\, s \in [0;m]\}$,
- for all $i \in [n]$ and $s \in [m]$, the letter α_i^{s-1} appears before α_i^s,
- for all $i, j \in [n]$ and $s, t \in [m]$, if α_i^{s-1} appears before α_j^{t-1} then α_i^s appears before α_j^t.

An (m,n)-sketch w indicates the order on the set $\{x_i + s \mid i \in [n],\ s \in [0;m]\}$ for a point $(x_1, \dots, x_n)$ in an m-Catalan region, with α_i^s representing the number $x_i + s$.

Definition 2.5. Let w be an (m,n)-sketch. We define an ordering $\leq_w$ on the letters of w, that is, on the set $\{\alpha_i^s \mid i \in [n], s \in [0;m]\}$ by $\alpha_i^s <_w \alpha_j^t$ if and only if α_i^s is before α_j^t in w.

We define $\beta(w)$ to be the m-Catalan region made of the points $x \in \mathbb{R}^n$ such that $x_i - x_j < s$ if and only if $\alpha_i^0 <_w \alpha_j^s$.

Lemma 2.6. *[2] The map β is a bijection between (m,n)-sketches and m-Catalan regions.*

The $\mathbf{S}$-braid arrangement is a subarrangement of the m-Catalan arrangement for $m = \max \bigcup_{i,j \in [n]} S_{i,j}^+$. Hence, any $\mathbf{S}$-braid region is a union of m-Catalan regions. We further define $\beta_{\mathbf{S}}(w)$ to be the region of $\mathcal{A}_{\mathbf{S}}$ containing the m-Catalan region $\beta(w)$.

Definition 2.7. We define $\mathrm{Triple}_{\mathbf{S}} = \{(i,j,s) \mid i,j \in [n], (s \in S_{i,j}^+) \text{ or } (s = 0 \text{ and } i < j)\}$.

The $\mathbf{S}$-braid arrangement $\mathcal{A}_{\mathbf{S}}$ is called *transitive* if for all $i, j \in [n]$ and $s, t \in \mathbb{N}$,

$$(i,j,s) \notin \mathrm{Triple}_{\mathbf{S}} \text{ and } (j,k,t) \notin \mathrm{Triple}_{\mathbf{S}} \implies (i,k,s+t) \notin \mathrm{Triple}_{\mathbf{S}}.$$

In [2], Bernardi further defined the notion of *local maximality* on the (m,n)-sketches.

Definition 2.8. Let $\mathbf{S} = (S_{i,j})_{1 \leq i < j \leq n}$ be a collection of finite sets of integers and let $m = \max \bigcup_{i,j \in [n]} S_{i,j}^+$. A (m,n)-sketch is called ***S**-locally maximal* if for all $j \in [n]$, if s is maximum such that α_j^s is immediately followed by α_i^0 for some $i \in [n]$, then $(i,j,s) \in \mathrm{Triple}_{\mathbf{S}}$. We denote the set of $\mathbf{S}$-locally maximal sketches by $\mathcal{L}_{\mathbf{S}}$.

The following result was proved in [2, Sec. 8].

Theorem 2.9. *[2] For any tuple $\mathbf{S} = (S_{i,j})_{i<j}$, the map $\beta_{\mathbf{S}} : \mathcal{L}_{\mathbf{S}} \to R(\mathcal{A}_{\mathbf{S}})$ is surjective. Moreover, if $\mathcal{A}_{\mathbf{S}}$ is transitive, then $\beta_{\mathbf{S}}$ is bijective.*

3 Right inverse to the GPS labeling

In this section, we first define a map from $\mathbf{S}$-locally maximal sketches to $D_{\mathbf{S}}$-parking functions that is consistent with the GPS labeling of a region. We then define a map from $D_{\mathbf{S}}$-parking functions to $\mathbf{S}$-locally maximal sketches and show that when composed with $\beta_{\mathbf{S}}$, this is a right inverse to the GPS labeling.

Let $\mathbf{S} = (S_{i,j})_{1 \le i < j \le n}$ be a tuple of finite sets of integers, and let $m = \max \bigcup_{i,j \in [n]} S_{i,j}^+$. By definition, the GPS labeling $\lambda(R)$ of a region of R of $\mathcal{A}_{\mathbf{S}}$ is $p = (p_1, \ldots, p_n)$, where

$$p_i = |\{(j,s) \mid j \in [n],\ s \in S_{i,j}^+ \text{ such that } s < x_i - x_j \}|.$$

For a (m,n)-sketch w, we let $\Phi(w)$ be the $D_{\mathbf{S}}$-parking function $p = (p_1, \ldots, p_n)$, where

$$p_i = |\{(j,s) \mid j \in [n],\ s \in S_{i,j}^+ \text{ such that } \alpha_j^s <_w \alpha_i^0 \}|.$$

It is clear from the definition of $\beta_{\mathbf{S}}$ that $\Phi = \lambda \circ \beta_{\mathbf{S}}$.

We will now define a right inverse of Φ. Precisely, we will define a map Ψ from the set of $D_{\mathbf{S}}$-parking functions to the set of $\mathbf{S}$-local maximum (m,n)-sketches.

Let $p = (p_1, \ldots, p_n)$ be a $D_{\mathbf{S}}$-parking function. We define the word $w = w_1 \ldots w_\ell = \Psi(p)$ by determining its letters w_r one at a time according to the following process which is illustrated in Figure 2:

We maintain a n-tuple $P_r \in \mathbb{Z}^n$ which we initialize at $P_1 = (p_1, \ldots, p_n)$, and an ordered list O_r of indices in $[n]$ which we initialize to be empty: $O_1 = \emptyset$. We determine the letter w_r of $\Psi(p)$, starting from $r = 1$, according to the following rule:

Case 1: The tuple P_r has a zero entry.

Then, the letter w_r is α_k^0, where k is the rightmost zero entry of P_r. We obtain P_{r+1} from P_r by subtracting 1 from coordinate k and every coordinate i with a positive entry such that $0 \in S_{i,k}^+$. We obtain O_{r+1} from O_r by appending k at the end of the list O_r.

Case 2: The tuple P_r does not have a zero entry, but O_r is non-empty.

Then, the letter w_r is α_k^s, where k is the first element of O_r and $-s$ is the k^{th} entry of P_r. We obtain P_{r+1} from P_r by subtracting 1 from coordinate k and every coordinate i with a positive entry such that $s \in S_{i,k}^+$. We obtain the list O_{r+1} from O_r by removing the first element k of the list, and if $s < m$ appending k at the end of the list O_r.

This process terminates when neither condition is satisfied, and returns the word $w_1 \dots w_\ell$ constructed thus far. See Figure 2 for some examples.

P_r	$1,\underline{0},1$	$1,-1,\underline{0}$	$1,\underline{-1},-1$	$\underline{0},-2,-1$	$-1,-2,\underline{-1}$	$\underline{-1},-2,-2$	$-2,-2,-2$
O_r	$\emptyset$	2	$2,3$	3	$3,1$	1	$\emptyset$
w_r	α_2^0	α_3^0	α_2^1	α_1^0	α_3^1	α_1^1	

(a) $\Psi(1,0,1)$ for the 1-Shi arrangement in dimension 3.

P_r	$1,\underline{0},1$	$1,-1,\underline{0}$	$1,\underline{-1},-1$	$1,-2,\underline{-1}$	$\underline{0},-2,-2$	$\underline{-1},-2,-2$	$-2,-2,-2$
O_r	$\emptyset$	2	$2,3$	3	$\emptyset$	1	$\emptyset$
w_r	α_2^0	α_3^0	α_2^1	α_3^1	α_1^0	α_1^1	

(b) $\Psi(1,0,1)$ for the 1-Shi arrangement minus the hyperplane $\{x_1 - x_2 = 1\}$ (see Figure 1)

.

Figure 2: Computing $\Psi(p)$ for the parking function $p = (1,0,1)$ for two different $\mathbf{S}$.

Lemma 3.1. *For any $D_{\mathbf{S}}$-parking function p, the word $\Psi(p)$ is an $\mathbf{S}$-locally maximal sketch.*

Proof. Let $p = (p_1, \dots, p_n) \in \mathrm{Park}_{\mathbf{S}}$ and let $w = \Psi(p)$.

We first show that w is an (m,n)-sketch. It is easy to see that when the procedure defining Ψ terminates, the entries $(q_1, \dots, q_n)$ of P_r are all either positive or equal to $-m$ (for the indices that ever entered the list O_r). Let $U \subset [n]$ be the set of indices of the positive entries of P_r, and let $V = [n] \setminus U$. Note that for $i \in U$, we have subtracted 1 from position i for every element of $S_{i,j}^+$ for every $j \in V$. Hence, $p_i = q_i + \sum_{j \in V} |S_{i,j}^+| \geq 1 + \sum_{j \in V} |S_{i,j}^+|$. Moreover, in the digraph $D_{\mathbf{S}}$, for $i \in U$, one has $|\{(i,j) \mid j \notin U\}| = 1 + \sum_{j \in V} |S_{i,j}^+|$. This gives $p_i \geq |\{(i,j) \mid j \notin U\}|$ for all $i \in U$. This contradicts $p \in \mathrm{Park}_{\mathbf{S}}$ unless $U = \emptyset$. We conclude that $U = \emptyset$ and $V = [n]$. This implies that all the letters α_i^s for $i \in [n]$ and $s \in [0;m]$ appear once in $w = \Psi(p)$. It is then clear from the definition that w is an (m,n)-sketch (since the list O_r is "first in first out").

It remains to show that w is $\mathbf{S}$-locally maximal. Suppose the letter $w_r = \alpha_j^s$ is immediately followed by α_i^0 in w. We will show $(i,j,s) \in \mathrm{Triple}_{\mathbf{S}}$.

If $s = 0$ and $i < j$, then clearly $(i,j,s) \in \mathrm{Triple}_{\mathbf{S}}$. If $s = 0$ and $i > j$, as $w_r = \alpha_j^0$, the rightmost zero in P_r has to be at coordinate j. In other words, for $k > j$, the k^{th} coordinate of P_r is non-zero. But, for the next letter to be α_i^0, the i^{th} coordinate of P_{r+1} must be zero. As we obtain P_{r+1} by subtracting 1 from every coordinate k such that $0 \in S_{k,j}^+$, we must have $0 \in S_{i,j}^+$, so $(i,j,s) \in \mathrm{Triple}_{\mathbf{S}}$.

If $s \neq 0$, then P_r had no zeros. Now, if the next letter of $\Psi(p)$ is α_i^0, we must have that P_{r+1} has a zero at position i and that this is in fact the rightmost zero in P_{r+1}. By definition of Ψ, P_{r+1} is obtained from P_r by subtracting 1 from every coordinate k such that $s \in S^+_{k,j}$. Hence $s \in S^+_{i,j}$, and $(i,j,s) \in \mathrm{Triple}_{\mathbf{S}}$. Thus, w is $\mathbf{S}$-locally maximal. □

The main result of this section is the following:

Theorem 3.2. *For every collection of sets $\mathbf{S} = (S_{i,j})_{i<j}$, one has $\Phi \circ \Psi = \mathrm{Id}_{Park_{\mathbf{S}}}$.*

We first need some definitions and a lemma.

Definition 3.3. We define $\mathcal{M}_{\mathbf{S}}$ to be the set of (m,n)-sketches such that if α^s_j is immediately followed by α^0_i, then $(i,j,s) \in \mathrm{Triple}_{\mathbf{S}}$.

We define $\mathcal{N}_{\mathbf{S}} = \mathrm{Im}(\Psi)$, that is, the image by Ψ of $\mathrm{Park}_{\mathbf{S}}$.

Remark 3.4. It is clear that $\mathcal{M}_{\mathbf{S}} \subseteq \mathcal{L}_{\mathbf{S}}$. Moreover we have shown $\mathcal{N}_{\mathbf{S}} \subseteq \mathcal{M}_{\mathbf{S}}$ in the proof of Lemma 3.1.

Lemma 3.5. *Let $p = (p_1, \ldots, p_n) \in Park_{\mathbf{S}}$, and let α^s_k be the r^{th} letter of $w = \Psi(p)$. Let $P_r = (q_1, \ldots, q_n)$. For all $i \in [n]$ such that $q_i \geq 0$, one has $p_i - q_i = |\{\alpha^t_j <_w \alpha^s_k \mid t \in S^+_{i,j}\}|$.*

Proof. By definition of Ψ, the tuple P_r is determined by p and the first $r-1$ letters $w_1, \ldots, w_{r-1}$ of $\Psi(p)$. For $\ell \in [r-1]$, we consider how the letter w_ℓ affects the i^{th} coordinate of P_r.

Let $w_\ell = \alpha^t_j$. By definition of Ψ, for such a letter, 1 is subtracted from every coordinate k with a positive entry such that $t \in S^+_{k,j}$ that is, 1 is subtracted from p_i for every letter of the form α^t_j with $t \in S^+_{i,j}$ in the first $r-1$ letters of $\Psi(p)$.

Hence, $p_i - q_i = |\{\alpha^t_j <_w \alpha^s_k \mid t \in S^+_{i,j}\}|$. □

We now prove Theorem 3.2.

Proof of Theorem 3.2. Let $p = (p_1, \ldots, p_n) \in \mathrm{Park}_{\mathbf{S}}$. Let $w = \Psi(p)$ and $\Phi(w) = p' = (p'_1, \ldots, p'_n)$. By definition of Φ,

$$p'_i = |\{(j,s) \mid \alpha^s_j <_w \alpha^0_i,\ j \in [n],\ s \in S^+_{i,j}\}| = |\{\alpha^s_j <_w \alpha^0_i \mid s \in S^+_{i,j}\}|.$$

We wish to show that $p = p'$. Suppose α^0_i is the r^{th} letter of w. Then, for $P_r = (q_1, \ldots, q_n)$, we have $q_i = 0$, and further, by Lemma 3.5,

$$p_i = p_i - q_i = |\{\alpha^t_j <_w \alpha^0_i \mid t \in S^+_{i,j}\}| = p'_i.$$

Hence $p = p'$, that is, $\Phi \circ \Psi = \mathrm{Id}_{\mathrm{Park}_{\mathbf{S}}}$. □

Remark 3.6. As $\Phi = \lambda \circ \beta_{\mathbf{S}}$, Theorem 3.2 implies that $\lambda \circ \beta_{\mathbf{S}} \circ \Psi = \mathrm{Id}_{\mathrm{Park}_{\mathbf{S}}}$. Hence the map $\beta_{\mathbf{S}} \circ \Psi$ is a right inverse to the GPS labeling.

Note that this implies Mazin's surjectivity result [3] for $\mathbf{S}$-braid arrangements, and hence for all deformations of the braid arrangement by Remark 1.2.

4 Bijectivity of the GPS labeling for $(\mathbf{m}, \varepsilon)$-arrangements

In this section, we establish certain sufficient conditions for the bijectivity of the GPS labeling, and use them to prove the following theorem.

Theorem 4.1. *The GPS labeling λ is bijective for $(\boldsymbol{m}, \varepsilon)$-arrangements with inverse $\beta_{\mathbf{S}} \circ \Psi$.*

The proof of Theorem 4.1 starts with the following lemma.

Lemma 4.2. *Let $\mathcal{A}_{\mathbf{S}}$ be an $\mathbf{S}$-braid arrangement. If $\mathcal{N}_{\mathbf{S}} = \mathcal{M}_{\mathbf{S}} = \mathcal{L}_{\mathbf{S}}$, then the GPS labeling λ is bijective. Conversely, if λ and $\beta_{\mathbf{S}} : \mathcal{L}_{\mathbf{S}} \to R(\mathcal{A}_{\mathbf{S}})$ are both bijective then $\mathcal{N}_{\mathbf{S}} = \mathcal{M}_{\mathbf{S}} = \mathcal{L}_{\mathbf{S}}$.*

Proof. Recall that $\mathrm{Im}(\Psi) = \mathcal{N}_{\mathbf{S}} \subseteq \mathcal{M}_{\mathbf{S}} \subseteq \mathcal{L}_{\mathbf{S}}$. Let $\beta^N_{\mathbf{S}}$ and $\beta^L_{\mathbf{S}}$ be the restriction of the map $\beta_{\mathbf{S}}$ to $\mathcal{N}_{\mathbf{S}}$ and $\mathcal{L}_{\mathbf{S}}$ respectively.

By Theorem 3.2, $\lambda \circ \beta^N_{\mathbf{S}} \circ \Psi = \mathrm{Id}_{\mathrm{Park}_{\mathbf{S}}}$. Hence $\beta^N_{\mathbf{S}}$ is injective, and λ is bijective if and only if $\beta^N_{\mathbf{S}} : \mathcal{N}_{\mathbf{S}} \to R(\mathcal{A}_{\mathbf{S}})$ is surjective. By Theorem 2.9, $\beta^L_{\mathbf{S}} : \mathcal{L}_{\mathbf{S}} \to R(\mathcal{A}_{\mathbf{S}})$ is surjective. Hence if $\mathcal{N}_{\mathbf{S}} = \mathcal{L}_{\mathbf{S}}$ then $\beta^N_{\mathbf{S}}$ is surjective and λ is bijective. Conversely, if $\beta^L_{\mathbf{S}}$ is bijective, then $\beta^N_{\mathbf{S}}$ is surjective if and only if $\mathcal{N}_{\mathbf{S}} = \mathcal{L}_{\mathbf{S}}$. □

We now want to prove that $\mathcal{N}_{\mathbf{S}} = \mathcal{M}_{\mathbf{S}} = \mathcal{L}_{\mathbf{S}}$ for every $(\mathbf{m}, \varepsilon)$-arrangement $\mathcal{A}_{\mathbf{S}}$. To this aim, we will establish some sufficient conditions for the identities $\mathcal{N}_{\mathbf{S}} = \mathcal{M}_{\mathbf{S}}$ and $\mathcal{M}_{\mathbf{S}} = \mathcal{L}_{\mathbf{S}}$ to hold.

Definition 4.3. We say an $\mathbf{S}$-braid arrangement $\mathcal{A}_{\mathbf{S}}$ has *Property Y*, or (Y) holds, if

$$\forall i, j, k \in [n],\ \forall s \geq 0 \text{ with } s > 0 \text{ or } j < i, \text{ if } s \notin S^+_{i,j}, \text{ then } s + t \notin S^+_{k,j} \text{ for all } t > 0, k \neq i. \tag{Y}$$

Lemma 4.4. *Let $\mathcal{A}_{\mathbf{S}}$ be an $\mathbf{S}$-braid arrangement. Then (Y) holds if and only if $\mathcal{M}_{\mathbf{S}} = \mathcal{L}_{\mathbf{S}}$.*

Proof. Suppose (Y) holds. Assume for contradiction that there exists $w \in \mathcal{L}_{\mathbf{S}} \setminus \mathcal{M}_{\mathbf{S}}$. Then, there is $(i, k, s) \notin \mathrm{Triple}_{\mathbf{S}}$ such that α^s_k is followed immediately by α^0_i in w. Now, as $w \in \mathcal{L}_{\mathbf{S}}$, we will have α^{s+t}_k followed immediately by α^0_j for some $t > 0$, with $(j, k, s + t) \in \mathrm{Triple}_{\mathbf{S}}$. As $(i, k, s) \notin \mathrm{Triple}_{\mathbf{S}}$, we have $s \notin S^+_{i,k}$ with $s > 0$ or $k < i$. By (Y), $s + t \notin S^+_{j,k}$. Hence, $s + t = 0$ with $j < k$, which contradicts $t > 0$. Hence $\mathcal{L}_{\mathbf{S}} = \mathcal{M}_{\mathbf{S}}$.

Suppose now $\mathcal{M}_{\mathbf{S}} = \mathcal{L}_{\mathbf{S}}$. Suppose (Y) does not hold. Then two cases may arise.

Case 1: There exists $s > 0$ such that $s \notin S^+_{i,k}$ and $s + t \in S^+_{j,k}$ for some $i, j, k \in [n]$, $t > 0$.

Then, consider the (m, n)-sketch

$$w = \alpha^0_n \ldots \alpha^0_k \ldots \alpha^0_1 \ldots \alpha^s_k \alpha^0_i \ldots \alpha^{s+t}_k \alpha^0_j \ldots$$

where the first $(n-2)$ letters are α^0_p with $p \neq i, j$ in descending order of p. Then, w is in $\mathcal{L}_{\mathbf{S}} \setminus \mathcal{M}_{\mathbf{S}}$, which is a contradiction.

Case 2: There exists $i > k$ such that $0 \notin S^+_{i,k}$ and $t \in S^+_{j,k}$ for some $j \in [n]$, $t > 0$.

Then, consider the (m,n)-sketch

$$w = \alpha_n^0 \dots \alpha_k^0 \alpha_i^0 \dots \alpha_1^0 \dots \alpha_k^t \alpha_j^0 \dots$$

where the letters before α_k^0 are α_p^0, $p > k$, $p \neq j$ in descending order of p and the letters after α_i^0 are α_p^0, $p < k < i$, $p \neq j$ in descending order of p. Again, w is in $\mathcal{L}_{\mathbf{S}} \setminus \mathcal{M}_{\mathbf{S}}$, which is a contradiction. □

Definition 4.5. We say a $\mathbf{S}$-braid arrangement $\mathcal{A}_{\mathbf{S}}$ has *Property* X, or (X) holds, if

$$\forall 1 \leq i < j \leq n, k \in [n], \forall s \in S^+_{j,k} \text{ either } s \in S^+_{i,k} \text{ or } (s = 0 \text{ and } i < k). \tag{X}$$

Lemma 4.6. *Let $\mathcal{A}_{\mathbf{S}}$ be an $\mathbf{S}$-braid arrangement. If* (X) *holds, then $\mathcal{N}_{\mathbf{S}} = \mathcal{M}_{\mathbf{S}}$.*

Proof. We suppose that (X) holds. We know $\mathcal{N}_{\mathbf{S}} \subseteq \mathcal{M}_{\mathbf{S}}$ and want to prove $\mathcal{N}_{\mathbf{S}} = \mathcal{M}_{\mathbf{S}}$. By Theorem 3.2, the map $\Phi : \mathcal{N}_{\mathbf{S}} \to \mathrm{Park}_{\mathbf{S}}$ is surjective, thus it suffices to show that $\Phi : \mathcal{M}_{\mathbf{S}} \to \mathrm{Park}_{\mathbf{S}}$ is injective.

Suppose for contradiction that there are distinct sketches $w = w_1 \dots w_{(m+1)n}$ and $w' = w'_1 \dots w'_{(m+1)n}$ both in $\mathcal{M}_{\mathbf{S}}$ such that $\Phi(w) = \Phi(w')$. Let $a \in [(m+1)n]$ be the smallest index such that $w_a \neq w'_a$.

Claim: We cannot have $w_a = \alpha_i^s$ and $w'_a = \alpha_j^t$ with $s \neq 0$ and $t \neq 0$.

Suppose $w_a = \alpha_i^s$ and $w'_a = \alpha_j^t$ with $s \neq 0$, $t \neq 0$. Since $s \neq 0$, and w is an (m,n)-sketch, α_i^{s-1} will be one of the first $a-1$ letters of w. Similarly, since $t \neq 0$, α_j^{t-1} will be one of the first $a-1$ letters of w'. As $w_k = w'_k$ for all $k \in [a-1]$, α_i^{s-1} and α_j^{t-1} are two of the first $a-1$ letters of both w and w'.

Now in w, we have α_i^s appears before α_j^t, and hence α_i^{s-1} must appear before α_j^{t-1} as w is an (m,n)-sketch. But, in w', α_i^s appears after α_j^t and hence α_i^{s-1} must appear after α_j^{t-1} as w' is an (m,n)-sketch. This contradicts the fact that $w_k = w'_k$ for all $k \in [a-1]$. Hence, our claim holds.

From this claim, we can assume without loss of generality (up to exchanging the role of w and w') that $w_a = \alpha_i^0$ and $w'_a = \alpha_j^s$ with either $s > 0$ or ($s = 0$ and $j < i$).

By looking at the i^{th} coordinate of the parking function $\Phi(w) = \Phi(w')$ we get

$$|\{\alpha_k^t <_w \alpha_i^0 \mid k \neq i,\ t \in S^+_{i,k}\}| = |\{\alpha_k^t <_{w'} \alpha_i^0 \mid k \neq i,\ t \in S^+_{i,k}\}|. \tag{4.1}$$

Let $b > a$ be such that $w'_b = \alpha_i^0$, and let us denote $w'_{b-\ell} = \alpha_{k_\ell}^{t_\ell}$ for all $\ell \in [0; b-a]$. By (4.1) we get $t_\ell \notin S^+_{i,k_\ell}$ for all $\ell \in [b-a]$.

Let us now prove that $t_\ell = 0$ and $i \leq k_\ell$ for all $\ell \in [0; b-a]$, by induction on ℓ. The base case $\ell = 0$ holds since $w'_b = \alpha_i^0$. Suppose, by induction, that $t_{\ell-1} = 0$ and

$i \leq k_{\ell-1}$ for some $\ell \in [b-a]$. Since $w' \in \mathcal{M}_{\mathbf{S}}$ we have $(k_{\ell-1}, k_\ell, t_\ell) \in \mathrm{Triple}_{\mathbf{S}}$. Thus either $t_\ell \in S^+_{k_{\ell-1},k_\ell}$ or ($t_\ell = 0$ and $k_{\ell-1} < k_\ell$). In the second case we get $t_\ell = 0$ and $i \leq k_\ell$, as wanted. In the case $t_\ell \in S^+_{k_{\ell-1},k_\ell}$, Property (X) implies that either $t_\ell \in S^+_{i,k_\ell}$ or ($t_\ell = 0$ and $i \leq k_\ell$). Since we have established above that $t_\ell \notin S^+_{i,k_\ell}$, we get $t_\ell = 0$ and $i \leq k_\ell$, as wanted.

For $\ell = b-a$, this gives $w'_a = \alpha^s_j$ with $s = t_{b-a} = 0$ and $j = k_{b-a} \geq i$, which gives a contradiction. □

Lemmas 4.2, 4.4 and 4.6 immediately imply the following.

Corollary 4.7. *If a* **S***-braid arrangement satisfies* (X) *and* (Y)*, then the GPS labeling* λ *is bijective.*

Finally, it is easy to check that $(\mathbf{m}, \varepsilon)$-arrangements satisfy (X) and (Y) which completes the proof of Theorem 4.1.

5 Bijectivity for transitive arrangements

In this section, we focus on *transitive* **S**-braid arrangements and prove the following result.

Theorem 5.1. *Let* $\mathcal{A}_{\mathbf{S}}$ *be a transitive* **S***-braid arrangement. Then the following are equivalent:*

(a) the GPS labeling λ *is bijective,*
(b) $\mathcal{N}_{\mathbf{S}} = \mathcal{L}_{\mathbf{S}} = \mathcal{M}_{\mathbf{S}}$*,*
(c) (X) *and* (Y) *hold,*
(d) $\mathcal{A}_{\mathbf{S}}$ *is a* $(\boldsymbol{m}, \varepsilon)$*-arrangement.*

For a graphical arrangement $\mathcal{A}$ (a **S**-braid arrangement for which $S_{i,j} \in \{\{0\}, \emptyset\}$ for all i, j), (Y) always holds, while (X) can be rewritten as

$$\text{there are no indices } i < j < k \text{ in } [n] \text{ such that } H_{j,k,0} \notin \mathcal{A} \text{ and } H_{i,k,0} \in \mathcal{A}. \tag{5.1}$$

Hence, Theorem 5.1 implies the following.

Corollary 5.2. *The GPS labeling* λ *is bijective for a transitive graphical arrangement* $\mathcal{A}$ *if and only if* (5.1) *holds.*

This corollary could also be obtained from a result of Mazin and Miller [4], which states that λ is bijective for a (not necessarily transitive) graphical arrangement if and only if

$$\text{there are no indices } i < j < k \text{ in } [n] \text{ such that } H_{i,j,0} \in \mathcal{A},\ H_{j,k,0} \notin \mathcal{A} \text{ and } H_{i,k,0} \in \mathcal{A}. \tag{5.2}$$

Indeed, for graphical arrangements, the transitivity property can be rewritten as:

$$\text{there are no indices } i<j<k \text{ in } [n] \text{ such that } H_{i,j,0}\notin\mathcal{A},\ H_{j,k,0}\notin\mathcal{A} \text{ and } H_{i,k,0}\in\mathcal{A}, \tag{5.3}$$

and it is clear that the arrangements satisfying (5.2) and (5.3) are those satisfying (5.1).

The rest of this section is dedicated to the proof of Theorem 5.1. The equivalence between (a) and (b) is a direct consequence of Theorem 2.9 and Lemma 4.2. Next we prove the equivalence between (b) and (c).

Lemma 5.3. *Let $\mathcal{A}_{\mathbf{S}}$ be a transitive $\mathbf{S}$-braid arrangement such that* (Y) *holds and $\mathcal{N}_{\mathbf{S}}=\mathcal{M}_{\mathbf{S}}$. Then* (X) *holds.*

Proof. Suppose for contradiction that (X) does not hold.

Suppose first that there exist $k<i<j$ such that $0\in S^+_{j,k}\setminus S^+_{i,k}$. We take the minimal such i given k and j. The transitivity of $\mathcal{A}_{\mathbf{S}}$ implies $0\in S^+_{j,i}$. Now, consider the sketches

$$w=\alpha^0_n\ldots\alpha^0_{i+1}\underline{\alpha^0_k}\alpha^0_j\underline{\alpha^0_i}\alpha^0_{i-1}\ldots\alpha^0_1\ldots \text{ and } w'=\alpha^0_n\ldots\alpha^0_{i+1}\underline{\alpha^0_i}\alpha^0_j\underline{\alpha^0_k}\alpha^0_{i-1}\ldots\alpha^0_1\ldots,$$

where the letters from α^0_n to α^0_1 (except α^0_i,α^0_j, and α^0_k) are in decreasing order of subscript. We have underlined the letters which have moved between w and w' (the omitted suffix in w and w' corresponds to repeating the indicated prefix but with superscripts $1,2,\ldots,m$, so as to get (m,n)-sketches). It is easy to see that $w,w'\in\mathcal{M}_{\mathbf{S}}$ (using the minimality assumption on i), and $p_\ell=p'_\ell$ for all $\ell\in[n]$. So $\Phi(w)=\Phi(w')$. Since Φ is injective on $\mathcal{N}_{\mathbf{S}}=\mathcal{M}_{\mathbf{S}}$, we reach a contradiction.

So there must exist $s>0$ such that $s\in S^+_{j,k}\setminus S^+_{i,k}$ for some $i<j$ and k. Note that $0\in S^+_{j,i}$ as if not, by transitivity, as $s\notin S^+_{i,k}$ we have $s\notin S^+_{j,k}$, which would be a contradiction. Let $i^*=\min\{\ell\in[n]\mid 0\in S^+_{i,\ell}\}\cup\{i+1\}$ and $j^*=\min\{\ell\in[i]\mid 0\in S^+_{j,\ell}\}$. By (Y), we have $s-1\in S^+_{\ell,k}$ for all $\ell\neq j$, otherwise we would have $s\notin S^+_{j,k}$. If $i^*>j^*$, we consider the sketches

$$w=\alpha^0_k\ldots\alpha^{s-1}_k\alpha^0_{n^*}\ldots\alpha^0_1\alpha^s_k\underline{\alpha^0_j}\underline{\alpha^0_i}\ldots \text{ and } w'=\alpha^0_k\ldots\alpha^{s-1}_k\alpha^0_{n^*}\ldots\alpha^0_{i^*}\underline{\alpha^0_i}\ldots\alpha^0_{j^*}\underline{\alpha^0_j}\ldots\alpha^0_1\alpha^s_k\ldots,$$

where $n^*=\max[n]\setminus\{i,j,k\}$, and $\alpha^0_{n^*}\ldots\alpha^0_1$ represent all the letters α^0_ℓ with $\ell\neq i,j,k$ in decreasing order of ℓ. We have again underlined the letters which move between w and w' and omitted the suffixes (which are uniquely determined by the indicated prefixes for (m,n)-sketches). If $i^*\leq j^*$, then we instead consider the sketch

$$w'=\alpha^0_k\ldots\alpha^{s-1}_k\alpha^0_{n^*}\ldots\alpha^0_{i^*}\underline{\alpha^0_i}\underline{\alpha^0_j}\ldots\alpha^0_1\alpha^s_k\ldots.$$

It is easy to see that (in either case) the sketches w,w' are in $\mathcal{M}_{\mathbf{S}}$. Let $\Phi(w)=(p_1,\ldots,p_n)$ and $\Phi(w')=(p'_1,\ldots,p'_n)$. Clearly, $p_\ell=p'_\ell$ for all $\ell\neq i,j$. Further, by choice of i^*, as

$s \notin S^+_{i,k}$ and $j > i$, $p_i = p'_i$. Finally, by choice of j^*, as $s \in S^+_{j,k}$ and $0 \in S^+_{j,i}$, $p_j = p'_j$. Hence $\Phi(w) = \Phi(w')$ for $w \neq w'$. Since Φ is injective on $\mathcal{N}_{\mathbf{S}} = \mathcal{M}_{\mathbf{S}}$, we again reach a contradiction. Hence (X) holds. □

The equivalence of (b) and (c) follows directly from Lemmas 4.4, 5.3 and 4.6. It only remains to prove the equivalence of (c) and (d).

First, it is easy to check (X) and (Y) hold for $(\mathbf{m}, \varepsilon)$-arrangements hence (d) implies (c). We now suppose that (X) and (Y) hold for an arrangement $\mathcal{A}_{\mathbf{S}}$. We want to show that $\mathcal{A}_{\mathbf{S}}$ is a $(\mathbf{m}, \varepsilon)$-arrangement. Fix $k \in [n]$. Let s be the least non-negative integer such that $s \notin S^+_{i,k}$ with $s > 0$ or $i > k$. Then, as (Y) holds, for all $j \neq i, k$, one has $s + t \notin S^+_{j,k}$ for all $t > 0$. Hence, $[1, s-1] \subseteq S^+_{j,k} \subseteq [0, s]$ for all $j \neq i, k$.

Next, suppose $s + t \in S^+_{i,k}$ for some $t > 0$. Then, $s \in S^+_{j,k}$ for all $j \neq i$, as otherwise, as (Y) holds, $s + t \notin S^+_{i,k}$. Further, as (X) holds, $s + t \in S^+_{j,k}$ for all $j < i$, a contradiction unless $i = 1$. But $s \in S^+_{j,k}$ and $s \notin S^+_{1,k}$ contradicts (X). Hence, $[1, s-1] \subseteq S^+_{i,k} \subseteq [0, s-1]$.

Finally, suppose $s \in S^+_{j,k}$ for some $j \neq i$. Then we define $m_k = s$ and for all $\ell \neq k$ we define $\varepsilon_{\ell,k} = 1$ if $s \notin S_{\ell,k}$ (and either $s > 0$ or $k < \ell$) and $\varepsilon_{\ell,k} = 0$ otherwise. As (X) holds, for $p > \ell$, if $s \in S^+_{p,k}$, $s \in S^+_{\ell,k}$, hence we must have $\varepsilon_{p,k} = 1$ for all $p > \ell$. If $s \notin S^+_{j,k}$ for all $j \in [n]$, we define $m_k = s - 1$ and $\varepsilon_{\ell,k} = 0$ for all $\ell \neq k$. Hence $\mathcal{A}_{\mathbf{S}}$ is an $(\mathbf{m}, \varepsilon)$-arrangement. This completes the proof of Theorem 5.1.

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