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Billey–Postnikov posets

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Abstract. *Billey–Postnikov (BP) decompositions* govern when Schubert varieties $X(w)$ are bundles of smaller Schubert varieties. We further develop the theory of BP decompositions and show that, in finite type, they can be recognized by pattern conditions and are indexed by the order ideals of a poset $\mathsf{bp}(w)$ that we introduce and study. We then give three new applications of BP decompositions: we show that when W is of finite simply laced type and $X(w)$ is rationally smooth, Schubert structure constants have a triangularity property which gives a canonical bijection realizing rank-symmetry of $[e, w]$; we classify the finite types (other than E) for which rationally smooth Bruhat intervals admit generalized *Lehmer codes*, answering questions and conjectures of Billey–Fan–Losonczy, Bolognini–Sentinelli, and Bishop–Milićević–Thomas; and we show that rationally smooth Schubert varieties in infinite type need not have Grassmannian BP decompositions, disproving conjectures of Richmond–Slofstra and Oh–Richmond.

Keywords: BP decomposition, smooth Schubert variety, Lehmer code

1 Introduction

Let G be a complex reductive group (or, more generally, a Kac–Moody group) with Borel subgroup B. The B-orbit closures in the generalized flag variety G/B are called *Schubert varieties* $X(w)$ and are indexed by elements w of the Weyl group W. These varieties are important for many reasons. For one, the classes $\{[X(w)] \mid w \in W\}$ give a basis for $H^*(G/B)$, the study of whose structure constants c_{uv}^w has long defined the field of *Schubert calculus*. On the other hand, in geometric representation theory, *Kazhdan–Lusztig theory* [20] relates singularities of the $X(w)$, characters of Verma modules of $\mathfrak{g}$, and distinguished bases of the Hecke algebra $\mathcal{H}(W; q)$.

The parabolic subgroups $B \subset P_J$ are indexed by subsets J of the simple reflections S generating W. Under the natural projection $\pi^J : G/B \twoheadrightarrow G/P_J$ it is known that $X(w)$ maps onto a B-orbit closure $X^J(w^J) := \overline{Bw^J P_J/P_J} \subset G/P_J$; the $X^J(w^J)$ are also known as (parabolic) Schubert varieties. The fibers of $\pi^J|_{X(w)}$ in general vary in dimension. However, if they are of constant dimension, then all fibers are isomorphic to

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$X(w_J)$, so that $X(w)$ is the total space of a fiber bundle whose base and fiber are both smaller Schubert varieties; in this case the decomposition $w = w^J w_J$ is called a *Billey–Postnikov (BP) decomposition*. Aside from the geometric applications discussed below, BP decompositions have also found applications to enumeration, Bruhat combinatorics, and Kazhdan–Lusztig polynomials [14, 15, 16, 25]; see Oh–Richmond for a survey [22].

For $G = \mathrm{SL}_n$, the varieties G/P_J and G/B are naturally identified with the classical (partial) flag varieties. In this setting, Lakshmibai–Sandhya [21] famously characterized the smooth $X(w)$ as those for which w avoids the *permutation patterns* 3412 and 4231; the study of permutation patterns has since developed into an active subfield of combinatorics in its own right. This result was generalized to all finite types by Billey and Postnikov [3, 4], who (implicitly) introduced BP decompositions. Richmond and Slofstra [24] developed the theory of BP decompositions and used them to show that smooth Schubert varieties in finite type are iterated bundles of smooth Schubert varieties in G/P for P a *maximal* parabolic subgroup, generalizing a result of Ryan and Wolper [26, 28]. BP decompositions for P maximal are called *Grassmannian* BP decompositions, since in this case G/P is a Grassmannian when $G = \mathrm{SL}_n$.

The correct level of generality for these results is when $X(w)$ is *rationally smooth*, which can be defined in terms of the vanishing of certain local intersection cohomology groups. For our purposes, we take the equivalent condition, that the Kazhdan–Lusztig polynomial $P_{e,w}(q)$ is equal to 1, as the definition. This has the added benefit of allowing us to define rational smoothness for an element w in an arbitrary (not-necessarily-crystallographic) Coxeter group. In finite simply-laced type, it is a result of Carrell–Peterson and Carrell–Kuttler [9, 10] that smoothness and rational smoothness are equivalent. For any G, it is also known [7] that rational smoothness is equivalent to the palindromicity of the Poincaré polynomial of $X(w)$.

We now describe our main results. The first of these identifies surprising new structures in the BP decompositions of any (not-necessarily rationally smooth) w. We show, in finite type, how to detect that J induces a BP decomposition of $X(w)$ using *J-star patterns* and that the set of such J is closed under union and intersection, so that these decompositions are indexed by the order ideals of the *Billey–Postnikov poset* $\mathsf{bp}(w)$ that we construct. We then return to the setting of rationally smooth w and prove several kinds of results about Schubert structure constants c_{uv}^w, about *generalized Lehmer codes*, and about the failure of the Richmond–Slofstra result in infinite type.

1.1 Patterns for BP decompositions and BP posets

We first characterize BP decompositions by the avoidance of what we call *J-star patterns*; see Section 3.1 for precise definitions. This generalizes a result of Alland–Richmond [1] which applies for $W = S_n$ when J is maximal. We write $\mathsf{BP}(w)$ for the set of $J \subset S$ inducing a BP decomposition; see Definition 2.1 for a Coxeter-theoretic criterion.

Theorem 1.1. *Let W be a finite Weyl group and $w \in W$. Then $J \in \mathsf{BP}(w)$ if and only if w avoids all J-star patterns.*

Theorem 1.2. *Let W be a finite Weyl group and $w \in W$. If $J, K \in \mathsf{BP}(w)$, then $J \cap K, J \cup K \in \mathsf{BP}(w)$. In particular, $\mathsf{BP}(w)$ is a distributive lattice.*

Theorem 1.2 implies that there is a set partition $\mathcal{S}_w$ of S and a partial order on its blocks, which we call the *reduced BP poset* $\mathsf{bp}(w)$, such that the elements of $\mathsf{BP}(w)$ are precisely the unions of the blocks lying in the order ideals of $\mathsf{bp}(w)$.

We can use Theorem 1.1 to characterize when w admits a chain of *maximal* BP decompositions and therefore when $X(w)$ is an iterated bundle of Grassmannian Schubert varieties. This recovers a classification due to Alland and Richmond [1] and includes the well-studied case of smooth Schubert varieties. When w admits such a chain, all blocks of $\mathcal{S}_w$ are singletons, so $\mathsf{bp}(w)$ is a partial order on S. The poset $\mathsf{bp}(w)$ also generalizes aspects of the *staircase diagram* construction [25] which applies in the smooth case. Theorem 1.1 is used to prove Theorem 1.2. It is not clear how one could prove Theorem 1.2 directly from the geometric or Coxeter-theoretic characterizations of BP decompositions. We conjecture that Theorem 1.2 in fact holds in any Coxeter group.

Conjecture 1.3. *Let w be in a Coxeter group W. If $J, K \in \mathsf{BP}(w)$, then $J \cap K, J \cup K \in \mathsf{BP}(w)$.*

Theorem 1.4. *Conjecture 1.3 holds for Coxeter groups of rank three.*

1.2 Poincaré duality in smooth Schubert varieties

The Schubert variety $X(w)$ is paved by open affine Schubert cells $\Omega_v := BvB/B$ for $v \leq w$ in *Bruhat order* on W. The geometric context of the BP poset is seen as follows:

Proposition 1.5. *Let W be a finite Weyl group and let $w \in W$ be a rationally smooth element. Any linear extension of the BP poset $\mathsf{bp}(w)$ determines a decomposition of $X(w)$ as an iterated bundle of rationally smooth Grassmannian Schubert varieties.*

A handy application of the BP poset is that we can determine many Schubert structure constants under a rationally smooth element. Write $[e, w]_k := \{u \leq w \mid \ell(u) = k\}$.

Theorem 1.6. *Let W be a finite Weyl group of simply-laced type, and let $w \in W$ be a (rationally) smooth element. For any $0 \leq k \leq \ell(w)$, we can linearly order $[e, w]_k$ and $[e, w]_{\ell(w)-k}$ so that the matrix $(c^w_{u,v})$, whose rows are indexed by $u \in [e, w]_k$ and columns are indexed by $v \in [e, w]_{\ell(w)-k}$, is upper triangular with 1's on the diagonal.*

A well-known characterization of rational smoothness for $w \in W$ is that the Bruhat interval $[e, w]$ is rank-symmetric. It is therefore natural to seek, for each rationally smooth $w \in W$ and $0 \leq k \leq \ell(w)$, a bijection between $[e, w]_k$ and $[e, w]_{\ell(w)-k}$. In type A for example, such bijections are not hard to construct artificially via the factorization of the rank generating function of $[e, w]$ ([19]). Thus, an important corollary to Theorem 1.6 is that in finite simply-laced types, there is a *canonical* such bijection:

Corollary 1.7. *Let W be a finite Weyl group of simply-laced type, and let $w \in W$ be a (rationally) smooth element. There is a* unique *bijection $\phi : [e, w]_k \to [e, w]_{\ell(w)-k}$ such that $c^w_{u,\phi(u)} \neq 0$ for all $u \in [e, w]_k$.*

1.3 Generalized Lehmer codes

As a result of the affine paving by Schubert cells, we can express the Poincaré polynomial $\mathcal{P}(w) := \sum_{i=0}^{\ell(w)} \dim(H^{2i}(X(w)))q^i$ of $X(w)$ as $\mathcal{P}(w) = \sum_{v \leq w} q^{\ell(v)}$; we can take this as the definition of $\mathcal{P}(w)$ for W non-crystallographic. It is a classical fact that for W a finite Coxeter group with rank r and longest element w_0 we have $\mathcal{P}(w_0) = \prod_{i=1}^{r} [d_i]_q$, where the $d_1, \ldots, d_r \in \mathbb{N}$ are the *degrees* of W and $[d]_q$ denotes a q-integer. For $W = S_n$, this formula can be realized explicitly via the *Lehmer code*, which can be interpreted as giving an order preserving bijection from a product $C_{d_1} \times \cdots \times C_{d_r}$ of chains of cardinalities $d_1, \ldots, d_r$ to Bruhat order on W. Gasharov showed [19] that, far less obviously, for $W = S_n$, the Bruhat interval $[e, w]$ below any rationally smooth element w also admits an order preserving bijection from a product of chains (which we will henceforward just call a Lehmer code). Billey–Fan–Losonczy conjectured that Gasharov's result could be extended to all finite Weyl groups.

Conjecture 1.8 (Billey–Fan–Losonczy [5])**.** *Let W be a finite Weyl group and $w \in W$ rationally smooth, then $[e, w]$ admits a Lehmer code.*

Conjecture 1.8 was resolved in the affirmative by Billey in types A and B [4]. In [6] Bishop–Milićević–Thomas report on some computations showing several cases, namely F_4, H_4, and E_6, in which $[e, w_0]$ does *not* have a Lehmer code (there under the name *cubulation*); the authors also ask to what extent these observations can be extended to rationally smooth intervals. Recent results of Bolognini–Sentinelli and Sentinelli–Zatti [8, 27] construct a Lehmer code for $[e, w_0]$ in D_n and show one does not exist for F_4; those authors likewise wonder about the general case.

In Theorem 1.9 we resolve Conjecture 1.8 in all types except E, and indeed also for the finite non-crystallographic groups. The finite irreducible W fall into two classes from this perspective. We resolve the conjecture in the affirmative for the infinite families as well as for H_3, and for the exceptional groups of types H_4 and F_4, we show that Lehmer codes exist for rationally smooth w *except* w_0. We use BP decompositions to construct many of these Lehmer codes and to construct the examples in part (3).

Theorem 1.9. *Let W be an irreducible finite Coxeter group not of type E and suppose w is rationally smooth.*

(1) *If W is of type $A_n, B_n, D_n, I_2(n)$, or H_3 then w admits a Lehmer code;*

(2) *If W is of type H_4 or F_4, then w admits a Lehmer code if and only if $w \neq w_0$.*

1.4 Grassmannian BP decompositions in infinite type

In light of the utility of iterated BP decompositions to the understanding of rationally smooth Schubert varieties in finite type, it is natural to hope that it could be extended to infinite type. Whether this is possible was posed as a question by Oh and Richmond [22, Q. 8.1]; an affirmative answer was conjectured by Richmond and Slofstra [24, Conj. 6.5].

Conjecture 1.10 (Richmond–Slofstra [24]; Oh–Richmond [22])**.** *Let W be any Coxeter group and suppose that $w \in W$ is rationally smooth. Then w has a Grassmannian BP decomposition.*

Conjecture 1.10 is known to hold when W is finite [24, 26, 28], of affine type $\widetilde{A}_n$ (see [2] and the discussion in [24, § 6.1]), or in a certain large family of groups of indefinite type [23]. In Theorem 1.11 we give the first known counterexample. The conjecture clearly holds for any element w in any Coxeter group such that $|\mathrm{Supp}(w)| = 2$, since $\{s\} \in \mathsf{BP}(w)$ for any $s \in D_R(w)$. We should thus consider groups W of rank three. Many of these are either finite, or of type $\widetilde{A}_2$, or among the groups from [23]. One of the "smallest" remaining sources of a counterexample is the affine Weyl group of type $\widetilde{C}_2$. We reasoned using BP decompositions to find the following counterexample.

Theorem 1.11. *Conjecture 1.10 fails for $w = srstrsr$ in the affine Weyl group of type $\widetilde{C}_2$.*

1.5 Outline

In Section 2 we recall some background on Coxeter groups, root systems, and rational smoothness. In Section 3 we give some exposition on BP posets and Theorems 1.1 and 1.2. Section 4 gives some examples of our Schubert calculus results from Section 1.2. We lack the space to give further exposition on the results from Sections 1.3 and 1.4, but proofs of all of our results can be found in [17].

2 Preliminaries

2.1 Coxeter groups and BP decompositions

Let W be a Coxeter group with simple reflection set S. We write ℓ for the length function and $\leq$ for the Bruhat order on W. For $J \subset S$, we write W_J for the parabolic subgroup generated by J, and W^J for the set of minimal length representatives of the cosets W/W_J. The subgroup W_J is a Coxeter group in its own right, with simple reflection set J. Each

$w \in W$ has a unique parabolic decomposition $w = w^J w_J$ with $w^J \in W^J$ and $w_J \in W_J$, and these elements satisfy $\ell(w) = \ell(w^J) + \ell(w_J)$. The *(left) descent set* of $w \in W$ is $D_L(w) := \{s \in S \mid \ell(sw) < \ell(w)\}$ and the *support* $\mathrm{Supp}(w)$ of w is $\{s \in S \mid s \leq w\}$.

Definition 2.1 (See [3, 24])**.** A parabolic decomposition $w = w^J w_J$ is called a *Billey–Postnikov* decomposition, or *BP* decomposition, if $\mathrm{Supp}(w^J) \cap J \subset D_L(w_J)$. In this case, we also say that w is *BP at* J or w is J*-BP*.

Previous investigations of BP decompositions have often been satisfied to prove the *existence* of a suitable BP decomposition for *certain* (e.g. *rationally smooth*) elements w. A perspective of this paper is that it is valuable to consider the set of *all* BP decompositions of an arbitrary element $w \in W$. To this end, for any $w \in W$, we define

$$\mathsf{BP}(w) := \{J \subset S \mid w = w^J w_J \text{ is a Billey–Postnikov decomposition}\}.$$

We write $[v, w]$ for a closed Bruhat interval. If $v, w \in W^J$, we write $[v, w]^J$ for $[v, w] \cap W^J$. We write $\mathcal{P}^J(w)$ for the length generating function of the Bruhat interval below $w \in W^J$: $\mathcal{P}^J(w) := \sum_{v \in [e,w]^J} q^{\ell(v)}$. We write $\mathcal{P}(w)$ for $\mathcal{P}^{\emptyset}(w)$. We have:

Proposition 2.2 ([24])**.** *Let W be any Coxeter group, let $w \in W$, and let $J \subset S$. Then $J \in \mathsf{BP}(w)$ if and only if $\mathcal{P}(w) = \mathcal{P}^J(w^J)\mathcal{P}(w_J)$.*

2.2 Schubert varieties and rational smoothness

For $v, w \in W$, and $J \subset S$ we write $P^J_{v,w}(q)$ for the corresponding *parabolic Kazhdan–Lusztig polynomial* [12, 20], a polynomial in $\mathbb{N}[q]$ [13].

Theorem 2.3 (See [11, 20])**.** *Let W be any Coxeter group, $J \subset S$, and $w \in W^J$. Then TFAE:*

(RS1) $P^J_{e,w}(q) = 1$ *and*

(RS2) the polynomial $\mathcal{P}^J(w)$ is palindromic: $q^{\ell(w)}\mathcal{P}^J(w)(q^{-1}) = \mathcal{P}^J(w)(q)$.

We say an element w satisfying these equivalent conditions is J*-rationally smooth* (or just *rationally smooth* if $J = \emptyset$).The reason for this terminology is the following. Suppose W is crystallographic (but not necessarily finite) and let G be the corresponding Kac–Moody group. Let $T \subset B$ be a maximal torus inside a Borel subgroup of G. For $w \in W^J$, the *Schubert variety* is $X^J(w) := \overline{BwP_J}/P_J$, a finite-dimensional subvariety of the generalized flag variety G/B (which may be an infinite-dimensional ind-variety). In this case, there is a third equivalent condition for rational smoothness:

(RS3) the local intersection cohomology $IH^i(X^J(w))_v$ at the T-fixed point vP_J vanishes for all $W^J \ni v \leq w$ and $i > 0$.

Finally, if W is of finite simply-laced type, then by a result of [9] we also have:

(RS4) $X^J(w)$ is a smooth variety.

2.3 Root system conventions

For much of the paper we work in the case W is a finite Weyl group. Let Φ be a crystallographic root system of rank r, with a choice of positive roots Φ^+ and the corresponding simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$. The *root poset* on Φ^+ is defined so that $\beta \leq \gamma$ if $\gamma - \beta$ can be written as a nonnegative linear (necessarily integral) combination of Δ. For $\beta \geq \alpha$ where $\alpha \in \Delta$ is a simple root, we will also say that β is *supported on* α. Let $W = W(\Phi)$ be the Weyl group of Φ. For $w \in W$, its *(right) inversion set* is $I(w) = \{\beta \in \Phi^+ \mid w\beta \in \Phi \setminus \Phi^+\}$. We slightly abuse notation to also consider $J \subset S$ as a subset of Δ while discussing BP decompositions. For $J \subset S$ we write $\Delta_J \subset \Delta$ for the corresponding simple roots and let $\Phi_J^+ \subset \Phi^+$ be the set of positive roots that can be written as a linear combination of simple roots in Δ_J. For any $w \in W$, we have $I(w^J) \cap \Phi_J^+ = \emptyset$.

3 The Billey–Postnikov poset

3.1 J-star patterns and distributivity of BP decompositions

Definition 3.1. We say $(\beta, c_1\gamma_1, \ldots, c_k\gamma_k)$ form a *J-star* if $\beta \in \Phi_J^+$, $\gamma_1, \ldots, \gamma_k \in \Phi^+ \setminus \Phi_J^+$, and $c_1, \ldots, c_k \in \mathbb{N}$ are such that $\beta + \sum_{i \in I} c_i\gamma_i \in \Phi^+$ for all subsets $I \subset [k]$. We say that $w \in W$ *contains* a J-star $(\beta, c_1\gamma_1, \ldots, c_k\gamma_k)$ if $\beta \notin I(w)$ and $\beta + \sum_{i=1}^k c_i\gamma_i \in I(w)$.

In a finite crystallographic root system, the multi-set $\{c_1, \ldots, c_k\}$ can only be $\{1\}$, $\{2\}$, $\{3\}$, $\{1,1\}$, $\{1,2\}$ or $\{1,1,1\}$. We have the following technical lemma.

Lemma 3.2. *Fix $\alpha \in \Delta_J \subset \Delta$. Suppose $\tau \in \Phi^+ \setminus \Phi_J^+$ is supported on α. Then either*

1. *there exists $\alpha_j \in \Delta_J$ such that $\tau - \alpha_j \in \Phi^+$ is also supported on α; or*
2. *there exists a J-star $(\alpha, c_1\gamma_1, \ldots, c_k\gamma_k)$ such that $\alpha + \sum_{i=1}^k c_i\gamma_i = \tau$.*

The lemma requires some work, but, given the lemma, we can prove Theorem 1.1:

Proof of Theorem 1.1. First, assume that w is not BP at J. We will then show that w contains a certain J-star. By definition, there exists $\alpha \in \mathrm{Supp}(w^J) \cap J$ such that $\alpha \notin D_L(w_J)$. Since $\alpha \in \mathrm{Supp}(w^J)$, there exists $\tau \in I(w^J)$ supported on α (by e.g. [18, Lemma 3.2]). Find the minimal such τ in the root poset. Recall that w^J does not have any inversions in Φ_J^+. If there exists $\alpha_j \in \Delta_J$ such that $\tau - \alpha_j \in \Phi^+$ is supported on α, then as $\alpha_j \notin I(w^J)$, we must have $\tau - \alpha_j \in I(w^J)$, contradicting the minimality of τ. Thus by Lemma 3.2, there exists a J-star $(\alpha, c_1\gamma_1, \ldots, c_k\gamma_k)$ such that $\alpha + \sum_{i=1}^k c_k\gamma_k = \tau$. We claim that w contains the J-star $(w_J^{-1}\alpha, c_1w_J^{-1}\gamma_1, \ldots, c_kw_J^{-1}\gamma_k)$.

To see that this is indeed a J-star, $\alpha \notin D_L(w_J)$ means precisely that $w_J^{-1}\alpha \in \Phi^+$ and further in Φ_J^+ as $w_J^{-1} \in W(J)$ and $\alpha \in \Delta_J$. Each γ_i is supported on some $\alpha_i \notin J$ so

$u\gamma_i \in \Phi^+ \setminus \Phi_J^+$ for any $u \in W(J)$. Moreover, $w_J^{-1}\alpha + \sum_{i\in I} c_i w_J^{-1}\gamma_i = w_J^{-1}(\alpha + \sum_{i\in I} c_i\gamma_i) \in w_J^{-1}(\Phi^+ \setminus \Phi_J^+) \subset \Phi^+$. Finally, $w(w_J^{-1}\alpha) = w^J\alpha \in \Phi^+$ and $w(w_J^{-1}\alpha + \sum_{i=1}^k c_i w_J^{-1}\gamma_i) = w^J\tau \in \Phi^-$, which means that w contains this J-star.

For the other direction, assume that w contains a J-star at $(\beta, c_1\gamma_1, \ldots, c_k\gamma_k)$. By definition, $\beta \in \Phi_J^+$ and $w\beta \in \Phi^+$. We must have $w_J\beta \in \Phi_J^+$ since otherwise, $w^J(-w_J\beta) \in \Phi^-$, contradicting w^J has no inversions in Φ_J^+. As each $\gamma_i \in \Phi^+ \setminus \Phi_J^+$, $w_J\gamma_i \in \Phi^+$ as well. Write $\tau = w_J(\beta + \sum_{i=1}^k c_i\gamma_i) \in \Phi^+ \setminus \Phi_J^+$ and write $w_J\beta = \sum d_j\alpha_j$ as a combination of roots in Δ_J. Since $w_J^{-1}(w_J\beta) > 0$, there is some $\alpha_j \leq \beta$ such that $w_J^{-1}\alpha_j \in \Phi^+$. Moreover, $\alpha_j \leq w_J\beta < \tau \in I(w^J)$ so $\alpha_j \in \mathrm{Supp}(w^J)$. So w is not J-BP at J. $\square$

Theorem 1.1 in hand, it is possible to prove Theorem 1.2 by tracking J-star patterns.

3.2 Properties of the BP poset

For a poset P, let $\mathcal{L}(P)$ denote the lattice of its order ideals. Consider $\mathsf{BP}(w)$ ordered by inclusion. It is a distributive lattice, with the meet and join operations given by taking the intersection and union (Theorem 1.2). By the fundamental theorem of finite distributive lattices, $\mathsf{BP}(w)$ is isomorphic to the lattice of order ideals of some poset.

Definition 3.3. For $w \in W$, its *Billey–Postnikov (BP) preposet* $\widetilde{\mathsf{bp}}(w)$ is defined on $[n-1]$ such that $\mathsf{BP}(w) = \mathcal{L}(\widetilde{\mathsf{bp}}(w))$. Denote its relations as $\leq_w$. The *Billey–Postnikov poset* $\mathsf{bp}(w)$ is obtained from $\widetilde{\mathsf{bp}}(w)$ via the equivalence relation $a \sim b$ if $a \leq b$ and $b \leq a$. Elements of $\mathsf{bp}(w)$ are given by subsets of S.

Example 3.4. For $w = 4231$ and $w' = 3412$ we have $\mathsf{BP}(w) = \{\emptyset, \{1\}, \{3\}, \{1,3\}, [3]\}$ and $\mathsf{BP}(w') = \{\emptyset, \{2\}, [3]\}$. The posets $\mathsf{bp}(w)$ and $\mathsf{bp}(w')$ are shown in Figure 1. Notice that we have $1 = 3$ in the BP preposet $\widetilde{\mathsf{bp}}(w')$, so $\mathsf{bp}(w')$ contains a non-singleton.

Figure 1: The BP posets of 4231 (left) and 3412 (right).

We now introduce *closure operators*, which can be used to efficiently compute $\mathsf{bp}(w)$.

Definition 3.5. For $w \in W$, the *closure* of $A \subset S$ is $\mathrm{cl}_w(A) := \bigcap\{J \in \mathsf{BP}(w) \mid J \supseteq A\}$. For simplicity, denote the closure of a singleton $\mathrm{cl}_w(\{i\})$ as $\mathrm{cl}_w(i)$.

Proposition 3.6. *Let $w \in W$.*

1. *For $A \subset S$, $A \subset \mathrm{cl}_w(A)$. For $A \subset B$, $\mathrm{cl}_w(A) \subset \mathrm{cl}_w(B)$.*

2. *For* $A \subset S$, $\mathrm{cl}_w(\mathrm{cl}_w(A)) = \mathrm{cl}_w(A)$.

3. *In* $\widetilde{\mathsf{bp}}(w)$, $i \leq_w j$ *if and only if* $i \in \mathrm{cl}_w(j)$.

The following result relates $\mathsf{bp}(w)$ and $\mathsf{bp}(w_J)$. This guarantees the BP poset behaves well with respect to iterated BP decompositions.

Proposition 3.7. *Let* $w \in W$ *and* $J \in \mathsf{BP}(w)$. *Then* $\mathsf{bp}(w_J)$ *is the order ideal in* $\mathsf{bp}(w)$ *consisting of blocks contained in* J.

3.3 Specialization to type A

In type A, the J-stars are of the form $(e_i - e_j, e_j - e_k)$, $(e_j - e_k, e_i - e_j)$ or $(e_j - e_k, e_i - e_j, e_k - e_l)$ with $i < j < k < l$. Thus we can characterize BP decompositions more explicitly.

Definition 3.8. We say that $(w, [a, b])$, where $w \in \mathfrak{S}_n$ and $1 \leq a < b \leq n$, *contains the pattern* $(u, [c, d])$, where $u \in \mathfrak{S}_k$, $1 \leq c < d \leq k$, if there exists indices $1 \leq i_1 < \cdots < i_k \leq n$ such that $w(i_x) < w(i_y)$ if and only if $u(x) < u(y)$ for all $1 \leq x < y \leq k$, and $i_x \in [a, b]$ if and only if $x \in [c, d]$. We say that $(w, [a, b])$ *avoids* $(u, [c, d])$ if $(w, [a, b])$ does not contain $(u, [c, d])$. We also write $(u, [c, d])$ in one-line notation as $u_1 \cdots u_{c-1}\underline{u_c \cdots u_d}u_{d+1} \cdots u_k$.

Corollary 3.9. *Let* $J = \{a, \ldots, b-1\} \subset S$ *be connected and* $w \in \mathfrak{S}_n$. *Then* w *is BP at* J *if and only if* $(w, [a, b])$ *avoids* $\underline{23}1$, $3\underline{12}$ *and* $3\underline{14}2$.

Remark. We can deduce [1, Thm. 1.1], which states that w is BP at $[n-1] \setminus \{k\}$ if and only if w avoids $23|1$ and $3|12$, where the values to the left of the bar "|" appear in the first k indices of w while the values to the right appear in the last $n-k$ indices. In our setup, w is BP at $[n-1] \setminus \{k\}$ if and only if w is BP at both $\{1, \ldots, k-1\}$ and $\{k+1, \ldots, n-1\}$. The rest is straightforward by Corollary 3.9. When a connected set J contains the leaf 1 or $n-1$, we do not need to worry about $3\underline{14}2$.

4 Poincaré dual Schubert classes

We first note that smoothness behaves well with respect to iterated bundles.

Proposition 4.1. *Suppose* $w \in W$ *is rationally smooth. If* $J \subset S$, *then* w_J *is rationally smooth. Furthermore, if* $J \in \mathsf{BP}(w)$ *then* w^J *is* J*-rationally smooth.*

Such iterated bundles are used to prove Theorem 1.6. We give a brief illustration of this theorem and Corollary 1.7.

Example 4.2. Consider the smooth permutation $w = 65178432$ with its BP poset shown in Figure 2. Choose a linear extension $a = (3, 1, 4, 6, 2, 7, 5)$, and write $J^{(i)} = J^{(i-1)} \setminus \{a_i\}$ with $J^{(0)} = S$ and $J^{(7)} = \emptyset$. We compute each $w^{(i)} = w_{J^{(i-1)}}^{J^{(i)}}$ to obtain $w^{(1)} = 15623478$, $w^{(2)} = 31245678$, $w^{(3)} = 12374568$, $w^{(4)} = 12347856$, $w^{(5)} = 13245678$, $w^{(6)} = 12345687$, $w^{(7)} = 12346578$ so that $w = w^{(1)} w^{(2)} \cdots w^{(7)}$. Visually, their corresponding rectangular Young's diagrams are shown in Figure 2. Here, for simplicity, at each node we draw the partition with respect to the connected component containing a_i inside $J^{(i-1)}$.

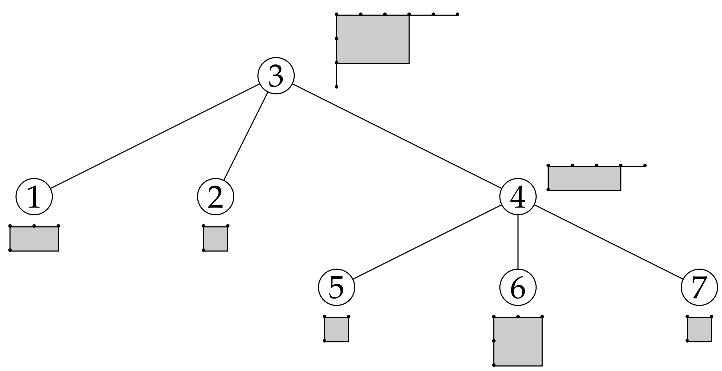

Figure 2: The BP poset of a smooth $w = 65178432$ in type A_7. In the smooth case, BP posets are closely related to the staircase diagrams of [25].

The bijection $\phi : [e, w]_k \to [e, w]_{\ell(w)-k}$ in Corollary 1.7 can now be constructed as follows. Given $u \in [e, w]_k$, write down $u^{(i)} = u_{J^{(i-1)}}^{J^{(i)}}$, which corresponds to a partition inside the rectangle $w^{(i)}$. Let $v^{(i)} = w_0(\mathrm{Supp}(w^{(i)})) \cdot u \cdot w_0(\mathrm{Supp}(w^{(i)}) \setminus \{a_i\})$ be the dual shape. Then $\phi(u) = v^{(1)} v^{(2)} \cdots v^{(7)}$. Figure 3 gives an example of this duality with

$$u = 13624578 \cdot 21345678 \cdot 12364578 \cdot 12345768 \cdot s_2 \cdot s_7 \cdot e = 36152784,$$
$$v = 12534678 \cdot 21345678 \cdot 12354678 \cdot 12346857 \cdot e \cdot e \cdot s_5 = 21548637.$$

Remark. The idea of constructing a *canonical* bijection via the matrix of Schubert structure constants does not generalize to all finite Weyl groups. For $w = s_2s_3s_1s_2s_1s_3s_2s_4s_3$ in type F_4 indexing a smooth Schubert variety in the Grassmannian G/P_J where $J = \{1, 2, 4\}$, has matrix of structure constants c_{uv}^w for ranks 4 and 5 shown below:

$u \backslash v$	$s_2s_3s_1s_2s_3$	$s_3s_4s_1s_2s_3$	$s_2s_3s_4s_2s_3$
$s_4s_1s_2s_3$	1	1	1
$s_3s_1s_2s_3$	1	1	0
$s_3s_4s_2s_3$	0	1	1

.

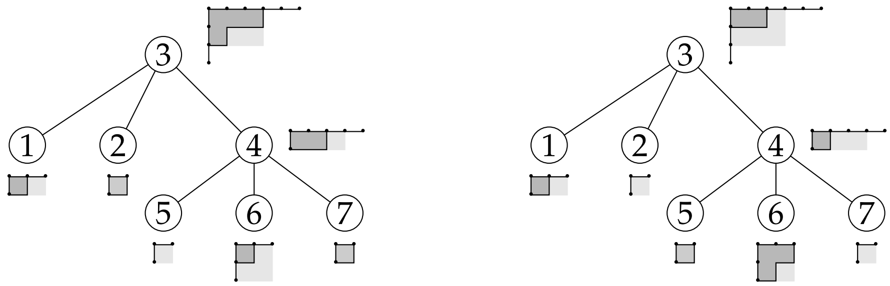

Figure 3: The duality map in $[e, w]$ for w in Figure 2.

Note that there are multiple bijections $\phi : [e, w]^J_4 \to [e, w]^J_5$ such that all $c^w_{u,\phi(u)}$ are nonzero, and all such ϕ's can be extended to G/B, under the group element $w \cdot w_0(J)$.

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