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Introducing the Coxeter flag variety

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Abstract. Let W be a finite Weyl group and fix a Coxeter element $c \in W$. We construct a complex of toric varieties in the complete flag variety by setting all extremal generalized Plucker coordinates which are not c-noncrossing to zero. We describe the structure of this complex, including the corresponding moment polytopes for each c, which reflect the structure of c-clusters. We give a GKM description of the (T-equivariant) cohomology rings and show that its isomorphism type of this depends only on W.

Résumé. Soit W un groupe de Weyl fini et $c \in W$ un élément de Coxeter fixé. Nous construisons un complexe de variétés toriques en annulant toutes les coordonnées de Plücker généralisées extrémales qui ne sont pas c-non-croisées. Nous décrivons la structure de ce complexe, ainsi que les polyèdres du moment correspondants pour chaque c, lesquels reflètent la structure des c-clusters. Nous donnons ensuite une description GKM des anneaux de cohomologie (équivariante par rapport à T) et montrons que leur type d'isomorphisme dépend uniquement de W, et non de c.

Keywords: flag variety, noncrossing partition, Schubert, Richardson, Plucker, equivariant cohomology

1 Introduction

Let G be a reductive complex algebraic group of finite rank n and fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. The *generalized flag variety* is the coset space G/B. The starting point for any question about the geometry of G/B is the *Weyl group* $W = N_G(T)/T$. The Bruhat decomposition

$$G/B = \bigcup_{w \in W} \mathring{X}^w \qquad \text{with} \qquad \mathring{X}^w = BwB/B$$

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gives rise to *Schubert varieties* $X^w = \overline{BwB}$ and *Richardson varieties* $X_u^v = X^v \cap w_\circ X^{w_\circ u}$, where $w_\circ \in W$ the longest element. Describing the interactions of these varieties using the combinatorics of W is a major focus of modern algebraic combinatorics, including in Schubert calculus, Kazhdan–Lusztig theory, and beyond.

A *(standard) Coxeter element* $c \in W$ is any product of all simple reflections in W. *Coxeter–Catalan combinatorics* is the study of a number of equinumerous families naturally generated by the choice of a pair (W, c), including:

(A) The lattice of *c-noncrossing partitions* $\mathrm{NC}(W,c) \subseteq W$ comprises all prefixes $\tau_1 \cdots \tau_i$ of minimal length reflection factorizations $c = \tau_1 \cdots \tau_n$ [4].

(B) The *(combinatorial) c-clusters* Cl_c are subsets of the almost positive roots that index the (algebraic) clusters in a cluster algebra [17].

(C) The *c-Cambrian congruence* $\equiv_c$ is a lattice quotient of the right weak order $\leqslant_R$ [16].

This abstract is an invitation to study a space $\mathrm{CFl}_c \subseteq G/B$ that enriches the Coxeter–Catalan combinatorics of (W, c) in much the same way that G/B enriches the general combinatorics of W. Complete results and proofs can be found in our paper [3].

We call CFl_c the *c-Coxeter flag variety*.[1] Of the possible definitions of CFl_c—we have one for each of (A)-(C) above—this abstract will use the *generalized Plücker embedding*

$$\begin{aligned} \mathrm{Pl} : G/B &\hookrightarrow \mathbb{P}(V^\lambda) \\ gB &\mapsto [gv_\lambda], \end{aligned} \tag{1.1}$$

where $\lambda \in \hom_{\text{alg. grp.}}(T, \mathbb{C}^\times)$ is fixed to be regular and dominant, V^λ is the simple G-module with highest weight λ, and $v_\lambda \in V^\lambda$ is a highest weight vector. Each $w \in W$ indexes a one-dimensional *extremal weight space*

$$V^\lambda_{w\lambda} = \{v \in V^\lambda \mid hv = (\lambda)(wh) \text{ for all } h \in T\}.$$

The *extremal Plücker coordinates* Pl_w are the projective coordinates on G/B associated with each $V^\lambda_{w\lambda}$. For any $\mathcal{A} \subseteq W$, we can define a *$\mathcal{A}$-Plücker vanishing variety*

$$\mathrm{PV}_{\mathcal{A}} = \bigcap_{w \in \mathcal{A}} \{gB \in G/B \mid \mathrm{Pl}_w(gB) = 0\} \subseteq G/B.$$

Understanding arbitrary Plücker vanishing subvarieties is intrinsically impossible because of the key role they play in Mnëv's universality theorem [14].

Definition 1.1. The *c-Coxeter flag variety* is $\mathrm{CFl}_c \coloneqq \mathrm{PV}_{\mathrm{NC}(W,c)} \subseteq G/B$.

[1] We use the term "flag variety" to emphasize the remarkable combiantorial parallels between CFl_c and G/B. The inclusion $\mathrm{CFl}_c \subseteq G/B$ is also commensurate with the fact that the Coxeter–Catalan combinatorics of W are a proper subset of arbitrary Weyl group combinatorics.

We show that each c-Coxeter flag variety is an equidimensional complex of toric Richardson varieties indexed by c-Cambrian classes and, with affine pacing indexed by c-clusters. Moreover, the associated moment polytopes form a contractible polytopal complex of (W, c)-*polypositroids* in the sense of Lam–Postnikov [13].

In the special case of $G = \mathrm{GL}_{n+1}$, $W = S_{n+1}$, and $c = (n+1 \cdots 2\,1)$, this variety was studied in [1, 2] as the *quasisymmetric flag variety* because of the following result.

Theorem 1.2 ([2, Theorem A])**.** If $G = \mathrm{GL}_{n+1}$, $W = S_{n+1}$, and $c = (n+1 \cdots 2\,1)$, then

$$H^{\bullet}(\mathrm{CFl}_c) \cong \mathbb{Z}[x_1, \ldots, x_{n+1}]/\langle f(x_1, \ldots, x_{n+1}) - f(0, \ldots, 0) \mid f \in \mathrm{QSym}_{n+1} \rangle$$

where QSym_{n+1} is the quasisymmetric polynomial ring of Gessel and Stanley.

In this case we constructed a near-complete geometric and combinatorial analogue of Schubert calculus, replacing symmetric polynomials with quasisymmetric polynomials and Schubert polynomials with the *(equivariant) forest polynomials* of [1, 15].

The reader should consider how the results of [1, 2] might generalize to arbitrary Coxeter flag varieties, including analogues of Theorem 1.2 (see Section 7). However, even if one keeps $G = \mathrm{GL}_{n+1}$, varying "c" produces new and surprising structures which are not yet understood; see Appendix A for a concrete example.

2 Preliminaries

We review some standard preliminary material on Weyl groups and algebraic groups, directing the reader to [6, 20] for details. Let $\mathrm{Char}(T)$ be the group of algebraic characters of T. The Weyl group W acts on $\mathbb{E} = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Char}(T)$ by reflections. The adjoint action of T decomposes the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ into weight spaces

$$\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

and the set of nonzero weights is a root system Φ for W. The choice of B determines a set of simple roots Δ and positive roots Φ^+ in Φ.

For $\alpha \in \Phi^+$, let $s_\alpha \in W$ be the corresponding reflection. Let $\mathsf{S} = \{s_\alpha \mid \alpha \in \Delta\}$, the set of simple reflections, and $\mathsf{T} = \{s_\alpha \mid \alpha \in \Phi^+\}$, the set of all reflections. For $\tau \in \mathsf{T}$, let $r(\tau)$ be the associated root. For $w \in W$, a *reduced word* $\boldsymbol{w} = (s_1^{\boldsymbol{w}}, \ldots, s_\ell^{\boldsymbol{w}})$ is any minimal-length factorization

$$w = s_1^{\boldsymbol{w}} \cdots s_\ell^{\boldsymbol{w}} \text{ with } s_1^{\boldsymbol{w}}, \ldots, s_\ell^{\boldsymbol{w}} \in \mathsf{S}.$$

The *length* of w is $\ell(w)$, the number of simple reflections in any such factorization. The *inversions* of w are elements of $\mathrm{Inv}(w) = \{\tau \in \mathsf{T} \mid \ell(\tau w) < \ell(w)\}$. Let $\leqslant_B$ and $\leqslant_R$ be the strong and (right) weak Bruhat orders on W. Multiplication by the longest element $w_\circ \in W$ gives an antiautomorphism of these orders.

For $w \in W$ and any subset $C \subseteq r(\mathrm{Inv}(w))$ that is closed under addition in Φ, let

$$U_C = \prod_{\alpha \in C} \exp(\mathfrak{g}_\alpha) \qquad \text{where} \qquad \exp(X) = 1 + X + \frac{1}{2}X^2 + \cdots \tag{2.1}$$

and the product is taken in any fixed order. This gives a (non-canonical) T-equivariant isomorphism of varieties from U_C to $\bigoplus_{\alpha \in C} \mathfrak{g}_\alpha$. We also have a (canonical) isomorphism

$$U_{r(\mathrm{Inv}(w))} \cong \mathring{X}^w \qquad \text{given by} \qquad u \mapsto uwB,$$

so in all we have a T-equivariant coordinatization of each open Schubert cell $\mathring{X}^w$.

A *minimal reflection factorization* for $w \in W$ is any minimal factorization

$$w = \tau_{i_1} \cdots \tau_{i_k} \text{ with } \tau_{i_1}, \ldots, \tau_{i_k} \in \mathsf{T}.$$

The *absolute length* of w is $\ell_{\mathsf{T}}(w) = k$. The *absolute order* on W is the prefix order with respect to minimal reflection factorizations. For a Coxeter element $c \in W$, the *c-noncrossing partitions* are

$$\mathrm{NC}(W, c) = \{u \in W \mid u \leqslant_{\mathsf{T}} c\},$$

and the *positive noncrossing partitions* are

$$\mathrm{NC}(W, c)^+ = \{u \in \mathrm{NC}(W, c) \mid c \leqslant_B u\}.$$

The size of these sets are the *W-Catalan numbers* and the *positive W-Catalan numers*:

$$|\,\mathrm{NC}(W, c)| = \mathrm{Cat}_W \qquad \text{and} \qquad |\,\mathrm{NC}(W, c)^+| = \mathrm{Cat}_W^+.$$

For $\leqslant_B$, $\leqslant_R$, or $\leqslant_{\mathsf{T}}$, we use a dot to indicate a covering relation, as in $\lessdot_B$, $\lessdot_R$, or $\lessdot_{\mathsf{T}}$. Every such relation corresponds to an edge in the Cayley graph $\mathrm{Cayley}(W)$ with respect to right multiplication by the generating set T. For an interval I in the Bruhat or absolute order, we write $\mathrm{Hasse}_{\mathsf{T}}(I)$ and $\mathrm{Hasse}_B(I)$ for the edge-labelled subgraph of $\mathrm{Cayley}(W)$ with vertex set I and edge set given by the covering relations of $\leqslant_{\mathsf{T}}$ or $\leqslant_B$, respectively.

A *Coxeter matroid* is a subset $\mathcal{M} \subset W$ such that $w\mathcal{M}$ has a unique Bruhat-maximum element for every $w \in W$. Recall that we have fixed a regular dominant weight $\lambda \in \mathbb{E}$. The Gelfand–Serganova theorem [7, Theorem 6.3.1] gives a bijection

$$\mathcal{M} \mapsto P_{\mathcal{M}} := \mathrm{conv}(\{w \cdot \lambda \mid w \in \mathcal{M}\}).$$

between Coxeter matroids and *Coxeter matroid polytopes*; we take the above map as our definition for Coxeter matroid polytopes.

For $Y \subseteq G/B$ a T-invariant subvariety, the *moment polytope* of Y is

$$\mu(Y) = \mathrm{conv}(\{w \cdot \lambda \mid w \in Y^T\}) \subseteq \mathbb{E}.$$

Proposition 2.1 ([9]). The moment polytope $\mu(Y)$ of an irreducible T-invariant variety $Y \subset G/B$ is a Coxeter matroid polytope.

The T-fixed points of G/B are wB/B for $w \in W$, so $\mu(G/B) = P_W$, the *W-permutahedron*. For a Richardson variety X_u^v, which has T-fixed point set $[u,v] = \{w \mid u \leqslant_B w \leqslant_B v\}$, the moment polytope is $\mu(X_u^v) = P_{[u,v]}$, the *twisted Bruhat interval polytope*.

Finally, say that a T-invariant subvariety $X \subset G/B$ is *rigid* if, for any T-invariant subvariety $Y \subseteq G/B$, $Y^T \subset X^T$ implies $Y \subset X$. The following is well-known to experts.

Theorem 2.2. If $\mathcal{A} \subseteq W$, then $\mathrm{PV}_{\mathcal{A}}$ is the union of all T-orbit closures $Y \subset G/B$ with $Y^T \subset \mathcal{A}$. As such it is the unique rigid variety with T-fixed point set $\mathcal{A}$.

3 Coxeter Richardson Varieties

A *c-Coxeter Richardson variety* is a Richardson variety of the form X_w^{wc}. These are n-dimensional toric varieties, which have been studied by a number of authors [8, 11, 12].

Fact 3.1 ([3, Theorem 7.1]). For $w \in W$, X_w^{wc} is nonempty if and only if $\ell(wc) = \ell(w) + \ell(c)$.

A *translated c-Coxeter Richardson variety* is the translation $w^{-1}X_w^{wc}$. In this section we investigate when two $w, w' \in W$ give the same translated variety; we find that this is determined by Reading's *c-Cambrian equivalence* [18].

Fix a reduced word $\boldsymbol{c}$ for c and let $\boldsymbol{c}^\infty$ denote the ∞-fold concatenation of $\boldsymbol{c}$. For $x \in W$, we let $\boldsymbol{x}_{c^\infty}$ denote the lexicographically first reduced word for x in $\boldsymbol{c}^\infty$. For any subword $\boldsymbol{x}$ of $\boldsymbol{c}^\infty$, we refer to the part of $\boldsymbol{x}$ in the ith copy of $\boldsymbol{c}$ in $\boldsymbol{c}^\infty$ as the *ith syllable of $\boldsymbol{x}$*.

Definition 3.2. A *c-sortable permutation* is a permutation x such that each syllable of $\boldsymbol{x}_{c^\infty}$ contains every letter in the following syllable. We denote the set of c-sortable permutations in W by $\mathrm{Sort}(W, c)$.

Example 3.3. Let $W = S_5$ and $c = s_2s_1s_3s_4 \in S_5$. Using one-line notation, the permutations $x = 35421$ and $y = 53421$ have c-sorting words

$$\begin{aligned} \boldsymbol{x} &= s_2\ s_1\ s_3\ s_4 \mid s_2\ \cancel{s_1}\ s_3\ s_4 \mid s_2\ \cancel{s_1}\ \cancel{s_3}\ \cancel{s_4} \mid \cdots, \qquad \text{and} \\ \boldsymbol{y} &= s_2\ s_1\ s_3\ s_4 \mid s_2\ \cancel{s_1}\ s_3\ s_4 \mid s_2\ s_1\ \cancel{s_3}\ \cancel{s_4} \mid \cdots. \end{aligned}$$

Then x is c-sortable, but y is not, as $\{s_2, s_3, s_4\} \not\supseteq \{s_1, s_2\}$.

In [18, Theorem 1.1], Reading shows that for any $x \in W$ there is a unique $\leqslant_R$-maximum c-sortable element $\pi_\downarrow(x) \leqslant_R x$. The *$c$-Cambrian congruence* is the relation on W defined by $x \equiv_c y$ if $\pi_\downarrow(x) = \pi_\downarrow(y)$.

Example 3.4. Continuing Example 3.3 with $W = S_5$ and $c = s_2s_1s_3s_4 \in S_5$, $x = 35421$, and $y = 53421$, we have $x \equiv_c y$: x is covered by y in $\leqslant_R$, so it must be $\pi_\downarrow(y)$.

For $w \in W$, let $w_{op} := w^{-1}w_\circ$, and note that $X_w^{wc} \neq \varnothing$ if and only if $c \leqslant_R w_{op}$.

Theorem 3.5 ([3, Theorem 14.3 and Corollary 6.9])**.** If $u, v \in W$ have $c \leqslant_R u_{op}, v_{op}$, then the following are equivalent.

1. $u_{op} \equiv_c v_{op}$
2. $u^{-1}X_u^{uc} = v^{-1}X_v^{vc}$

Theorem 3.5 is proved independently as [8, Theorem 1.9]; in [3] we extend it to describe equivalence among $u^{-1}X_u^{uc'}$ for any $c' \leqslant_B c$. We sketch a few details from the proof that will be useful in later sections. The essential definition is that of a *translated Bruhat interval*: for $u, v \in W$, let

$$\overleftarrow{[u,v]} := u^{-1}[u,v] \qquad \text{where} \qquad [u,v] = \{w \mid u \leqslant_B w \leqslant_B v\}. \tag{3.1}$$

Proposition 3.6. For $u, v, w \in W$, the Plücker vanishing variety associated to $u[w,v]$ is uX_w^v, and therefore uX_w^v is rigid. In particular, this holds for $u = w^{-1}$ and $v = wc$.

Proof sketch. Translating by u in G/B also translates each T-fixed point and extremal Plücker coordinate by u. Moreover, if $Y \subseteq G/B$ is a T-invariant variety, then $Y^T = \{z \in W \mid Y \cap BzB \neq \varnothing\}$, so assuming $Y^T \subseteq [w,v]$ implies that $Y \subseteq X^v$ and $Y \subseteq w_\circ X^{w_\circ w}$. □

Say that two Bruhat intervals $[u,v]$ and $[u',v']$ are *shape equivalent* if multiplication by $u'u^{-1}$ gives an order homomorphism from $[u,v]$ to $[u',v']$. A consequence of the rigidity in Proposition 3.6 is the following geometric implication of shape equivalence.

Corollary 3.7. For $u, v, u', v' \in W$, the following are equivalent.

1. $[u,v]$ and $[u',v']$ are shape equivalent.
2. $\overleftarrow{[u,v]} = \overleftarrow{[u',v']}$
3. $u^{-1}X_u^v = (u')^{-1}X_{u'}^{v'}$

Proof sketch for Theorem 3.5. It is sufficient to show that $\overleftarrow{[u,uc]} = \overleftarrow{[v,vc]}$ if and only if $u_{op} \equiv_c v_{op}$. We show that $\overleftarrow{[u,vc]}$ is determined by a distinguished chain in $[u,uc]$, and that this chain is uniquely determined by the c-Cambrian equivalence class of u_{op}. □

4 Translated Richardsons and noncrossing partitions

In this section we explore how translated c-Coxeter Richardson varieties interact with noncrossing partitions.

Theorem 4.1 ([3, Theorem 7.1])**.** For any $w \in W$ we have an injective homomorphism

$$\begin{array}{rcl}\mathrm{Sh}_{w^{-1}} : \mathrm{Hasse}_B([w, wc]) & \to & \mathrm{Hasse}_\mathsf{T}(\mathrm{NC}(W, c)) \\ u & \mapsto & w^{-1}u\end{array}$$

Moreover $\mathrm{Hasse}_\mathsf{T}(\mathrm{NC}(W, c))$ is the union of the images of these maps.

Sketch of proof. Every maximal Bruhat chain $w \lessdot_B w\tau_1 \lessdot_B \cdots \lessdot_B wc$ corresponds to a maximal absolute chain $\mathrm{id} \lessdot_\mathsf{T} \tau_1 \lessdot_\mathsf{T} \cdots \lessdot_\mathsf{T} c$, proving the homomorphism. To see it is surjective, we do the reverse, finding intervals $[u, uc]$ that contain Bruhat chians mapping to every absolute chain, which is nontrivial. □

We now describe Reading's map from Cambrian classes to noncrossing partitions [17, Theorem 6.1] . The *cover reflections* for $x \in \mathrm{Sort}(W, c)$ are $\mathrm{Covers}(x) = \{xsx^{-1} \mid xs \lessdot_R x\} \subset \mathrm{Inv}(x)$. Then there is a unique order (up to trivial commutation) such that

$$\mathrm{nc}_c(x) := \prod_{\tau \in \mathrm{Covers}\, x} \tau \in \mathrm{NC}(W, c),$$

Moreover the map $\mathrm{nc}_c : \mathrm{Sort}(W, c) \to \mathrm{NC}(W, c)$ is a bijection, and restricts to a bijection

$$\{x \in \mathrm{Sort}(W, c) \mid x \geqslant_R c\} \to \mathrm{NC}(W, c)^+.$$

Example 4.2. Continuing Example 3.3 with $W = S_5$ and $c = s_2s_1s_3s_4 \in S_5$, and $x = 35421 \in \mathrm{Sort}(S_5, c)$, the cover reflections of x are $(4\,5)$, $(2\,4)$, and $(1\,2)$. Moreover,

$$\mathrm{nc}_c(x) = (2\,4)(1\,2)(4\,5) = (2\,4)(4\,5)(1\,2) = 41352.$$

We find a novel characterization of Reading's map using c-Coxeter Richardson varieties. Given $w \in \mathrm{NC}(W, c)$, define the *noncrossing inversion set* to be

$$\mathrm{Inv}_{\mathrm{NC}}(w) := \{\tau \in \mathrm{Inv}(w) \mid \tau w \in \mathrm{NC}(W, c)\}.$$

Theorem 4.3 ([3, Theorem 15.1])**.** For $w \in W$ with $\ell(wc) = \ell(w) + \ell(c)$:

1. the Bruhat maximum element of $\overleftarrow{[w, wc]}$ is $u = \mathrm{nc}_c(\pi_\downarrow(w_{op}))$, and

2. the neighbors of u in $\mathrm{Sh}_{w^{-1}}\big(\mathrm{Hasse}_B([w, wc])\big)$ are the n elements of $\{\tau u \mid \tau \in \mathrm{Inv}_{\mathrm{NC}}(u)\}$.

Sketch of proof. By the Coxeter matroid property, it is sufficient to prove 2. We do so by explicitly constructing intervals $[v, vc]$ which are shape equivalent to $[w, wc]$ and contain, in a clear way, the covering relation $v\tau u \lessdot_B vu$. Key to our proof are results of Biane–Josuat-Vergès [5] and Reading–Speyer [19]. □

5 Cluster charts for CFl_c

We now characterize CFl_c using the combinatorics of clusters. As defined in [17, Section 7], the *c-cluster fan* is a complete simplicial fan in $\mathbb{R}\Phi$ whose ray generators comprise the *almost positive roots* $\Phi_{\geqslant -1} = \Phi^+ \cup -\Delta$. The *positive cluster fan* is the set

$$Cl_c^+ = \{\mathcal{F} \cap \Phi^+ \mid \mathcal{F} \in Cl_c\},$$

where as is customary we identify each cone $\mathcal{F} \in Cl_c$ with its collection of ray generators.

Biane and Josuat-Vergès [5] construct a bijection between $\mathrm{NC}(W,c)$ and Cl_c^+ by taking each $u \in \mathrm{NC}(W,c)$ to the roots corresponding to its *right* inversions. We exploit the fact that $Cl_c^+ = Cl_{c^{-1}}^+$ to create a "left" version of this map:

$$\mathrm{Clust} : \mathrm{NC}(W,c) \to Cl_c^+, \qquad \text{with} \qquad \mathrm{Clust}(u) = \{r(\tau) \mid \tau \in \mathrm{Inv}_{\mathrm{NC}}(u)\}.$$

Restricting to $\mathrm{NC}(W,c)^+$ gives a bijection to the maximal cones in Cl_c^+, which subdivides the positive root cone $\mathbb{R}_{\geqslant 0}\Delta$ and contains every positive root as a ray generator.

Definition 5.1. For $u \in \mathrm{NC}(W,c)$ we define the *Coxeter Schubert cell* by

$$\mathring{X}^u_{\mathrm{NC}} := U_{\mathrm{Clust}(u)}B/B \subset BuB/B.$$

Explicit examples of $\mathring{X}^u_{\mathrm{NC}}$ can be found in Appendix A. If we coordinatize $\mathring{X}^u$ as the T-module $\bigoplus_{\alpha \in r(\mathrm{Inv}(w))} \mathfrak{g}_\alpha$ as in Equation (2.1), then $\mathring{X}^u_{\mathrm{NC}} \subset \mathring{X}^u$ is the coordinate subspace $\bigoplus_{\alpha \in \mathrm{Clust}(u)} \mathfrak{g}_\alpha$, which is a T-submodule with characters $\mathrm{Clust}(u)$.

Theorem 5.2 ([3, Theorem 9.3])**.** We have $\displaystyle\bigcup_{\substack{w \in W \\ w \leqslant_B wc}} w^{-1}X_w^{wc} = \mathrm{CFl}_c = \bigsqcup_{u \in \mathrm{NC}(W,c)} \mathring{X}^u_{\mathrm{NC}}.$

As we already know that $w^{-1}X_w^{wc} \subseteq \mathrm{CFl}_c$, we prove Theorem 5.2 by establishing the second equality, and that each $\mathring{X}^u_{\mathrm{NC}}$ belongs to some $w^{-1}X_w^{wc}$.

Proposition 5.3. If $u \in \mathrm{NC}(W,c)$, then $\mathring{X}^u_{\mathrm{NC}} = BuB \cap \mathrm{CFl}_c$. Otherwise, $BuB \cap \mathrm{CFl}_c = \varnothing$.

The proof of this fact uses a key property of the cluster fan. Say that a subset $C \subset \Phi_W^+$ of roots is *strongly closed* if $\mathbb{R}_{\geqslant 0}C \cap \Phi = C$. A direct calculation ([3, Proposition 9.8]) shows that for any $w \in W$ and any subset $C \subset \mathrm{Inv}(w)$ such that $r(C)$ is strongly closed,

$$U_{r(C)}wB/B = BwB \cap \bigcap_{\alpha \in \mathrm{Inv}(w) \setminus C} \{\mathrm{Pl}_{s_\alpha w} = 0\}.$$

Fact 5.4 (*Cluster cone property*)**.** For $u \in \mathrm{NC}(W,c)$ the subset $\mathrm{Clust}(u)$ is strongly closed. Therefore Proposition 5.3 is true.

For the next result, recall the contents of Theorem 4.3.

Proposition 5.5. Take $w \in W$ with $\ell(wc) = \ell(w) + \ell(c)$, and let $u \in \mathrm{NC}(W,c)^+$ be the Bruhat-maximum element of $\overleftarrow{[w,wc]}$. Then the $w^{-1}X_w^{wc}$ is the closure of $\mathring{X}^u_{\mathrm{NC}}$.

To prove Theorem 5.2, we show an analogue of Proposition 5.5 for the $\mathring{X}^u_{\mathrm{NC}}$ for $u \in \mathrm{NC}(W,c) \setminus \mathrm{NC}(W,c)^+$ by applying the proposition to each parabolic subgroup of W.

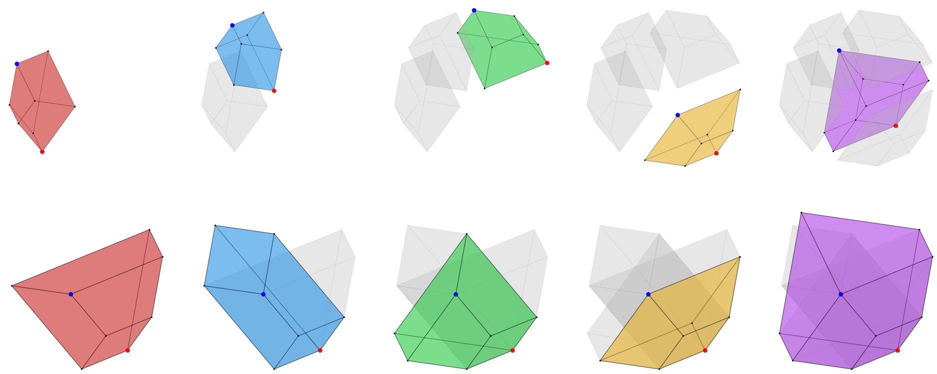

Figure 1: With $c = s_1s_3s_2 \in S_4$, the cumulative overlay of moment polytopes for untranslated (top) and translated (bottom) c-Coxeter Richardson varieties. Each translated polytope is distinct, as the c-Cambrian congruence is trivial on $\{w \mid w_{op} \leqslant_R c\}$.

6 The Coxeter Flag Variety as a toric complex

Using Theorem 4.1 and 5.2, we have an abstract polytopal complex whose 1-skeleton is $\mathrm{Hasse}_{\mathrm{T}}(\mathrm{NC}(W, c))$, namely

$$\mathrm{Complex}(\mathrm{CFl}_c) = \left(\bigsqcup_{w \leqslant_R wc} w^{-1} P_{[w,wc]} \right) \Big/ \left\{ \begin{array}{l} \text{identify any faces } F_1 \text{ and } F_2 \\ \text{having the same vertex sets.} \end{array} \right\}. \tag{6.1}$$

Corollary 6.1. CFl_c is a toric complex which is combinatorially equivalent under the moment map to $\mathrm{Complex}(\mathrm{CFl}_c)$.

By decomposing $\mathrm{Complex}(\mathrm{CFl}_c)$ according to our affine paving, one can show that the $\mathrm{Complex}(\mathrm{CFl}_c)$ is contractable. However, the type of $\mathrm{Complex}(\mathrm{CFl}_c)$ depends heavily on c; for instance if $G = \mathrm{GL}_3$ and $c = s_3s_2s_1$, every constituent will be a cube, but if $c = s_2s_1s_3$, one constituent will be a tetragonal trapezohedron; see Appendix A and Figure 1.

Recall that a Coxeter matroid polytope is a (W, c)-*polypositroid* if its defining inequalities have the form $\mathbf{x} \cdot \vec{n} \leqslant a$ with $(\mathrm{id} - c)\vec{n}$ parallel to a root [13].

Theorem 6.2 ([3, Theorem 8.5])**.** All faces in $\mathrm{Complex}(\mathrm{CFl}_c)$ are (W, c)-polypositroids.

7 The cohomology of CFl_c

We now describe the equivariant cohomology of CFl_c with respect to the *adjoint torus* $T_{ad} = T/Z(G)$; this choice is made to resolve divisibility issues that arise for general T. In this cohomology theory we have $H^{\bullet}_{T_{ad}}(pt) = \mathbb{Z}[t_\alpha \mid \alpha \in \Delta]$.

Our main tool is a framework known as GKM theory [10]; we now describe the relevant details. For $\mathcal{A} \subseteq W$, let the *GKM cohomology ring* be the $H^\bullet_{T_{ad}}(pt)$-algebra

$$H^\bullet_{T_{ad}}(\mathcal{A}) = \left\{(f_w)_{w\in\mathcal{A}} \in \left(H^\bullet_{T_{ad}}(pt)\right)^{\mathcal{A}} \mid t_\alpha \text{ divides } f_w - f_{s_\alpha w} \text{ whenever } w, s_\alpha w \in \mathcal{A}\right\}.$$

Under a certain condition, which we will refer to as having a *good affine paving* (see [2, §11.1] for applicable criteria), we have an isomorphism $H^\bullet_{T_{ad}}(\mathrm{PV}_{\mathcal{A}}) \cong H^\bullet_{T_{ad}}(\mathcal{A})$. This is a free $H^\bullet_{T_{ad}}(pt)$-module, and the ordinary (singular) cohomology ring can be obtained as

$$H^\bullet(\mathrm{PV}_{\mathcal{A}}) \cong H^\bullet_{T_{ad}}(\mathrm{PV}_{\mathcal{A}})/\mathfrak{m} \qquad \text{where} \qquad \mathfrak{m} = (t_\alpha \mid \alpha \in \Delta).$$

Theorem 7.1 ([3, Theorem 9.3])**.** The decomposition $\mathrm{CFl}_c = \bigsqcup_{u\in\mathrm{NC}(W,c)} \mathring{X}^u_{\mathrm{NC}}$ from Theorem 5.2 is a good affine paving. As a consequence,

$$H^\bullet_{T_{ad}}(\mathrm{CFl}_c) \cong H^\bullet_{T_{ad}}(\mathrm{NC}(W,c)) \qquad \text{and} \qquad H^\bullet(\mathrm{CFl}_c) \cong H^\bullet_{T_{ad}}(\mathrm{CFl}_c)/\mathfrak{m}.$$

Corollary 7.2. For Coxeter elements $c, c' \in W$, we have $H^\bullet(\mathrm{CFl}_c) \cong H^\bullet(\mathrm{CFl}_{c'})$

Proof. All Coxeter elements are conjugate so $c = uc'u^{-1}$ for some $u \in W$. The map

$$H^\bullet_{T_{ad}}(\mathrm{CFl}_c) \to H^\bullet_{T_{ad}}(\mathrm{CFl}_{c'}) \qquad \text{defined by} \qquad (f_w)_{w\in\mathrm{NC}(W,c)} \mapsto \left(f_{uwu^{-1}}(t^{u\cdot\alpha})\right)_{uwu^{-1}\in\mathrm{NC}(W,c')}$$

is a ring isomorphism that maps $\mathfrak{m}H^\bullet_{T_{ad}}(\mathrm{CFl}_c)$ to $\mathfrak{m}H^\bullet_{T_{ad}}(\mathrm{CFl}_{c'})$. □

Thus Theorem 1.2 also describes $H^\bullet(\mathrm{CFl}_c)$ for any $c \in S_{n+1}$. However, the isomorphism does not map classes $[w^{-1}X^{wc}_w]$ to classes $[v^{-1}X^{vc'}_v]$, so the forest polynomials in [1, 2] are no longer a Schubert-type basis. In [3, §12], we find a family of "flow-up bases" for each $H^\bullet(\mathrm{CFl}_c)$ and $H^\bullet_{T_{ad}}(\mathrm{CFl}_c)$ with a Schubert-like property, but a combinatorial formula for these elements is unknown outside the quasisymmetic case.

Question 7.3. What are the forest polynomials for each Coxeter element $c \in S_{n+1}$? Is there an analogue of Theorem 1.2 and related results for other reductive groups?

A Examples in type A

The article [3] gives examples from all classical groups; here we consider $G = \mathrm{GL}_{n+1}(\mathbb{C})$. In this case $W = S_{n+1}$, the symmetric group generated by $s_1, \ldots, s_n$ with $s_i = (i, i+1)$. Further $\Phi = \{e_i - e_j \mid i \neq j\} \subseteq \mathbb{E} = \mathbb{R}^{n+1}$ and the weight lattice is $\mathbb{Z}^{n+1}$. The Lie algebra is $\mathfrak{g} = \mathrm{Mat}_{n+1}(\mathbb{C})$, with weight space $\mathfrak{g}_{e_i - e_j}$ being the i, j coordinate subspace. Choosing B to be all upper triangular matrices in G gives $\Phi^+ = \{e_i - e_j \mid i < j\}$. We can choose regular dominant weight $\lambda = (n, n-1, \ldots, 0) \in \mathbb{E}$, which has extremal Plücker functions

$$\mathrm{Pl}_w(M) = \prod_{i=1}^{n} \det\left(M_{w(i),j}\right)_{1\leq i,j\leq k}.$$

The $U_{r(\mathrm{Inv}(w))}$ chart of a Schubert cell $\mathring{X}^w$ is obtained from the permutation matrix of w by replacing each 0 that is not right or below a 1 with an $\ast$ (free entry). For example,

$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{123}, \quad \underbrace{\begin{bmatrix} \ast & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}}_{213}, \quad \underbrace{\begin{bmatrix} 1 & & \\ & \ast & 1 \\ & 1 & \end{bmatrix}}_{132}, \quad \underbrace{\begin{bmatrix} \ast & \ast & 1 \\ 1 & & \\ & 1 & \end{bmatrix}}_{231}, \quad \underbrace{\begin{bmatrix} \ast & 1 & \\ \ast & & 1 \\ 1 & & \end{bmatrix}}_{312}, \quad \text{and} \quad \underbrace{\begin{bmatrix} \ast & \ast & 1 \\ \ast & 1 & \\ 1 & & \end{bmatrix}}_{321}.$$

To illustrate the extremal Plücker functions, let a, b, and c to be the $(1,2)$, $(2,1)$, and $(1,1)$ entries of $\mathring{X}^{321}$. Then the extremal Plücker functions on this chart are

$$(\mathrm{Pl}_{123}, \mathrm{Pl}_{213}, \mathrm{Pl}_{132}, \mathrm{Pl}_{231}, \mathrm{Pl}_{312}, \mathrm{Pl}_{321}) = (c(c-ab), b(c-ab), -ac, -b, -a, -1).$$

We now give two example of Coxeter flag varieties. First, we describe the Coxeter flag variety for $c = s_1 s_2 = 231$, which is isomorphic to the "quasisymmetric" case, in which the closure of every $\mathring{X}^u_{\mathrm{NC}}$ chart has a cubical moment polytope:

$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{123}, \quad \underbrace{\begin{bmatrix} \ast & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}}_{213}, \quad \underbrace{\begin{bmatrix} 1 & & \\ & \ast & 1 \\ & 1 & \end{bmatrix}}_{132}, \quad \underbrace{\begin{bmatrix} \ast & \ast & 1 \\ 1 & & \\ & 1 & \end{bmatrix}}_{231}, \quad \text{and} \quad \underbrace{\begin{bmatrix} \ast & 0 & 1 \\ \ast & 1 & \\ 1 & & \end{bmatrix}}_{321}.$$

Taking $c = s_1 s_3 s_2 = 2413 \in S_4$ departs from the "quasisymmetric" case. The charts are

$$\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{1234}, \quad \underbrace{\begin{bmatrix} \ast & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{2134}, \quad \underbrace{\begin{bmatrix} 1 & & & \\ & \ast & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}}_{1324}, \quad \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ast & 1 \\ & & 1 & \end{bmatrix}}_{1243}, \quad \underbrace{\begin{bmatrix} 1 & & & \\ & \ast & 1 & \\ & \ast & & 1 \\ & 1 & & \end{bmatrix}}_{1423}, \quad \underbrace{\begin{bmatrix} \ast & 1 & & \\ 1 & & & \\ & & \ast & 1 \\ & & 1 & \end{bmatrix}}_{2143}, \quad \underbrace{\begin{bmatrix} \ast & \ast & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{bmatrix}}_{2314},$$

$$\underbrace{\begin{bmatrix} 1 & & & \\ & \ast & \ast & 1 \\ & 0 & 1 & \\ & 1 & & \end{bmatrix}}_{1432}, \quad \underbrace{\begin{bmatrix} \ast & 0 & 1 & \\ \ast & 1 & & \\ 1 & & & \\ & & & 1 \end{bmatrix}}_{3214}, \quad \underbrace{\begin{bmatrix} \ast & \ast & 1 & \\ 1 & & & \\ & \ast & & 1 \\ & 1 & & \end{bmatrix}}_{2413}, \quad \underbrace{\begin{bmatrix} \ast & \ast & \ast & 1 \\ 1 & & & \\ & & 1 & \\ & 1 & & \end{bmatrix}}_{2431}, \quad \underbrace{\begin{bmatrix} \ast & 0 & 1 & \\ \ast & 1 & & \\ \ast & & & 1 \\ 1 & & & \end{bmatrix}}_{4213}, \quad \underbrace{\begin{bmatrix} \ast & 0 & 1 & \\ \ast & \ast & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}}_{3412}, \quad \text{and} \quad \underbrace{\begin{bmatrix} \ast & 0 & \ast & 1 \\ \ast & 1 & & \\ 0 & & 1 & \\ 1 & & & \end{bmatrix}}_{4231}.$$

The moment polytopes for the top-dimensional chart closures are shown in Figure 1, and we observe that $\mathring{X}^{3412}_{\mathrm{NC}}$ corresponds to the (purple) tetragonal trapezohedron, with vertices corresponding to the permutations 1234, 1243, 1423, 1432, 2134, 2143, 2314, 2413, 3214, and 3412.

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References

[1] N. Bergeron, L. Gagnon, P. Nadeau, H. Spink, and V. Tewari. "Equivariant quasisymmetry and noncrossing partitions". 2025. arXiv:2504.15234. Link.

[2] N. Bergeron, L. Gagnon, P. Nadeau, H. Spink, and V. Tewari. "The quasisymmetric flag variety: a toric complex on noncrossing partitions". 2025. arXiv:2508.12171. Link.

[3] N. Bergeron, L. Gagnon, H. Spink, and V. Tewari. "The Coxeter Flag Variety". 2026. arXiv: 2601.23111. Link.

[4] D. Bessis. "The dual braid monoid". *Ann. Sci. École Norm. Sup. (4)* **36**.5 (2003), pp. 647–683.

[5] P. Biane and M. Josuat-Vergès. "Noncrossing partitions, Bruhat order and the cluster complex". *Ann. Inst. Fourier (Grenoble)* **69**.5 (2019), pp. 2241–2289. DOI.

[6] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*. Vol. 231. Graduate Texts in Mathematics. Springer, New York, 2005, pp. xiv+363.

[7] A. V. Borovik, I. M. Gelfand, and N. White. *Coxeter matroids*. Vol. 216. Progress in Mathematics. Birkhäuser, Boston, MA, 2003.

[8] C. Defant, M. Sherman-Bennett, and N Williams. "Permutahedron Triangulations via Total Linear Stability and the Dual Braid Group". 2025. arXiv:2509.11497. Link.

[9] I. M. Gelfand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova. "Combinatorial geometries, convex polyhedra, and Schubert cells". *Adv. in Math.* **63**.3 (1987), pp. 301–316.

[10] M. Goresky, R. Kottwitz, and R. MacPherson. "Equivariant cohomology, Koszul duality, and the localization theorem". *Invent. Math.* **131**.1 (1998), pp. 25–83. DOI.

[11] M. Harada, T. Horiguchi, M. Masuda, and S. Park. "The volume polynomial of regular semisimple Hessenberg varieties and the Gelfand-Zetlin polytope". *Proc. Steklov Inst. Math.* **305** (2019), pp. 318–344.

[12] A. Knutson, M. Sanchez, and M. Sherman-Bennett. "Permutahedral subdivisions and class formulas from Coxeter elements". *Sém. Lothar. Combin.* **93B** (2025), Art. 8, 12.

[13] T. Lam and A. Postnikov. "Polypositroids". *Forum Math. Sigma* **12** (2024), Paper e42, 67 pp.

[14] N. E. Mnëv. "The universality theorems on the classification problem of configuration varieties and convex polytopes varieties". *Topology and geometry—Rohlin Seminar*. Vol. 1346. Lecture Notes in Math. Springer, Berlin, 1988, pp. 527–543.

[15] P. Nadeau and V. Tewari. "Forest polynomials and the class of the permutahedral variety". *Adv. Math.* **453** (2024), Paper No. 109834, 33 pp. DOI.

[16] N. Reading. "Cambrian lattices". *Adv. Math.* **205**.2 (2006), pp. 313–353. DOI.

[17] N. Reading. "Clusters, Coxeter-sortable elements and noncrossing partitions". *Trans. Amer. Math. Soc.* **359**.12 (2007), pp. 5931–5958. DOI.

[18] N. Reading. "Sortable elements and Cambrian lattices". *Algebra Universalis* **56**.3-4 (2007), pp. 411–437. DOI.

[19] N. Reading and D. E. Speyer. "Sortable elements in infinite Coxeter groups". *Trans. Amer. Math. Soc.* **363**.2 (2011), pp. 699–761. DOI.

[20] T. A. Springer. *Linear algebraic groups*. Second. Vol. 9. Progress in Mathematics. Birkhäuser, Boston, MA, 1998.