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Parity patterns meet Genocchi numbers: four labelings and three bijections

Qi Fang[1], Shishuo Fu*[1], Haijun Li[1], and Quan Yuan[1]

[1]*College of Mathematics and Statistics, Chongqing University, Chongqing 401331, P.R. China*

Abstract. Hetyei introduced in 2019 the homogenized Linial arrangement and showed that its regions are counted by the median Genocchi numbers. In the course of devising a different proof of Hetyei's result, Lazar and Wachs considered another hyperplane arrangement that is associated with certain bipartite graph called Ferrers graph. We bijectively label the regions of this latter arrangement with permutations whose ascents are subject to a parity restriction. This labeling not only establishes the equivalence between two enumerative results due to Hetyei and Lazar–Wachs, respectively, but also motivates us to derive and investigate a Seidel-like triangle that interweaves Genocchi numbers of both kinds.

Applying similar ideas, we introduce three more variants of permutations with analogous parity restrictions. We provide labelings for regions of the aforementioned arrangement using these three sets of restricted permutations as well. Furthermore, bijections from our first permutation model to two previously known permutation models are established.

Keywords: Median Genocchi numbers, Genocchi numbers, Permutation patterns, Hyperplane arrangements, Seidel triangle, Bijective proof, Equidistribution.

1 Introduction

The (signless) *Genocchi numbers* $\{g_n\}_{n\geq 1} = \{1, 1, 3, 17, 155, \ldots\}$ [23, A110501] and their close relative the (signless) *median Genocchi numbers* $\{h_n\}_{n\geq 0} = \{1, 2, 8, 56, 608, \ldots\}$ [23, A005439] are two sequences that bear number theoretical, combinatorial, and most recently geometrical interest. Due to their direct relation with the even-indexed Bernoulli numbers B_{2n}, namely $g_n = 2(1-2^{2n})(-1)^n B_{2n}$, the Genocchi numbers have a neat exponential generating function given by [9]

$$\sum_{n\geq 1} g_n \frac{x^{2n}}{(2n)!} = x \tan \frac{x}{2}.$$

*fsshuo@cqu.edu.cn. Shishuo Fu was partially supported by the National Natural Science Foundation of China grant 12171059.

The two kinds of Genocchi numbers are related via the following identity that holds for all $n \in \mathbb{N}$.

$$h_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n+1}{2j+1} g_{n+1-j}. \tag{1.1}$$

Based on Gandhi's generation of Genocchi numbers, Dumont [9] discovered in 1974 the first ever combinatorial interpretations of Genocchi numbers in terms of certain permutations subject to parity conditions on their descents and ascents (see Definition 2). Since then, a wealth of combinatorial models whose enumerations yield either Genocchi numbers or median Genocchi numbers were found; see for example [5, 4, 14, 17, 15, 13, 8] and the references contained in the aforementioned two OEIS entries.

A quite recent development stems from geometric considerations. Firstly, Hetyei [14] showed in 2019, applying Athanasiadis's [1] finite field method of counting regions created by hyperplane arrangements, that certain tournaments are enumerated by the median Genocchi numbers (see Theorem 5). In a follow up paper, Lazar and Wachs [17] made direct use of Zaslavsky's formula to relate the intersection lattice associated with a hyperplane arrangement to the number of its regions. Consequently, they were able to reprove and refine Hetyei's result. As an interesting byproduct of their work, Lazar and Wachs [17] introduced a new witness of median Genocchi numbers. In order to state their results, we recall some definitions. We write a permutation σ of $[n] := \{1, 2, \ldots, n\}$ using the one-line notation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ where $\sigma_i = \sigma(i)$ for $1 \leq i \leq n$. Following [17], we call the pair (i, σ_i) a *drop* whenever $i > \sigma_i$. Note that it goes by other names such as "anti-excedance" in the literature; see for instance [24]. In particular, an *even-odd drop* refers to a drop (i, σ_i) with i being even and σ_i being odd. Let $\mathfrak{S}_n$ denote the set of permutations of $[n]$. Lazar and Wachs derived the following new interpretation for the median Genocchi numbers.

Theorem 1 ([17, Coro. 6.2])**.** *For all $n \geq 1$, we have $h_n = |\{\sigma \in \mathfrak{S}_{2n} : \sigma$ has only even-odd drops$\}|$.*

They continued to make the following conjecture.

Conjecture 1 ([17, Conj. 6.4])**.** *For all $n \geq 1$, the n-th Genocchi number g_n is equal to the number of cycles on $[2n]$ with only even-odd drops.*

Conjecture 1 aroused a lot of interest. Within two years from the time when the arXiv version of [17] was released, there are at least three independent proofs of Conjecture 1 that we are aware of; see Lin and Yan [18], Pan and Zeng [20], and Chern [6].

Let $\mathcal{H}_{2n-1}$ denote the homogenized Linial arrangment as introduced by Hetyei [14] (see section 2 for the precise definition). One crucial step in Lazar-Wachs's derivation was to characterize the intersection lattice associated with $\mathcal{H}_{2n-1}$ as a bond lattice associated with certain bipartite graph, then calculate its Möbius function by counting non-broken-circuit (NBC) sets. Certain graphical arrangement $\mathcal{K}_{2n}$ was introduced as an intermediate object in [17, Thm. 3.2]. We are able to construct a direct bijection between the regions of $\mathcal{K}_{2n}$ and a set of restricted permutations in $\mathfrak{S}_{2n}$ that we denote as $\mathfrak{G}^{\mathrm{I}}_{2n}$. Its definition given below involves the notion of *parity patterns*, which will be introduced in Section 3.

Definition 1. *For every $n \geq 1$, we let $\mathfrak{G}^{\mathrm{I}}_{2n} := \{\pi \in \mathfrak{S}_{2n} : \pi$ avoids parity patterns* eE, eO, oO$\}$. *Alternatively, $\pi \in \mathfrak{G}^{\mathrm{I}}_{2n}$ if and only if every ascent $\pi_i < \pi_{i+1}$ of π satisfies that π_i is odd and π_{i+1} is even.*

For instance, $\mathfrak{G}^{\mathrm{I}}_{2} = \{12, 21\}$ and $\mathfrak{G}^{\mathrm{I}}_{4} = \{1432, 2143, 3142, 3214, 3412, 3421, 4312, 4321\}$.

The following labeling is one out of four distinct labelings for the regions of $\mathcal{K}_{2n}$, and could be viewed as the first main result of our paper. Given a hyperplane arrangement $\mathcal{A}$, denote by $\mathcal{R}(\mathcal{A})$ the set of regions carved out by all the hyperplanes in $\mathcal{A}$ and use $r(\mathcal{A})$ to denote its cardinality.

Theorem 2. *For any $n \geq 1$, the regions of $\mathcal{K}_{2n}$ are bijectively labeled, via the mapping Λ_{I}, by the permutations in $\mathfrak{G}^{\mathrm{I}}_{2n}$.*

A triangular array connecting and refining the Genocchi numbers of both kinds is known as the *Seidel triangle* [22]; see [10] and [5] for related works. The first few rows of the Seidel triangle are shown in Table 1 below, wherein the numbers on the rightmost diagonal are the Genocchi numbers, while the leftmost column consists of the Genocchi medians, with each number occurring twice. The alternating arrows on the margin are to indicate the generation of a specific number by adding together the number preceding it in the same row (using the arrow for the order) and the number above it in the same column.

$n\backslash k$		1	2	3	4	5	
1		1					
2		1	$\leftarrow$				
3	$\rightarrow$	1	1				
4		2	1	$\leftarrow$			
5	$\rightarrow$	2	3	3			
6		8	6	3	$\leftarrow$		
7	$\rightarrow$	8	14	17	17		
8		56	48	34	17	$\leftarrow$	
9	$\rightarrow$	56	104	138	155	155	
10		608	552	448	310	155	$\leftarrow$

Table 1: The first ten rows of the Seidel triangle

More precisely, the entry in the n-th row and k-th column of the Seidel triangle, denoted $S_{n,k}$, can be generated according to the following rules:

$$
\begin{aligned}
&S_{1,1} = 1, \quad S_{n,k} = 0 \text{ if } k \notin [1, \lceil \frac{n}{2} \rceil], \\
&S_{2n,k} = S_{2n-1,k} + S_{2n,k+1}, \quad S_{2n+1,k} = S_{2n,k} + S_{2n+1,k-1}. \qquad (1.2)
\end{aligned}
$$

In particular, for every $n \geq 1$ we have

$$S_{2n,1} = h_{n-1}, \qquad S_{2n,n} = g_n. \tag{1.3}$$

Our second main result is to refine the counting of $\mathfrak{G}^{\mathrm{I}}_{2n}$ by the first letter of the permutation and derive the following relations for the first Seidel-like triangle thus obtained; see Table 2 for the first six rows of this triangle. Let us denote $G^{\mathrm{I}}_{2n} := |\mathfrak{G}^{\mathrm{I}}_{2n}|$, $\mathfrak{G}^{\mathrm{I}}_{2n,k} := \{\sigma \in \mathfrak{G}^{\mathrm{I}}_{2n} : \sigma_1 = k\}$, and $G^{\mathrm{I}}_{2n,k} := |\mathfrak{G}^{\mathrm{I}}_{2n,k}|$ for all $n \geq 1$ and $1 \leq k \leq 2n$.

Theorem 3. *For all $n \geq k \geq 1$, we have*

$$G^{\mathrm{I}}_{2n,1} = G^{\mathrm{I}}_{2n,2}, \tag{1.4}$$

$$G^{\mathrm{I}}_{2n,3} = 2(G^{\mathrm{I}}_{2n,1} + G^{\mathrm{I}}_{2n-2,1}), \tag{1.5}$$

$$G^{\mathrm{I}}_{2n,2n} = \sum_{i=1}^{2n-2} G^{\mathrm{I}}_{2n-2,i} = G^{\mathrm{I}}_{2n-2}, \tag{1.6}$$

$$2G^{\mathrm{I}}_{2n,2n-1} = \sum_{i=1}^{2n} G^{\mathrm{I}}_{2n,i}, \tag{1.7}$$

$$G^{\mathrm{I}}_{2n,2k} = S_{2n,n+1-k}. \tag{1.8}$$

Theorem 4. *For all $n \geq 2, n > k \geq 0$, we have*

$$G^{\mathrm{I}}_{2n,2k+2} = G^{\mathrm{I}}_{2n,2k} + \sum_{i=k}^{n-1} G^{\mathrm{I}}_{2n-2,2i}. \tag{1.9}$$

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1										
2	1	1	4	2								
3	3	3	8	6	28	8						
4	17	17	40	34	92	48	304	56				
5	155	155	344	310	676	448	1472	552	4720	608		
6	2073	2073	4456	4146	8060	6064	14848	7672	31472	8832	99136	9440

Table 2: The Seidel-like triangle refining the set $\mathfrak{G}^{\mathrm{I}}_{2n}$

The rest of this extended abstract is organized as follows. We collect definitions and some preliminary results in Section 2. The labeling Λ^{I} that proves Theorem 2 is described in a pseudocode fashion in Section 3, where three more sets of parity pattern avoiding permutations, namely $\mathfrak{G}^{\mathrm{II}}_{2n}$, $\mathfrak{G}^{\mathrm{III}}_{2n}$, and $\mathfrak{G}^{\mathrm{IV}}_{2n}$ are introduced. Section 4 features three bijections, the first of which is constructed recursively and it amounts to give a second proof of (1.8). The remaining two bijections are utilized to link our first set of permutations $\mathfrak{G}^{\mathrm{I}}_{2n}$ to two previously considered permutation models in the literature. The reader is directed to the arXiv version of our manuscript [11] for all the omitted details.

2 Preliminaries

A *(real) hyperplane arrangement* $\mathcal{A} \subseteq \mathbb{R}^n$ is a finite collection of affine codimension one subspaces in $\mathbb{R}^n$. Clearly the complement $\mathbb{R}^n \backslash \mathcal{A}$ is disconnected, and we denote $r(\mathcal{A})$ the number of connected components (called *regions*) of $\mathbb{R}^n \setminus \mathcal{A}$. Motivated by the work of Postnikov and Stanley [21], wherein the *Linial arrangement* was bijectively labeled by semiacyclic tournaments, Hetyei [14] considered a wider class called alternation acyclic tournaments and introduced the following *homogenized Linial arrangement*

$$\mathcal{H}_{2n-1} := \{x_i - x_j = y_i : 1 \leq i < j \leq n+1\},$$

which is a hyperplane arrangement in

$$\{(x_1, \ldots, x_{n+1}, y_1, \ldots, y_n) : (x_i, y_j) \in \mathbb{R}^2 \text{ for } (i,j) \in [n+1] \times [n]\} = \mathbb{R}^{2n+1},$$

and he enumerated the regions of $\mathcal{H}_{2n-1}$, relying on a formula derived in [7, Corollary 5.5] that connects the Genocchi medians with the so-called Legendre-Stirling numbers.

Theorem 5 ([14, Thms. 4.3 & 4.4])**.** *For all $n \geq 1$, we have $r(\mathcal{H}_{2n-1}) = h_n$.*

To produce an alternative proof of Theorem 5, Lazar and Wachs [17, Thm. 3.2] introduced another hyperplane arrangement in $\mathbb{R}^{2n+1} = \{(x_1, \ldots, x_{2n+1}) : x_i \in \mathbb{R}\}$, namely

$$\mathcal{K}_{2n} := \{x_{2i-1} - x_{2j} = 0 : 1 \leq i \leq j \leq n\},$$

and they established the following equivalence between $\mathcal{K}_{2n}$ and $\mathcal{H}_{2n-1}$.

Proposition 1. *There exists a vector space isomorphism $\psi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ that takes $\mathcal{K}_{2n}$ to $\mathcal{H}_{2n-1}$. Consequently, the regions of $\mathcal{K}_{2n}$ corresponds bijectively to the regions of $\mathcal{H}_{2n-1}$ and we have in particular $r(\mathcal{K}_{2n}) = r(\mathcal{H}_{2n-1})$.*

Here we take a different look at the arrangement $\mathcal{K}_{2n}$. Interpreted appropriately, the parity restrictions in their definitions highly resemble the even-odd drop condition for permutations in $\mathcal{E}_{2n}$ (see Definition 2(6) below). This motivates us to look for a more direct connection between $\mathcal{K}_{2n}$ and $\mathcal{E}_{2n}$ and leads us to Theorem 2. Before we explain this further, let us shift our attention to various classes of restricted permutations that are related to Genocchi numbers and Genocchi medians. To give more context, we recall seven types of permutations.

Definition 2. *For every $n \geq 1$, we introduce the following seven kinds of permutation subject to various parity restrictions.*

1. *A permutation $\sigma \in \mathfrak{S}_{2n}$ is called a Dumont permutation of the first kind if it satisfies the condition that if σ_i is even then $\sigma_i > \sigma_{i+1}$ and $i < 2n$, otherwise $\sigma_i < \sigma_{i+1}$ or $i = 2n$. The set of such permutations is denoted as $\mathcal{D}^{\mathrm{I}}_{2n}$.*

$$\begin{tikzcd}
\mathcal{R}(\mathcal{H}_{2n-1}) \arrow[r, "|\cdot|", "\text{Thm. 5}"'] \arrow[d, leftrightarrow, "\psi"', "\text{Prop. 1}"] & h_n & \mathcal{E}_{2n} \arrow[l, "|\cdot|"', "\text{Thm. 1}"] \arrow[d, leftrightarrow, "\Psi"] \\
\mathcal{R}(\mathcal{K}_{2n}) \arrow[rr, leftrightarrow, "\text{Thm. 2}", "\Lambda^{\mathrm{I}}"'] & & \mathfrak{G}^{\mathrm{I}}_{2n}
\end{tikzcd}$$

Figure 1: The equivalence between Theorems 5 and 1

2. *A permutation* $\sigma \in \mathfrak{S}_{2n}$ *is called a Dumont permutation of the second kind if* $\sigma_{2i} < 2i$ *and* $\sigma_{2i-1} \geq 2i-1$ *for all* $i \in [n]$*. The set of such permutations is denoted as* $\mathcal{D}^{\mathrm{II}}_{2n}$*.*

3. *A permutation* $\sigma \in \mathfrak{S}_{2n}$ *is called a Dumont permutation of the third kind if* $\sigma_i > \sigma_{i+1}$ *then* σ_i *and* σ_{i+1} *are both even. The set of such permutations is denoted as* $\mathcal{D}^{\mathrm{III}}_{2n}$*.*

4. *A permutation* $\sigma \in \mathfrak{S}_{2n}$ *is called a Dumont permutation of the fourth kind if* $\sigma_i < i$ *then* i *and* σ_i *are both even. The set of such permutations is denoted as* $\mathcal{D}^{\mathrm{IV}}_{2n}$*.*

5. *A permutation* $\sigma \in \mathfrak{S}_{2n}$ *is called a D-permutation if* $\sigma_{2i} \leq 2i$ *and* $\sigma_{2i-1} \geq 2i-1$ *for all* $i \in [n]$*. The set of such permutations is denoted as* $\mathfrak{D}_{2n}$*.*

6. *Denote the permutations of* $[2n]$ *with only even-odd drops by* $\mathcal{E}_{2n}$ *and call them the E-permutations.*

7. *Denote by* $\mathcal{X}_{2n}$ *the set of permutations* $\sigma \in \mathfrak{S}_{2n}$ *satisfying that if* $\sigma_i > \sigma_{i+1}$ *then* σ_i *is even and* σ_{i+1} *is odd.*

The enumeration of all seven classes of permutations above gives rise to the following relations.

$$|\mathcal{D}^{\mathrm{I}}_{2n}| = |\mathcal{D}^{\mathrm{II}}_{2n}| = |\mathcal{D}^{\mathrm{III}}_{2n}| = |\mathcal{D}^{\mathrm{IV}}_{2n}| = g_{n+1}, \ |\mathfrak{D}_{2n}| = |\mathcal{E}_{2n}| = |\mathcal{X}_{2n}| = h_n.$$

The history for these seven classes of permutations is omitted here. We end this section with a diagram that sets our Theorem 2 in the context of existing results. It is evident from Fig. 1 that in view of Theorem 2, the two enumerative results due respectively to Hetyei (Theorem 5) and Lazar-Wachs (Theorem 1) are essentially equivalent. Here Ψ refers to a variant of Foata's first fundamental transformation [19, Chap. 10.2].

3 Parity pattern and the labelings with all four models

Given a permutation $\pi = \pi_1 \cdots \pi_n$, each pair $\pi_i > \pi_{i+1}$ (resp. $\pi_i < \pi_{i+1}$) is called a *descent* (resp. an *ascent*) of π. Let σ be a permutation of length k. We say π **contains** the pattern σ if in π there exists a subsequence of length k that is order isomorphic to

σ, otherwise we say π **avoids** σ. From this perspective, descents and ascents are simply consecutive 21 and 12 patterns, respectively. For background and further information on the topic of pattern avoidance, we refer the reader to Kitaev's monograph [16] and the voluminous amount of references therein.

We are going to restrict the parities of both numbers π_i and π_{i+1} that form either a descent or an ascent. By a *parity pattern* we refer to one of the following eight consecutive patterns:

$$\text{eE, eO, oE, oO, Ee, Eo, Oe, and Oo.}$$

Here the parity pattern, say eO, refers to an ascent $\pi_i < \pi_{i+1}$ in π such that π_i is even and π_{i+1} is odd. The remaining seven patterns are understood analogously.

Given a point $\mathbf{x} = (x_1, \ldots, x_{2n}, x_{2n+1})$ that belongs to a region, say R, of $\mathcal{K}_{2n}$, note that the coordinate x_{2n+1} is unrestricted and can be ignored. For the remaining $2n$ coordinates, according to the region R we know the sign of $x_i - x_j$ for each pair (i, j) such that $i < j$ and $i + 1 \equiv j \equiv 0 \pmod 2$. In what follows, we refer to such a pair as an "oE pair".

Definition 3. *A permutation π in $\mathfrak{S}_{2n}$ is said to be compatible with a given region $R \in \mathcal{R}(\mathcal{K}_{2n})$, if for every* oE *pair (i, j) and $\mathbf{x} \in R$, we have $\pi^{-1}(i) < \pi^{-1}(j)$ if and only if $x_i < x_j$.*

The main objective of this section is to give a proof Theorem 2 by constructing the mapping $\Lambda^{\mathrm{I}} := \Lambda^{\mathrm{I}}_n : \mathcal{R}(\mathcal{K}_{2n}) \to \mathfrak{G}^{\mathrm{I}}_{2n}$. Here we only give the description of this first labeling and omit all the proofs. Fix a given region $R \in \mathcal{R}(\mathcal{K}_{2n+2})$, we first project[1] it to a region $\tilde{R} \in \mathcal{R}(\mathcal{K}_{2n})$ as follows:

$$\begin{aligned} \mathrm{proj}:\ & R \to \tilde{R} \\ \mathbf{x} = (x_1, \cdots, x_{2n+1}, x_{2n+2}, x_{2n+3}) \mapsto\ & \tilde{\mathbf{x}} = (x_1, \cdots, x_{2n+1}). \end{aligned}$$

Now by inductive hypothesis we can find the unique permutation $\tilde{\pi} := \Lambda^{\mathrm{I}}_n(\tilde{R}) \in \mathfrak{G}^{\mathrm{I}}_{2n}$ such that $\tilde{\pi}$ is compatible with $\tilde{R}$. Next, we insert $2n+1$ and $2n+2$ into $\tilde{\pi}$ while rearranging smaller letters when needed to derive a unique permutation $\pi \in \mathfrak{G}^{\mathrm{I}}_{2n+2}$. Finally, we set $\Lambda^{\mathrm{I}}_{n+1}(R) := \pi$. Pick any vector $\mathbf{x} \in R$, there are two cases to consider and the core of the construction is described as the *Insertion Algorithm I* and boxed below.

- If $x_{2n+1} < x_{2n+2}$ in $\mathbf{x}$, then input $y = 2n+2$ and run the insertion algorithm I to produce the word w. We define the permutation $\pi := (2n+1)w$.

- If $x_{2n+1} > x_{2n+2}$ in $\mathbf{x}$, then input the concatenated $y = (2n+2)(2n+1)$ and run the insertion algorithm I to produce the word w. We define the permutation $\pi := w$.

[1] Note that going through this projection, we are deprived of the information on the magnitude comparisons between the coordinates x_{2n+2} and x_j for all $j = 1, 3, \ldots, 2n+1$. This observation explains why we also need $\mathbf{x}$ as an input for the insertion algorithm.

A pair of odd integers (i, j) is said to be *bad* with respect to $\mathbf{x}$, if i precedes j in the one-line expression of $\tilde{\pi}$ but $x_j < x_{2n+2} < x_i$ in $\mathbf{x}$. The algorithm below tells us how to remedy all the bad pairs by rearranging certain odd entries in $\tilde{\pi}$, and then insert $2n + 1, 2n + 2$ into the adjusted permutation.

Insertion Algorithm I (IA-I)

0. (Initiate) Input y, $\tilde{\pi}$, and $\mathbf{x}$.
1. (Step 1) If there are no bad pairs in $\tilde{\pi}$ with respect to $\mathbf{x}$, then let $\tilde{w} := \tilde{\pi}$ and jump to Step 4. Otherwise, find the rightmost j in $\tilde{\pi}$ such that (i, j) forms a bad pair with respect to $\mathbf{x}$ for a certain i.
2. (Step 2) Find all bad pairs ending with j, say $(i_1, j), \dots, (i_m, j)$ with $i_1 > \cdots > i_m$. Remove $i_1, \dots, i_m$ from $\tilde{\pi}$ and form the subword $u := i_1 i_2 \cdots i_m$ by concatenation.
3. (Step 3) Insert u to the immediate right of j to get $\tilde{w} := \cdots ju \cdots$.
4. (Step 4) Find the rightmost odd number k in $\tilde{w}$ such that $x_k < x_{2n+2}$. Insert y to the immediate right of k to obtain $w := \cdots ky \cdots$. Note that if the algorithm goes through Steps 2 and 3, then $k = j$. In the particular case that no such k exists, simply set $w := y\tilde{w}$.
5. (End) Output w.

Example 1. *To illustrate the first labeling Λ^{I} constructed in the proof of Theorem 2, we start with a specific region $R \in \mathcal{R}(\mathcal{K}_{18})$ defined by its Hasse diagram in Fig. 2 (where x_a connected with x_b and lying above x_b means that $x_a > x_b$), and go through IA-I step-by-step to find its image π in $\mathfrak{G}^{\mathrm{I}}_{18}$.*

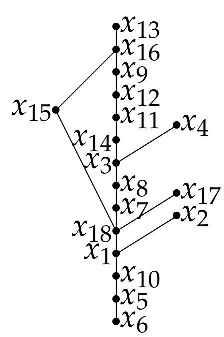

Figure 2: The Hasse diagram of a region $R \in \mathcal{R}(\mathcal{K}_{18})$

Define $\tilde{R}_i = \mathrm{proj}^{8-i}(R)$ for $i = 0, \dots, 7$ and $\tilde{\pi}^{(j)} = \Lambda^{\mathrm{I}}_j(\tilde{R}_{j-1})$ for $j = 1, \dots, 8$. One reads from the diagram above that $\tilde{R}_0 = \{\mathbf{x} \in \mathbb{R}^3 : x_1 - x_2 < 0\}$, so initially $\tilde{\pi}^{(1)} = \Lambda^{\mathrm{I}}_1(\tilde{R}_0) = 12$, *and we will insert* 3 *and* 4*. We see* $x_3 < x_4$ *in the diagram so we set* $y = 4$ *and start IA-I. There are no bad pairs in* $\tilde{\pi}^{(1)}$ *with respect to* $\mathbf{x}$ *and* 1 *is the rightmost* k *in* $\tilde{\pi}^{(1)}$ *such that* $x_k < x_4$*, so*

$$k = 1,\ w = 142,\ \textit{and}\ \tilde{\pi}^{(2)} = 3142.$$

We refer the reader to the arXiv version of this work [11] for the remaining steps, which lead up to finding the image $\Lambda^{\mathrm{I}}(R) = 65(10)1(18)(17)(15)783(14)(11)(12)9(16)(13)42 \in \mathfrak{G}^{\mathrm{I}}_{18}$*.*

A closer look at the set $\mathfrak{G}^{\mathrm{I}}_{2n}$ (see Definition 1) and our first labeling Λ^{I} led us to consider three more subsets of $\mathfrak{S}_{2n}$ that are enumerated by the median Genocchi numbers. Their definitions not only vary in the choices of parity patterns to be avoided, but also require some additional restrictions. Note that however, eO is one common pattern to be avoided by all four subsets of permutations. We include below the definition of the second set $\mathfrak{G}^{\mathrm{II}}$ to show the flavor, while omitting the definitions for the other two sets $\mathfrak{G}^{\mathrm{III}}$ and $\mathfrak{G}^{\mathrm{IV}}$.

Definition 4. *For every* $n \geq 1$*, let* $\mathfrak{G}^{\mathrm{II}}_{2n}$ *denote the set of permutations* σ *in* $\mathfrak{S}_{2n}$ *subject to the following conditions.*

1. σ *avoids the parity patterns* Ee, eO, oO*;*
2. *the only occurrence of* Oe *pattern that* σ *allows is as follows:* $\sigma_1 > \cdots > \sigma_m > \sigma_{m+1}$*, where* (σ_m, σ_{m+1}) *is an* Oe *pattern and all* σ_i $(1 \leq i \leq m)$ *are odd numbers. We may call such* Oe *pattern an initial* Oe *pattern.*

For example, the set $\mathfrak{G}^{\mathrm{II}}_4$ includes the following eight permutations, among which 3214 and 3241 each has an initial Oe pattern.

$$\mathfrak{G}^{\mathrm{II}}_4 = \{1243, 2143, 2431, 3124, 3214, 3241, 3412, 4312\}.$$

All three newly introduced sets of permutations could be utilized to label the regions of $\mathcal{K}_{2n}$, as indicated by the next theorem, the proof of which is omitted.

Theorem 6. *For any* $n \geq 1$*, the regions of* $\mathcal{K}_{2n}$ *are bijectively labeled, via the following three mappings, by the permutations in* $\mathfrak{G}^{\mathrm{II}}_{2n}$*,* $\mathfrak{G}^{\mathrm{III}}_{2n}$*, and* $\mathfrak{G}^{\mathrm{IV}}_{2n}$*, respectively.*

$$\Lambda^{\mathrm{II}}: \mathcal{R}(\mathcal{K}_{2n}) \to \mathfrak{G}^{\mathrm{II}}_{2n},\ \Lambda^{\mathrm{III}}: \mathcal{R}(\mathcal{K}_{2n}) \to \mathfrak{G}^{\mathrm{III}}_{2n},\ \Lambda^{\mathrm{IV}}: \mathcal{R}(\mathcal{K}_{2n}) \to \mathfrak{G}^{\mathrm{IV}}_{2n}.$$

4 Three bijections

Both kinds of Genocchi numbers have more than one combinatorial interpretation, as can be gleaned from Definition 2. This raises natural bijective questions. In this section, we present three bijections pertaining to our first permutation model $\mathfrak{G}^{\mathrm{I}}_{2n}$, the Dumont permutations of the third kind $\mathcal{D}^{\mathrm{III}}_{2n}$, and the collapsed permutations $\mathcal{CO}_{2n}$, which were first introduced by Ayyer, Hathcock, and Tetali [2] and can be defined as follows.

Definition 5 ([2, Definition 4.1])**.** *A permutation $\pi \in \mathfrak{S}_{2n}$ is said to be* collapsed*, if for every $k \in [2n]$ we have*

$$1 + \lfloor k/2 \rfloor \leq \pi^{-1}(k) \leq n + \lfloor k/2 \rfloor. \tag{4.1}$$

We denote the set of collapsed permutations of length $2n$ by $\mathcal{CO}_{2n}$ and its cardinality by CO_{2n}.

For example, the first three sets of collapsed permutations are listed below.

$$\mathcal{CO}_2 = \{12\}, \quad \mathcal{CO}_4 = \{1234, 1324\},$$
$$\mathcal{CO}_6 = \{123456, 123546, 124356, 125346, 132456, 132546, 134256, 135246\}.$$

Our first bijection $\Phi : \mathfrak{G}^{\mathrm{I}}_{2n} \to \mathcal{D}^{\mathrm{III}}_{2n}$ is constructed recursively, building on combinatorial interpretations of the recursion (1.9) and a similar recursion for the array formed by the even rows of the Seidel triangle; see Table 1. This approach leads to a bijective proof of (1.8). Our second and third bijections θ and ϑ can be combined to give a direct bijective proof of the following result.

Theorem 7. *For all $n \geq 0$, we have*

$$\mathrm{CO}_{2n+2} = h_n = G^{\mathrm{I}}_{2n}. \tag{4.2}$$

More precisely, these two mappings θ and ϑ give rise to the following triangle.

$$\begin{tikzcd}
& \mathfrak{D}_{2n} & \\
\mathcal{CO}_{2n+2} \arrow[ur, "\theta"] \arrow[rr, dashed, "\Theta"] & & \mathfrak{G}^{\mathrm{I}}_{2n} \arrow[ul, "\vartheta"']
\end{tikzcd}$$

Figure 3: The composed bijection $\Theta = \vartheta^{-1} \circ \theta : \mathcal{CO}_{2n+2} \to \mathfrak{G}^{\mathrm{I}}_{2n}$

Theorem 8. *For every $n \geq 1$, there exists a bijection*

$$\theta : \mathcal{CO}_{2n+2} \to \mathfrak{D}_{2n}$$
$$\pi = \pi_1 \pi_2 \cdots \pi_{2n+1} \pi_{2n+2} \mapsto \sigma = \sigma_1 \cdots \sigma_{2n},$$

where for $1 \leq i \leq n$, $\sigma_{2i-1} := \pi_{n+i+1} - 1$, and $\sigma_{2i} := \pi_{i+1} - 1$.

The third bijection ϑ is a composed mapping $\vartheta := \Psi_{\mathrm{LY}} \circ \Psi^{-1}$, where Ψ is Foata's first fundamental transformation and Ψ_{LY} is essentially a mapping introduced by Lin and Yan [18]. We omit the details here.

5 Concluding remarks

It is worth noting that the recurrence relation shown in (1.9) fully characterizes the entries in the even-indexed columns of the triangle in Table 2. There are further properties of this triangle that are worth exploring. Moreover, refining the counting of permutations in $\mathfrak{G}^{\mathrm{II}}$, $\mathfrak{G}^{\mathrm{III}}$, and $\mathfrak{G}^{\mathrm{IV}}$ according to their first letters gives rise to three more Seidel-like triangles comparable to Table 2.

Lastly, it is well known that the n-th Genocchi median h_n is divisible by 2^n (see [3, 4, 20]), and the sequence $\{h_n/2^n\}_{n\geq 0} = \{1,1,2,7,38,295,\ldots\}$ is registered as A000366 in the OEIS [23] and known as the *normalized median Genocchi numbers*. Given any combinatorial model enumerated by the median Genocchi numbers, it is then natural to wonder if this model allows a manifest explanation to the aforementioned divisibility of h_n by 2^n; see [2, Remark 4.4]. Initial computations suggest that the four permutation models introduced in this work, namely $\mathfrak{G}^{\mathrm{I}}$, $\mathfrak{G}^{\mathrm{II}}$, $\mathfrak{G}^{\mathrm{III}}$, and $\mathfrak{G}^{\mathrm{IV}}$ all furnish such an explanation. We plan to continue these investigations in our ensuing work [12].

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