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Algebraic cobordism rings of wonderful varieties and matroids

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Abstract. We prove a Feichtner–Yuzvinsky presentation and a simplicial presentation for the algebraic cobordism ring of the toric variety of the Bergman fan of any loopless matroid in terms of the lattice of flats of the matroid and the universal formal group law. For an essential hyperplane arrangement $\mathcal{H}$, we prove that the algebraic cobordism ring of the wonderful variety $W_{\mathcal{H}}$ of $\mathcal{H}$ and the algebraic cobordism ring of the toric variety of the matroid underlying $\mathcal{H}$ are isomorphic, and that both rings coincide with the complex cobordism ring of $W_{\mathcal{H}}$.

Keywords: matroids, wonderful varieties, formal group laws, algebraic cobordism

1 Introduction

The Bergman fan, defined by Ardila and Klivans [2], is a simplicial fan Σ_M associated to any loopless matroid M. To any fan, one can associate a toric variety, and by construction, the toric variety X_M of Σ_M is a *smooth* toric variety. The Chow ring of X_M depends only on the lattice of flats of M. Let $\mathcal{H}$ be an arrangement of hyperplanes in a vector space L defined over $\mathbb{C}$ whose intersection is the origin $\{0\} \in \mathbb{C}$. The arrangement $\mathcal{H}$ gives rise to a loopless matroid $M_{\mathcal{H}}$ whose flats are indexed by the various intersections of hyperplanes in $\mathcal{H}$. De Concini and Procesi defined a smooth compactification $W_{\mathcal{H}}$ of the complement of the hyperplane arrangement in L, known as the *wonderful compactification* of $\mathcal{H}$ [6], whose boundary is a simple normal crossings divisor. There is a natural inclusion $\iota\colon W_{\mathcal{H}} \hookrightarrow X_{M_{\mathcal{H}}}$, and Feichtner–Yuzvinsky showed in [7] that the inclusion ι induces an isomorphism of Chow rings via the pullback $\iota^*\colon \mathrm{CH}^*(X_{M_{\mathcal{H}}}) \xrightarrow{\sim} \mathrm{CH}^*(W_{\mathcal{H}})$. Recently, Larson–Li–Payne–Proudfoot [12] gave a presentation for the K-ring of X_M in terms of the combinatorial data of the lattice of flats of M, and they showed that the pullback in K-theory defines a ring isomorphism from $K^0(X_{M_{\mathcal{H}}})$ to $K^0(W_{\mathcal{H}})$.

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Algebraic cobordism is a functor that sends a smooth algebraic variety X defined over $\mathbb{C}$ to its *algebraic cobordism ring* $\Omega^*(X)$. Roughly, $\Omega^*(X)$ is the set of projective morphisms to X, modulo certain relations. The functor Ω^* was constructed by Levine–Morel in [14], where they showed that it is the universal oriented cohomology theory on the category of smooth varieties over $\mathbb{C}$. Examples of oriented cohomology theories include the *Chow theory* that sends a variety to its Chow ring of algebraic cycles modulo rational equivalence, and *Grothendieck's* K^0 functor that sends a variety to the Grothendieck ring of vector bundles on the variety. Since the discovery of algebraic cobordism, it has been computed for several families of varieties, including smooth toric varieties [11] and complete flag varieties [9], [5]. Computations of algebraic cobordism rings are important, as they generalize and unify computations of Chow rings and K-rings of smooth varieties.

The goal of this paper is to generalize some of the results in [12] to algebraic cobordism. As before, let M be a loopless matroid. In Lemma 4.6, we give a *Feichtner–Yuzvinsky* style presentation for the algebraic cobordism ring $\Omega^*(X_M)$ in terms of the lattice of flats of M and the *universal formal group law*. We will discuss formal group laws in detail in Section 3.1. The Feichtner–Yuzvinsky presentation can be deduced from work of Krishna and Uma [11], who give a presentation for the algebraic cobordism ring of any smooth toric variety. The presence of the universal formal group law in the Feichtner–Yuzvinsky presentation implies complicated relations among the generators in the algebraic cobordism ring. We give a much simpler *simplicial presentation* for $\Omega^*(X_M)$ in Lemma 4.9, which surprisingly does not involve the universal formal group law. We deduce from the simplicial presentation that $\Omega^*(X_M) \simeq CH^*(X_M) \otimes_{\mathbb{Z}} \Omega^*(\mathrm{pt})$ as $\Omega^*(\mathrm{pt})$-algebras, where $\mathrm{pt} = \mathrm{Spec}(\mathbb{C})$ is the point. We conclude that $\Omega^*(X_M)$ is a free module over $\Omega^*(\mathrm{pt})$ in Lemma 4.10, and we use the simplicial presentation in Lemma 4.11 to recover (and generalize) the integral isomorphism $K^0(X_M) \to CH^*(X_M)$ of [4], [12].[1] In Lemma 4.12, we show that for an essential hyperplane arrangement $\mathcal{H}$ defined over $\mathbb{C}$, the pullback in algebraic cobordism $\iota^*\colon \Omega^*(X_{M_{\mathcal{H}}}) \to \Omega^*(W_{\mathcal{H}})$ is an isomorphism of $\Omega^*(\mathrm{pt})$-algebras, and that $\Omega^*(W_{\mathcal{H}})$ agrees with the *complex cobordism ring* of $W_{\mathcal{H}}(\mathbb{C})$.

2 Recollection of wonderful varieties and matroids

2.1 Matroids and hyperplane arrangements

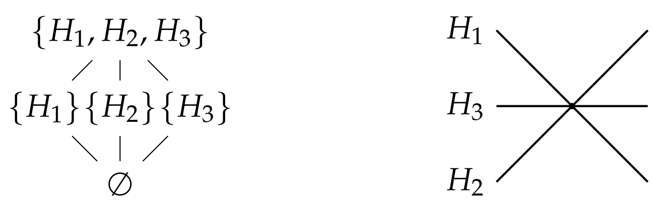
In this subsection, we will recall the definition of a matroid, and describe the matroid corresponding to an essential hyperplane arrangement, closely following the exposition [1].

[1] This integral isomorphism is known as the "exceptional isomorphism" in the literature. The exceptional isomorphism was first observed for the Boolean matroid in work of Berget, Eur, Spink and Tseng [4] and later generalized to all matroids by Larson, Li, Payne and Proudfoot [12].

A loopless **matroid** M is a pair $(E, \mathcal{F})$, where E is a finite set, called the ground set of M, and $\mathcal{F}$ is a set of subsets of E called **flats** such that $\emptyset, E \in \mathcal{F}$; if $G, H \in \mathcal{F}$, then $G \cap H \in \mathcal{F}$; and for any $G \in \mathcal{F}$ and any $i \in E \setminus F$, there is a unique $H \in \mathcal{F}$ containing i that **covers** G, i.e., $G \subseteq H$ and no other flat I satisfies $G \subsetneq I \subsetneq H$. Two flats H and G are called **incomparable** if $H \not\subseteq G$ and $G \not\subseteq H$; if H and G are not incomparable, then they are called **comparable**. When ordered by inclusion, the set $\mathcal{F}$ forms a "geometric lattice." The **rank** $\mathrm{rk}(G)$ of a flat G in this lattice equals the length of the largest chain of comparable flats properly contained in G; the **corank** of G is $\mathrm{rk}(E) - \mathrm{rk}(G)$. An **atom** is a flat of rank 1. Let r be the rank of E. In this case, we say that M is a **rank** r matroid.

Let $[n] := \{1, \ldots, n\}$. An **essential hyperplane arrangement** $\mathcal{H} = \{H_i\}_{i \in [n]}$ in $\mathbb{C}^r$ is a multiset of linear hyperplanes $H_1, \ldots, H_n$ in $\mathbb{C}^r$ such that $\bigcap_{i=1}^n H_i = \{0\}$. Every essential hyperplane arrangement $\mathcal{H} = \{H_i\}_{i \in [n]}$ determines a loopless matroid $M_{\mathcal{H}}$ on ground set $[n]$. For a subset $S \subseteq [n]$, let L_S be the linear subspace $L_S = \bigcap_{i \in S} H_i$. The set of flats of $M_{\mathcal{H}}$ is equal to $\mathcal{F}_{\mathcal{H}} = \{G \subseteq [n] : \forall j \in [n] \setminus G, L_{G \cup j} \subsetneq L_G\}$. In this case, the rank of a flat G is equal to the codimension of the intersection L_G.

Example 2.1. Consider the arrangement $\mathcal{H} = \{H_1, H_2, H_3\}$ in $\mathbb{C}^2$, defined by $H_1 = \mathrm{span}_{\mathbb{C}}\{(1, -1)\}$, $H_2 = \mathrm{span}_{\mathbb{C}}\{(1, 1)\}$, and $H_3 = \mathrm{span}_{\mathbb{C}}\{(1, 0)\}$. The matroid underlying $\mathcal{H}$ is $M_{\mathcal{H}} = U_{2,3}$. This matroid has a single flat of rank 2, corresponding to the intersection $H_1 \cap H_2 \cap H_3$, which is the origin in $\mathbb{C}^2$. There are three flats of rank 1, which correspond to the hyperplanes H_1, H_2, and H_3. Finally, there is a single flat of rank 0, corresponding to the empty set. The lattice of flats for $M_{\mathcal{H}} = U_{2,3}$ is illustrated below on the left, and the the arrangement $\mathcal{H}$ is illustrated below on the right:



2.2 Wonderful varieties of hyperplane arrangements

In this subsection, we will recall the definition of the wonderful compactification of an essential hyperplane arrangement of de Concini-Procesi [6], closely following [12].

Let $\mathcal{H} = \{H_i\}_{i \in [n]}$ be an essential hyperplane arrangement in $L = \mathbb{C}^n$. Projectivise each hyperplane H_i in $\mathcal{H}$ to obtain a set of hyperplanes $\{\mathbb{P}(H_i)\}_{i \in [n]}$ inside projective space $\mathbb{P}(L)$. The *wonderful variety* $W_{\mathcal{H}}$ of de Concini–Procesi [6] is an iterated blowup of $\mathbb{P}(L)$: first blow up the points $\mathbb{P}(L_G)$ for all corank 1 flats G, then blow up the strict transforms of the lines $\mathbb{P}(L_G)$ for all corank 2 flats G, and so on.

Theorem 2.2. *([6]) The iterated blow-up* $\pi \colon W_{\mathcal{H}} \to \mathbb{P}(L)$ *is a smooth compactification of* $\mathbb{P}(L) \setminus$

$\bigcup_{i\in\mathcal{H}} \mathbb{P}(H_i)$ *whose boundary is a simple normal crossings divisor. The irreducible components of this divisor are indexed by the proper non-empty flats of* $M_{\mathcal{H}}$.

Definition 2.3. For each flat G of $M_{\mathcal{H}}$, define the locus $U_G := \mathbb{P}(L_G) \setminus \bigcup_{G\subsetneq H\neq E} \mathbb{P}(L_H)$. Define the divisor D_G to be the closure of the preimage $\pi^{-1}(U_G)$ in $W_{\mathcal{H}}$. The divisor D_G is the strict transform of the exception divisor of $\mathbb{P}(L_G)$.

Example 2.4. Consider the arrangement $\mathcal{H}$ of Lemma 2.1, whose underlying matroid is $M_{\mathcal{H}} = U_{2,3}$. We have $L = \mathbb{C}^2$ and $\mathbb{P}(L) = \mathbb{P}^1$. The hyperplanes $\mathbb{P}(H_1)$, $\mathbb{P}(H_2)$, $\mathbb{P}(H_3)$ are three distinct points in $\mathbb{P}(L)$. As the blow-up of $\mathbb{P}^1$ at a point is isomorphic to $\mathbb{P}^1$, we have $W_{\mathcal{H}} \simeq \mathbb{P}^1$. Thus, $W_{\mathcal{H}}$ is a smooth compactification of $\mathbb{P}(L) \setminus \{\mathbb{P}(H_1), \mathbb{P}(H_2), \mathbb{P}(H_3)\}$, whose boundary is the simple normal crossings divisor $D_{G_1} + D_{G_2} + D_{G_3}$, $D_{G_i} := \mathbb{P}(H_i)$.

2.3 Toric varieties and the Bergman fan

In this subsection, we will briefly recall the definitions of cones, fans, toric varieties, and Bergman fans, closely following the conventions used in [3].

Let N be a lattice of rank n and $N_{\mathbb{R}}$ be the n-dimensional vector space $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. A **$k$-dimensional, rational, simplicial cone** $\sigma \subseteq N_{\mathbb{R}}$ is a subset that can be written as $\sigma = \operatorname{cone}\{v_1, \ldots, v_k\} := \{\lambda_1 v_1 + \ldots + \lambda_k v_k : \lambda_i \geq 0\} \subseteq N_{\mathbb{R}}$, where $v_1, \ldots, v_k$ are k linearly independent vectors in N. A cone τ is called a **face** of σ if it can be written as $\tau = \operatorname{cone}\{v_{i_1}, \ldots, v_{i_\ell}\}$ for some subset $\{v_{i_1}, \ldots, v_{i_\ell}\} \subseteq \{v_1, \ldots, v_k\}$. We declare $\operatorname{cone}(\emptyset) = \{0\}$ so that $\{0\}$ is a face of every cone. A **rational, simplicial fan** Σ in $N_{\mathbb{R}}$ is a non-empty collection of rational, simplicial cones $\sigma \in \Sigma$ in $N_{\mathbb{R}}$ such that every two cones $\sigma, \tau \in \Sigma$ intersect exactly along a common face, and if $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$. We say that Σ is **complete** if the union of all of its cones equals $N_{\mathbb{R}}$. Let $\Sigma(1)$ denote the set of **rays**, i.e. one dimensional cones, of Σ. For a ray $\rho \in \Sigma(1)$, the **primitive ray generator** $u_\rho \in N$ of ρ is the generator of the semigroup $N \cap \rho$. We say that Σ is **unimodular** if, for all cones $\sigma \in \Sigma$, the set of primitive ray generators of σ can be extended to a basis of N.

Definition 2.5. A **torus** is an algebraic group that is isomorphic to $(\mathbb{C}^\times)^n$ for some $n \geq 0$. Let T be a torus. A **toric variety** with respect to T is a normal algebraic variety X that contains T as a dense open subset, and such that the action of T on itself extends to an algebraic action of T on X.

Let $T = (\mathbb{C}^\times)^n$ be an n-dimensional torus, and let $N := \operatorname{Hom}(\mathbb{C}^\times, T)$ be its lattice of **one-parameter subgroups**. Denote the **lattice of characters** $\operatorname{Hom}(T, \mathbb{C}^\times)$ of T by $N^\vee$. A rational fan Σ in $N_{\mathbb{R}}$ gives rise to a toric variety X_Σ with torus T. There is a rich dictionary between the combinatorics of Σ and geometry of X_Σ, summarized in Table 1.

Let $M = (E, \mathcal{F})$ be a loopless matroid, and let $\mathbb{R}^E$ be the real vector space with basis $\{e_i : i \in E\}$. For a subset $G \subseteq E$, define $e_G := \sum_{i\in G} e_i \in \mathbb{R}^E$. The **Bergman fan** Σ_M of M

Table 1: Comparing properties of fans and toric varieties

Combinatorial data	Geometric data
Rational fan Σ	Toric variety X_Σ over $\mathbb{C}$
Unimodular rational fan Σ	Smooth toric variety X_Σ over $\mathbb{C}$
Complete rational fan Σ	Proper toric variety X_Σ over $\mathbb{C}$
Ray $\rho \in \Sigma(1)$	Divisor (i.e. codimension 1 subvariety) D_ρ in X_Σ

is the unimodular fan in $\mathbb{R}^E/\mathbb{R}e_E$ with cones $\sigma_{\mathcal{G}} = \mathrm{cone}\{e_G\}_{G\in\mathcal{G}}$, one for each flag $\mathcal{G}$ of proper and non-empty flats of M. We will denote the toric variety of Σ_M by X_M[2].

Example 2.6. Let e_1, e_2, e_3 be standard basis vectors in $\mathbb{R}^3$. The flags of flats for $U_{2,3}$ are

$$\varnothing \subseteq \{1\} \subseteq \{1,2,3\}, \quad \varnothing \subseteq \{2\} \subseteq \{1,2,3\}, \quad \varnothing \subseteq \{3\} \subseteq \{1,2,3\}.$$

Therefore, the Bergman fan of $U_{2,3}$ lies in $\mathbb{R}^3/(e_1+e_2+e_3)$, with cones $\sigma_1 = \mathrm{span}(e_1)$, $\sigma_2 = \mathrm{span}(e_2)$, $\sigma_3 = \mathrm{span}(e_3) = \mathrm{span}(-e_1-e_2)$. Here, $X_{U_{2,3}} \simeq \mathbb{P}^2 \setminus \{3 \text{ distinct points}\}$.

Remark 2.7. Let $\mathcal{H}$ be an essential hyperplane arrangement in $\mathbb{C}^n$. There is an embedding $\iota\colon W_{\mathcal{H}} \hookrightarrow X_{M_{\mathcal{H}}}$. If ρ_G is the cone in the Bergman fan of $M_{\mathcal{H}}$ corresponding to the single flat $\{G\}$, then the divisor D_{ρ_G} in $X_{\mathcal{H}}$ corresponding to the ray ρ_G satisfies $D_{\rho_G} \cap \iota(W_{\mathcal{H}}) = \iota(D_G)$, where D_G is the divisor in $W_{\mathcal{H}}$ of Lemma 2.3 corresponding to the flat G.

3 Formal group laws and algebraic cobordism

3.1 Formal group laws

We will discuss formal group laws in this subsection, closely following [8].

Definition 3.1. Let R be a commutative ring. A one-dimensional commutative **formal group law**[3] (R, F) is a formal power series $F(x,y) = \sum_{i,j\geq 0} a_{i,j}x^iy^j \in R[\![x,y]\!]$, satisfying (1) $F(0,v) = v$; (2) $F(w,v) = F(v,w)$; and (3) $F(w, F(v,z)) = F(F(w,v),z)$.

Remark 3.2. Let $F(x,y) = \sum_{i,j\geq 0} a_{i,j}x^iy^j \in R[\![x,y]\!]$ be a formal group law. Axiom (1) in Lemma 3.1 implies that $a_{0,0} = 0$ and $a_{1,0} = a_{0,1} = 1$. Axiom (2) imples $a_{i,j} = a_{j,i}$ for all i, j. The relations among the $a_{i,j}$ arising from Axiom (3) are complicated.

[2]As each set $\{e_G : G \in \mathcal{G}\}$ can be extended to a basis of $\mathbb{Z}^E$, Σ_M is a unimodular fan and the toric variety X_M is smooth. However, except for when M is the Boolean matroid, the set of vectors in Σ_E generates a strict subspace of $\mathbb{R}^E$. In these cases, Σ_M is *not* a complete fan, so X_M is *not* proper over $\mathbb{C}$.

[3]In this article, all formal group laws are one-dimensional and commutative. For brevity, we will drop the adjectives and refer to them simply as “formal group laws."

Lemma 3.3. *([8, Ch. I, §3, Proposition 1]) Given a formal group law (R, F), there is a unique formal power series $-_F x := -x - c_2x^2 - c_3x^3 - \cdots \in R[\![x]\!]$ called the **formal inverse** of (R, F) that satisfies $F(x, -_F x) = 0$.*

Example 3.4. The **additive formal group law** over $\mathbb{Z}$ is $F_A(x, y) = x + y$, and $-_{F_A} x = -x$. The **multiplicative formal group law** over $\mathbb{Z}$ is $F_M(x, y) = x + y - xy$, and $-_{F_M} x = \frac{-x}{1-x}$.

Definition 3.5. The **Lazard ring** $\mathbb{L}^*$[4] is the quotient of the free ring over $\mathbb{Z}$ with generators $a_{i,j}$, $i, j \geq 0$, modulo the relations imposed by the axioms of the formal group law. The **universal formal group law** is the formal group law $(\mathbb{L}^*, F_U)$ defined by $F_U(x, y) = \sum_{i,j\geq 0} a_{i,j}x^iy^j \in \mathbb{L}^*[\![x, y]\!]$.

Remark 3.6. A commutative ring R together with a ring homomorphism $f : \mathbb{L}^* \to R$ induces a formal group law (R, F) defined by $F(x, y) = \sum_{i,j\geq 0} f(a_{i,j})x^iy^j \in R[\![x, y]\!]$. Conversely, any formal group law over R is induced by a ring homomorphism $\mathbb{L}^* \to R$.

Given a formal group law (R, F), a power series ring $R[\![x_1, \ldots, x_k]\!]$, and $a \in \mathbb{Z}$, we define the notation:

$$x_1 +_F x_2 := F(x_1, x_2); \quad a \cdot_F x_1 := x_1 +_F x_1 +_F \cdots +_F x_1; \quad \sum_i^F x_i := x_1 +_F x_2 +_F \cdots +_F x_k.$$

3.2 Introduction to algebraic cobordism

Algebraic cobordism was constructed in [14]. In this subsection, we will give a very brief summary on the construction of the algebraic cobordism ring from [14].

Let $\mathbf{Sch}_{\mathbb{C}}$ be the category of separated schemes of finite type over $\mathbb{C}$, and let $\mathbf{Sm}_{\mathbb{C}}$ be the full subcategory of $\mathbf{Sch}_{\mathbb{C}}$ consisting of schemes smooth and quasi-projective over $\mathbb{C}$. Denote the category of graded, commutative rings by **Ring**. An “oriented cohomology theory" is a contravariant functor $\mathrm{h}^* : \mathbf{Sm}_{\mathbb{C}} \to \mathbf{Ring}$[5], satisfying several axioms that generalize the usual Eilenberg–Steenrod axioms. A “morphism” of oriented cohomology theories is a natural transformation of functors that commutes with “push-forwards." Examples of oriented cohomology theories include the *Chow theory* CH^* that sends a scheme X to its Chow ring $CH^*(X)$, the *Grothendieck* K^0 that sends a scheme X to the Grothendieck ring $K^0(X)$[6] of locally free coherent sheaves on X, and the universal oriented cohomology theory *algebraic cobordism* Ω^*.

[4] Originally proven in [13], the Lazard ring $\mathbb{L}^*$ is isomorphic to a graded polynomial ring in infinitely many variables $\mathbb{Z}[x_1, x_2, x_3, \ldots]$, where $\deg(x_i) = -2i$.

[5] In [14], Levine–Morel defined *oriented cohomology theories* for schemes over an arbitrary base field. For simplicity, we will focus our attention to schemes defined over the base field $\mathbb{C}$ in this paper.

[6] We are slightly abusing notation: although $K^0(X)$ is not a *graded* ring, one can form the graded ring $K^0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$, $\deg(\beta) = -1$. The functor $K^0(-) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$ is an oriented cohomology theory. Since K^0 is the evaluation of $K^0(-) \otimes \mathbb{Z}[\beta, \beta^{-1}]$ at $\beta = 1$, we will simply ignore the grading in this abstract.

For a smooth algebraic variety X, the algebraic cobordism ring $\Omega^*(X)$ is a ring that is generated by projective morphisms $Y \to X$ from smooth, quasi-projective, and irreducible varieties Y, modulo certain relations. The group $\Omega^k(X)$ is generated by the classes $[f\colon Y \to X]$ such that $k = \dim(X) - \dim(Y)$. The relations defining $\Omega^*(X)$ imply that $\Omega^k(X) = 0$ for all $k > \dim(X)$. The algebraic cobordism ring $\Omega^*(\mathrm{pt})$ of a point $\mathrm{pt} = \mathrm{Spec}(\mathbb{C})$ is isomorphic to the Lazard ring $\mathbb{L}^*$ and is generated by the classes of projective spaces $[\mathbb{P}^n \to \mathrm{pt}]$ and the classes of so-called "Milnor hypersurfaces" $[H_{n,m} \to \mathrm{pt}]$ ([14, §2.5.3]). The structure map $X \to \mathrm{pt}$ induces a pullback $\Omega^*(\mathrm{pt}) \to \Omega^*(X)$, giving $\Omega^*(X)$ the structure of an $\mathbb{L}^*$-module.

Given an oriented cohomology theory h^* and a vector bundle $E \to X$ of rank n in $\mathbf{Sm}_{\mathbb{C}}$, there are unique elements $c_i(E) \in \mathrm{h}^*(X)$, $i = 0, \ldots, n$, characterized by the axioms in [14, pp. 3], which include *Whitney sum formula* and the requirement that the c_i commute with pullbacks. The elements $c_i(E)$ are called **Chern classes**. Every oriented cohomology theory h^* comes with a formal group law F over $R = \mathrm{h}^*(\mathrm{pt})$. The formal group law is characterized by the following formula, called the *Quillen formula*: if $\mathcal{L}_1 \to X$ and $\mathcal{L}_2 \to X$ are line bundles in $\mathbf{Sm}_{\mathbb{C}}$, then $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = F(c_1(\mathcal{L}_1), c_1(\mathcal{L}_2))$.

Example 3.7. The formal group law associated to CH^* is the additive formal group law over $\mathbb{Z}$. The formal group law associated K^0 is the multiplicative formal group law over $\mathbb{Z}$. The formal group law associated to Ω^* is the universal formal group law over $\mathbb{L}^*$.

Example 3.8. Consider the complex cobordism functor MU^*. The functor MU^* takes an n-dimensional complex manifold X to its (graded) "complex cobordism ring" $MU^*(X)$ (see, e.g., [15]). Let $X \in \mathbf{Sm}_{\mathbb{C}}$. The set of complex points $X(\mathbb{C})$ is a complex manifold. The functor MU^* on $\mathbf{Sm}_{\mathbb{C}}$ which takes a scheme X to the complex cobordism ring $MU^*(X(\mathbb{C}))$ is an oriented cohomology theory. The formal group law arising from MU^* is the universal formal group law F_U over $\mathbb{L}^*$.[7]

Theorem 3.9. *([14, Theorem 1.2.6]) Given an oriented cohomology theory* h^**, there is a unique morphism of oriented cohomology theories* $\vartheta\colon \Omega^* \to \mathrm{h}^*$.

Theorem 3.10. *([14, p. 1.2.19]) The universal morphism* $\Omega^* \to CH^*$ *induces an isomorphism* $\Omega^*(-) \otimes_{\mathbb{L}^*} \mathbb{Z} \xrightarrow{\sim} CH^*(-)$*, where the map* $\mathbb{L}^* \to \mathbb{Z}$ *is induced by the additive group law* $F_A(x,y) = x + y$ *over* $\mathbb{Z}$ *(see Lemma 3.6).*

Theorem 3.11. *([14, p. 1.2.18]) The universal morphism* $\Omega^* \to K^0$ *induces an isomorphism* $\Omega^*(-) \otimes_{\mathbb{L}^*} \mathbb{Z} \xrightarrow{\sim} K^0(-)$*, where the map* $\mathbb{L}^* \to \mathbb{Z}$ *is induced by the multiplicative group law* $F_M(x,y) = x + y - xy$ *over* $\mathbb{Z}$ *(see Lemma 3.6).*

Remark 3.12. Let (R, F) be a formal group law. Any oriented cohomology theory of the form $\Omega^*(-) \otimes_{\mathbb{L}^*} R$ is called **free**, where $\mathbb{L}^* \to R$ is induced by (R, F) (see Lemma 3.6). By Lemma 3.10 and Lemma 3.11, both CH^* and K^0 are free.

[7] The theories MU^* and Ω^* have the same formal group law. As $\Omega \not\cong MU^*$, this tells us that the formal group law does *not* uniquely identify an oriented cohomology theory.

4 Algebraic cobordism rings of X_M and $W_{\mathcal{H}}$

4.1 Algebraic cobordism rings of smooth toric varieties

In [11, Theorem 1.2], the authors give a presentation for the algebraic cobordism ring of a smooth toric variety. We recall their presentation in Lemma 4.1 and compare it to the standard presentation for the Chow ring of a smooth toric variety in Lemma 4.3.

Let $T = (\mathbb{C}^\times)^n$ be an n-dimensional torus. Set $N := \mathrm{Hom}(\mathbb{C}^\times, T)$, and set $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$. Let Σ be a unimodular fan in $N_\mathbb{R}$ with smooth toric variety X_Σ. For each ray ρ in Σ, let v_ρ be the primitive vector of ρ, and let D_ρ be the divisor in X_Σ corresponding to ρ. The inclusion $D_\rho \hookrightarrow X_\Sigma$ defines a class $[D_\rho \hookrightarrow X_\Sigma]$ in $\Omega^*(X_\Sigma)$. Recall the notation for the universal formal group law $(\mathbb{L}^*, F_U)$. Consider the formal power series ring over $\mathbb{L}^*$ with variables indexed by the rays in Σ: $T_\Sigma := \mathbb{L}^*[\![x_\rho : \rho \text{ is a ray in } \Sigma]\!]$. Define two ideals in T_Σ: I_1^Σ is generated by elements $x_{\rho_1} \cdots x_{\rho_k}$, where the rays $\rho_1, \ldots, \rho_k$ do *not* generate a cone of Σ; I_2^Σ is generated by elements $\sum_{\rho \text{ ray}}^{F_U} \langle \chi, v_\rho \rangle \cdot_{F_U} x_\rho$, where $\chi \in N^\vee$.

Theorem 4.1. *([11, Theorem 1.2]) There is an $\mathbb{L}^*$-algebra isomorphism $\frac{T_\Sigma}{I_1^\Sigma + I_2^\Sigma} \to \Omega^*(X_\Sigma)$, defined on generators by $x_\rho \mapsto [D_\rho \hookrightarrow X_\Sigma]$.*

Remark 4.2. It is discussed in [11, Theorem 8.2] that the ideal $\langle x_\rho^{n+1} : \rho \text{ is a ray in } \Sigma \rangle$ in T_Σ is contained in $I_1^\Sigma + I_2^\Sigma$. Therefore, the generators x_ρ are *nilpotent* in $\Omega^*(X_\Sigma)$.

Remark 4.3. Using Lemma 4.1 and Lemma 3.10, one recovers the standard presentation for $\mathrm{CH}^*(X_M)$ as $\mathrm{CH}^*(X_M) \simeq \Omega^*(X_M) \otimes_{\mathbb{L}^*} \mathbb{Z}$. Heuristically, simply replace the pair $(\mathbb{L}^*, F_U)$ with the pair $(\mathbb{Z}, F_A)$ everywhere in sight: T_Σ becomes $\mathbb{Z}[\![x_\rho : \rho \text{ is a ray in } \Sigma]\!]$, I_2^Σ is generated by $\sum_{\rho \text{ ray}}^{F_A} \langle \chi, v_\rho \rangle \cdot_{F_A} x_\rho$, where $\chi \in N^\vee$, and I_1^Σ remains the same.

Next, we will show that the generators for the ideal I_2^Σ can be reduced to choosing a *finite set* of characters $\chi_1, \ldots, \chi_n$, as is true in the usual presentation for $CH^*(X_M)$.

Lemma 4.4. *Let $\chi_1, \ldots, \chi_n$ be generators for the $\mathbb{Z}$-module $N^\vee$. Then $I_2^\Sigma = I'$, where I' is the ideal $I' := \left\langle \sum_{\rho \text{ ray}}^{F_U} \langle \chi_i, v_\rho \rangle \cdot_{F_U} x_\rho : i = 1, \ldots, n \right\rangle$ in T_Σ.*

Proof. The inclusion $I' \subseteq I_2^\Sigma$ is clear, so we will prove that $I_2^\Sigma \subseteq I'$. Choose $\chi \in M$. We must show that $\sum_{\rho \text{ ray}}^{F_U} \langle \chi, v_\rho \rangle \cdot_{F_U} x_\rho \in I'$. Write $\chi = \sum_{i=1}^n a_i \chi_i$, where $a_i \in \mathbb{Z}$. The axioms of the formal group law imply

$$\sum_{\rho \text{ ray}}^{F_U} \langle \chi, v_\rho \rangle \cdot_{F_U} x_\rho = \sum_{\rho \text{ ray}}^{F_U} \left\langle \left(\sum_{1 \le i \le n} a_i \chi_i \right), v_\rho \right\rangle \cdot_{F_U} x_\rho$$

$$= \sum_{\rho \text{ ray}}^{F_U} \sum_{1 \le i \le n}^{F_U} \left(a_i \cdot_{F_U} \langle \chi_i, v_\rho \rangle \right) \cdot_{F_U} x_\rho = \sum_{1 \le i \le n}^{F_U} \left(a_i \cdot_{F_U} \sum_{\rho \text{ ray}}^{F_U} \langle \chi_i, v_\rho \rangle \cdot_{F_U} x_\rho \right).$$

As each x_ρ is nilpotent in T_Σ, it follows that the original expression lies in I'. □

4.2 The Feichtner–Yuzvinsky presentation

We will give a Feichtner–Yuzvinsky style presentation for the algebraic cobordism ring of the toric variety of the Bergman fan of a loopless matroid in Lemma 4.6.

Let $M = (E, \mathcal{F})$ be a loopless matroid of rank r. Recall the notation $X_M := X_{\Sigma_M}$ for the toric variety of the Bergman fan Σ_M of M. For a subset $G \subseteq E$, set $e_G := \sum_{i \in G} e_i \in \mathbb{R}^E$. The character lattice of Σ_M is the lattice $N_M^\vee := (\mathbb{Z}^E/\mathbb{Z}e_E)^*$ in $(\mathbb{R}^E/\mathbb{R}e_E)^*$, and the set $\mathcal{B}_M := \{e_i^* - e_j^* : i, j \in E\}$ contains a $\mathbb{Z}$-basis for the lattice $N_M^\vee$. Consider the formal power series ring $\mathcal{T}_M := \mathbb{L}^*[\![x_G : \; G \text{ a non-empty flat of } M]\!]$ with variables indexed by the non-empty flats G of M. Consider the following ideals in $\mathcal{T}_M$: $\mathcal{I}_1^M$ is generated by the elements $x_G x_{G'}$, where G and G' are incomparable flats; $\mathcal{I}_2^M$ is generated by elements ${\sum_{i_0 \in G}}^{F_U} x_G$, where $i_0 \in E$.

Definition 4.5. The **algebraic cobordism ring of** M is $\Omega^*(M) := \frac{\mathcal{T}_M}{\mathcal{I}_1^M + \mathcal{I}_2^M}$.

The following proposition can be deduced from Lemma 4.4, choosing $\mathcal{B}_M$ as a set of generators for the $\mathbb{Z}$-module $N_M^\vee$.

Proposition 4.6. *There is an* $\mathbb{L}^*$*-algebra isomorphism*

$$\varphi\colon \Omega^*(M) \to \frac{T_{\Sigma_M}}{I_1^{\Sigma_M} + I_2^{\Sigma_M}} = \Omega^*(X_M), \quad x_G \mapsto \begin{cases} x_{\rho_G}, & G \neq E; \\ {\sum_{i_0 \in H}}^{F_U} (-_{F_U} x_{\rho_H}), & G = E. \end{cases}$$

Remark 4.7. The Feichtner–Yuzvinsky presentation for $CH^*(X_M)$ from [7] can be recovered from Lemma 4.6 by Lemma 3.10, and similarly the Feichtner–Yuzvinsky presentation for $K^0(X_M)$ from [12] can be recovered from Lemma 4.6 by Lemma 3.11. Heuristically, as in Lemma 4.3, in the presentation $\frac{\mathcal{T}_M}{\mathcal{I}_1^M + \mathcal{I}_2^M}$, replace the pair $(\mathbb{L}^*, F_U)$ with $(\mathbb{Z}, F_A)$ to obtain the Feichtner–Yuzvinsky presentation for $CH^*(X_M)$, and replace $(\mathbb{L}^*, F_U)$ with $(\mathbb{Z}, F_M)$ to obtain the Feichtner–Yuzvinsky presentation for $K^0(X_M)$.

4.3 The simplicial presentation

In this subsection, we prove Lemma 4.9, which is a *simplicial presentation* for $\Omega^*(X_M)$. We also present two consequences of this theorem: Lemma 4.10 and Lemma 4.11.

First, define the following polynomial ring and formal power series ring:

$$S_M := \mathbb{Z}[h_G : \; G \text{ a non-empty flat of } M]; \qquad \mathcal{S}_M := \mathbb{Z}[\![h_G : \; G \text{ a non-empty flat of } M]\!].$$

We will consider the following ideals in S_M and $\mathcal{S}_M$, and for simplicity, we will use the same notation for these ideals in both rings: $\mathcal{J}_1^M$ is generated by the elements $(h_G - h_{G \vee H})(h_H - h_{G \vee H})$, where G, H are non-empty flats of M; and $\mathcal{J}_2^M$ is generated by elements h_G, where $\operatorname{rank}_M(\{G\}) = 1$.

Remark 4.8. The presentation $\frac{S_M}{\mathcal{J}_1^M+\mathcal{J}_2^M}$ is the **simplicial presentation** for the Chow ring of X_M, constructed in [3]. The elements in $CH^*(X_M)$ that correspond to the generators h_G are called **simplicial generators** in $CH^*(X_M)$. In [12], the authors showed that $K^0(X_M)$ can be presented using the *same simplicial presentation*. The elements in $K^0(X_M)$ corresponding to the h_G are called the **simplicial generators** in $K^0(X_M)$. The map that sends a simplicial generator h_G in $K^0(X_M)$ to the corresponding simplicial generator h_G in $CH^*(X_M)$ is the *integral isomorphism* $K^0(X_M) \to CH^*(X_M)$ of [12]. The simplicial generators h_G in $CH^*(X_M)$ satisfy the relation $h_G^r = 0$. Therefore, $\frac{S_M}{\mathcal{J}_1^M+\mathcal{J}_2^M} = \frac{\mathcal{S}_M}{\mathcal{J}_1^M+\mathcal{J}_2^M}$.

Theorem 4.9. *There is an* $\mathbb{L}^*$*-algebra isomorphism*

$$\Phi : \mathbb{L}^* \otimes_{\mathbb{Z}} \left(\frac{\mathcal{S}_M}{\mathcal{J}_1^M + \mathcal{J}_2^M} \right) \to \frac{\mathcal{T}_M}{\mathcal{I}_1^M + \mathcal{I}_2^M} = \Omega^*(X_M), \qquad h_G \mapsto \sum_{H \supseteq G}{}^{F_U} (-_{F_U} x_H).$$

In particular, $\Omega^*(X_M) \simeq \mathbb{L}^* \otimes_{\mathbb{Z}} CH^*(X_M)$ *as* $\mathbb{L}^*$*-algebras.*

Proof sketch. Let μ be the Möbius function on the lattice of flats $\mathcal{F}$. Consider the following two $\mathbb{L}^*$-algebra maps: $\phi\colon \mathbb{L}^* \otimes_{\mathbb{Z}} \mathcal{S}_M \to \mathbb{L}^* \otimes_{\mathbb{Z}} \mathcal{T}_M$, defined by $\phi(h_G) := \sum_{H \supseteq G}{}^{F_U} (-_{F_U} x_H)$, and $\psi\colon \mathbb{L}^* \otimes_{\mathbb{Z}} \mathcal{T}_M \to \mathbb{L}^* \otimes_{\mathbb{Z}} \mathcal{S}_M$, defined by $\psi(x_H) := \sum_{H \subseteq G}{}^{F_U} \mu(H,G) \cdot_{F_U} (-_{F_U} h_G)$. One can verify with Möbius inversion that ϕ and ψ are inverse to each other, so ϕ is an $\mathbb{L}^*$-algebra isomorphism. To show that Φ is an isomorphism, it is enough to show that $\phi(\mathcal{J}_1^M) = \mathcal{I}_1^M$ and $\phi(\mathcal{J}_2^M) = \mathcal{I}_2^M$. One can prove these equalities using the techniques of [12, §5 and Appendix A]. □

Corollary 4.10. *The algebraic cobordism ring* $\Omega^*(X_M)$ *is a free* $\mathbb{L}^*$*-module of finite rank.*

Proof. As $\Omega^*(X_M) \simeq \mathbb{L}^* \otimes_{\mathbb{Z}} CH^*(X_M)$ by Lemma 4.9, this follows since $CH^*(X_M)$ is a free $\mathbb{Z}$-module of finite rank ([7, Corollary 1]). □

Recall Lemma 3.12, which says that a formal group law (R, F) gives rise to a free oriented cohomology theory $\mathrm{h}^*_{(R,F)}(-) := \Omega^*(-) \otimes_{\mathbb{L}^*} R$. By definition, $\mathrm{h}^*_{(R,F)}(\mathrm{pt}) = R$.

Corollary 4.11. *If* $\mathrm{h}^*_{(R,F)}$ *is a free oriented cohomology theory with* $\mathrm{h}^*_{(R,F)}(\mathrm{pt}) = R$*, then*

$$\mathrm{h}^*_{(R,F)}(X_M) \simeq R \otimes_{\mathbb{Z}} CH^*(X_M) \quad \textit{as } R\textit{-algebras.}$$

As K^0 *is a free oriented cohomology theory with* $K^0(\mathrm{pt}) \simeq \mathbb{Z}$*, it follows that* $K^0(X_M) \simeq CH^*(X_M)$ *as rings, recovering the integral isomorphism of [12].*[8]

Proof. This follows from Lemma 4.9 and the sequence of isomorphisms

$$h^*_{(R,F)}(X_M) := \Omega^*(X_M) \otimes_{\mathbb{L}^*} R \simeq (\mathbb{L}^* \otimes_{\mathbb{Z}} CH^*(X_M)) \otimes_{\mathbb{L}^*} R \simeq R \otimes_{\mathbb{Z}} CH^*(X_M). \qquad \square$$

[8]The integral isomorphism of [12] is "exceptional" in the sense that it satisfies a "Hirzebruch–Riemann–Roch" type theorem. At this time, we do not have an explanation for the exceptionality from the point of view of algebraic cobordism. We hope to address this in future work.

4.4 Algebraic and complex cobordism rings of $W_{\mathcal{H}}$

In this subsection, let $\mathcal{H}$ be an essential hyperplane arrangement in $\mathbb{C}^n$, and let $W_{\mathcal{H}}$ be the associated wonderful variety. The matroid associated with $\mathcal{H}$ will be denoted $M_{\mathcal{H}}$, and the toric variety of this matroid is $X_{M_{\mathcal{H}}}$. As MU^* defines an oriented cohomology theory, there is a unique morphism $\vartheta\colon \Omega^* \to MU^*$. Denote the pullback of the inclusion $\iota\colon W_{\mathcal{H}} \hookrightarrow X_{M_{\mathcal{H}}}$ by $\iota^*\colon \Omega^*(X_{M_{\mathcal{H}}}) \to \Omega^*(W_{\mathcal{H}})$.

Theorem 4.12. *Both $\iota^*\colon \Omega^*(X_{M_{\mathcal{H}}}) \to \Omega^*(W_{\mathcal{H}})$ and $\vartheta\colon \Omega^*(W_{\mathcal{H}}) \to MU^*(W_{\mathcal{H}}(\mathbb{C}))$ are isomorphisms of $\mathbb{L}^*$-algebras.*

Proof sketch. It follows from [10, Appendix A] that the cycle class map $CH^*(W_{\mathcal{H}}) \to H^*(W_{\mathcal{H}}(\mathbb{C}))$ is an isomorphism.[9] In particular, $H^*(W_{\mathcal{H}}(\mathbb{C}))$ is concentrated in even degrees. Together with the fact that $CH^*(W_{\mathcal{H}})$ is a free $\mathbb{Z}$-module of finite rank ([7, Corollary 1]), this implies that the "Atiyah–Hirzebruch spectral sequence" degenerates on the second page ([16, Theorem 15.7]). Hence $MU^*(W_{\mathcal{H}}(\mathbb{C}))$ is a free $\mathbb{L}^*$-module whose rank equals the $\mathbb{Z}$-module rank of $CH^*(W_{\mathcal{H}})$. Consider the composition of $\mathbb{L}^*$-algebra maps:

$$\tau\colon \Omega^*(X_M) \xrightarrow{\iota^*} \Omega^*(W_{\mathcal{H}}) \xrightarrow{\vartheta} MU^*(W_{\mathcal{H}}(\mathbb{C})).$$

One can show using geometric arguments that ι^* and ϑ are both surjective. Hence, the composition $\tau = \vartheta \circ \iota^*$ is surjective as well. Recall from Lemma 4.10 that $\Omega^*(X_M)$ is a free $\mathbb{L}^*$-module whose rank equals the $\mathbb{Z}$-module rank of $CH^*(X_M)$. As $\mathbb{L}^*$ is an integral domain and τ is a surjective $\mathbb{L}^*$-algebra morphism between free $\mathbb{L}^*$-modules of the same rank, we conclude that τ is an isomorphism. In particular, $\tau = \vartheta \circ \iota^*$ is injective, which means that ι^* is injective. As ι^* is both injective and surjective, it must be an isomorphism. Thus, $\vartheta = \tau \circ (\iota^*)^{-1}$ is an isomorphism as well. □

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[9] It is *not* necessarily the case that the cycle class map $CH^*(X_M) \to H^*(X_M(\mathbb{C}))$ is an isomorphism for an arbitrary loopless matroid M. Indeed, it is *not* an isomorphism for $M = U_{2,3}$ from Lemma 2.6.

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