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Subgraphs Vs Orientations: infinite families of equidistributions

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Abstract. A classical enumerative result states that, given a graph G and a vertex u, the number of connected subgraphs of G is equal to the number of orientations of G such that every vertex can reach u by a directed path. We show that this result is an instance of a much broader set of enumerative identities between subgraphs and orientations corresponding to various connectivity constraints. Namely, given two sets of pairs of vertices $A = \{(u_i, v_i),\ i \in [k]\}$ and $B = \{(u'_i, v'_i),\ i \in [\ell]\}$, we consider the orientations α of G such that adding the elements of A and B as additional directed edges to α gives an orientation α' in which v_i cannot reach u_i for all $i \in [k]$, but v'_i can reach u'_i for all $i \in [\ell]$. We show that this set of orientations is equinumerous to a set of subgraphs satisfying the "same" connectivity constraints defined in terms of A and B.

We also extend our results to the enumeration of equivalence classes of orientations satisfying such connectivity constraints. Precisely, we consider the equivalence classes under cycle reversal, cocycle reversal, or cycle-cocycle reversal. We show that the equivalences classes are equinumerous to some sets of subgraphs defined by connectivity and acyclicity constraints.

1 Introduction

This article is concerned with a set of enumerative identities between subgraphs and orientations. Consider a finite undirected graph $G = (V, E)$ (loops and multiple edges are allowed). By *subgraph* of G we mean a graph of the form $H = (V, F)$ for some subset of edges $F \subseteq E$. A well-known result states that, given a vertex u, the number of orientations of G such that every vertex can reach u by a directed path is equal to the number of connected subgraphs of G [3]. Another classical result is that, given two vertices u, v, the number of orientations such that v can reach u is equal to the number of subgraphs such that u and v are connected [4]. And, of course, the total number of orientations, $2^{|E|}$, is equal to the number of subgraphs. What if these identities were all part of a big family? This is the main result we establish in the present article.

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Let us set the vocabulary needed to state our results. Consider a graph $G = (V, E)$ and two arbitrary sets of pairs of vertices $A, B \subseteq V^2$. Given an orientation α of G, we consider the oriented graph $\vec{G} \equiv \vec{G}(A, B; \alpha)$ obtained from α by adding the pairs in A and B as additional directed edges. We say that the orientation α is (A, B)-*valid* if

(a) for all (u, v) in A, the vertex v cannot reach u by a directed path of $\vec{G}$, and

(b) for all (u, v) in B, the vertex v can reach u by a directed path of $\vec{G}$.

Given a subgraph $H = (V, F)$ we consider the oriented graph $\vec{G} \equiv \vec{G}(A, B; F)$ obtained from H by replacing every edge in H by two directed edges (in opposite direction) and adding the pairs in A and B as additional directed edges. We say that the subgraph H is (A, B)-*valid* if $\vec{G}$ satisfies the conditions (a) and (b) above.

Our main result is that for every graph G and sets A and B, the (A, B)-valid orientations are equinumerous to the (A, B)-valid subgraphs. Let us examine some special cases for illustration. For $A = \{(u, v)\}$ and $B = \emptyset$, our result implies that the number of orientations of G such that there is no directed path from v to u is equal to the number of subgraphs for which u and v are not connected by a path. For $A = \emptyset$ and $B = \{(u, v) \mid v \in V\}$, our result implies the classical result mentioned above: the orientations of G such that every vertex can reach u by a directed path are equinumerous to the connected subgraphs of G. Figure 1 gives another illustration.

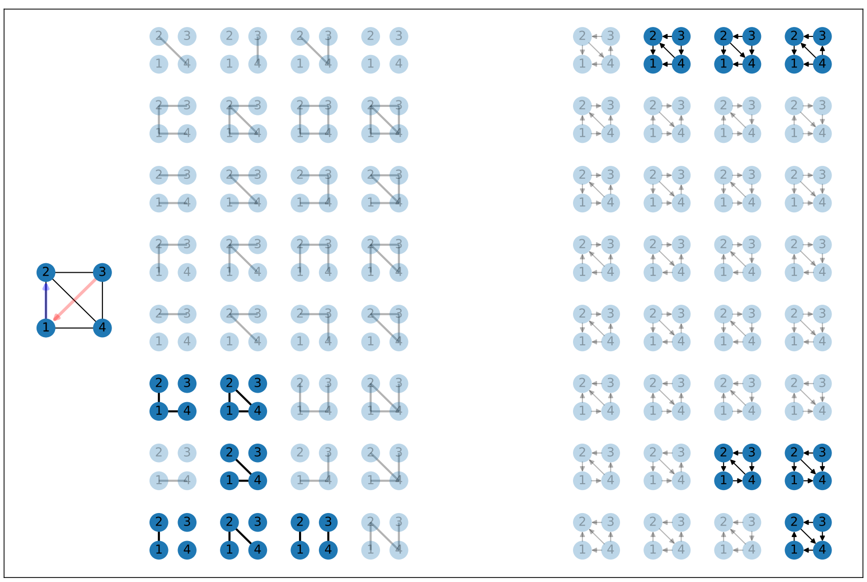

Figure 1: The (A, B)-valid subgraphs and orientations (6 among 32). The graph G has 5 edges on the vertex set $\{1, 2, 3, 4\}$, and $A = \{(3, 1)\}$ (red) and $B = \{(1, 2)\}$ (blue).

A good way to phrase the conditions (a) and (b) in a unified manner, and to interpolate between the case of orientations and the case of subgraphs is through the formalism of *fourientations* [2]. A *fourientation* of a graph $G = (V, E)$ is an assignment ϕ of one of four configurations for each edge $e \in E$ indicating how e can be used on a path from a vertex to another:

- either e can be traversed in both directions; we say that e is *2-way* in this case,
- or e can only be traversed in one direction; we say that e is *1-way* in this case,
- or e cannot be traversed at all; we say that e is *0-way* in this case.

We say that an edge in a fourientation is *solid* if it is either 0-way or 2-way. Clearly, orientations correspond to fourientations without solid edges, while subgraphs correspond to fourientations such that every edge is solid. We say that a fourientation ϕ of $G = (V, E)$ is (A, B)*-valid* if the fourientation $\vec{G}$ obtained from ϕ by adding the 1-way edges in A and B satisfies the conditions (a), (b) above. Our result above can then be generalized as follows: given any set of edges $S \subseteq E$, the number of (A, B)-valid fourientations having S as their set of solid edges is independent of the set S. Indeed, comparing these cardinalities for $S = \emptyset$ and $S = E$ yields the previously stated identities between (A, B)-valid orientations and (A, B)-valid subgraphs.

Another set of results we establish is about equivalence classes of (A, B)-valid orientations (and fourientations). Following [3], we say that two orientations of G are *cycle-reversal equivalent* if one can be obtained from the other by repeatedly flipping some directed cycles. A classical result is that the cycle-reversal equivalence classes of G are equinumerous to the number of forests of G (subgraphs without cycles). We show that for every sets A, B the cycle-reversal equivalence classes of (A, B)-valid orientations are equinumerous to the (A, B)-valid forests of G. Similar results are established for the cocycle-reversal equivalence, and the cycle-cocycle-reversal equivalence.

2 Notation

In this section we set our notation and review some definitions about fourientations.

We say *graph* to mean finite undirected graph, where loops and multiple edges are allowed. We say *digraph* to mean directed graph, and *arc* to mean directed edge. We say that a vertex v in a digraph can *reach* a vertex u if there is a directed path from v to u. We say that an arc $a = (u, v)$ in a digraph $G = (V, E)$ is *cyclic* if it belongs to a directed cycle (i.e. v can reach u) and *acyclic* otherwise (i.e. v cannot reach u). A non-empty set of arcs $S \subseteq A$ of the digraph G is a *directed cocycle* if there exists a partition $V = V_1 \cup V_2$ such that S is the set of arcs of G having their initial vertex in V_1 and terminal vertex in V_2, and G has no arc with initial vertex in V_2 and terminal vertex in V_1. It is well known that an arc is acyclic if and only if it belongs to a directed cocycle.

For a graph $G = (V, E)$, we let $\overleftrightarrow{G} = (V, \overleftrightarrow{E})$ be the digraph obtained from G by replacing each edge by two arcs in opposite direction. As explained above, a *fourientation* of G is an assignment of one among four possible configurations for each edge, either

0-way, 1-way (two possibilities) or 2-way. If we fix a conventional orientation $\overrightarrow{e}$ for an edge $e \in E$ of G, then we denote by $\overrightarrow{e}$ and $\overleftarrow{e}$ the two 1-way configurations of e, by $\overset{\circ}{e}$ the 0-way configuration, and by $\overleftrightarrow{e}$ the 2-way configuration. Note that the fourientations of G are naturally in correspondence with the digraphs of the form $D = (V, F)$ with $F \subseteq \overleftrightarrow{E}$. For a fourientation ϕ, we denote by $\overrightarrow{\phi}$ the corresponding digraph $D = (V, F)$. By *directed paths* (resp. *directed cycles*, *directed cocycles*) of a fourientation ϕ, we mean directed paths (resp. directed cycles, directed cocycles) in the digraph $\overrightarrow{\phi}$.

Given a fourientation ϕ of a graph $G = (V, E)$, and two sets of pairs of vertices $A \subseteq V^2$ and $B \subseteq V^2$ (which can be thought as arcs on V), we denote by $\vec{G}(A, B; \phi)$ the digraph obtained from $\overrightarrow{\phi}$ by adding the arcs in A and B.

Definition 2.1. Given a graph $G = (V, E)$, and two sets of pairs of vertices $A \subseteq V^2$ and $B \subseteq V^2$, we say that a fourientation ϕ of G is (A, B)-valid if the digraph $\vec{G} = \vec{G}(A, B; \phi)$ satisfies the conditions (a) and (b) stated in the introduction.

Remark 2.2. A fourientation ϕ of G is (A, B)-valid if and only if every arc in A is acyclic and every arc in B is cyclic in $\vec{G}(A, B; \phi)$.

Recall that the *solid edges* of a fourientation ϕ of $G = (V, E)$ are the edges which are either 0-way or 2-way. For a set $S \subseteq E$ we denote by $\mathrm{Four}_{G,S}(A, B)$ the set of (A, B)-valid fourientations with set of solid edges equal to S. Recall that the orientations of G correspond to fourientations without solid edges, while the subgraphs of G correspond to fourientations such that every edge is solid.

3 Identities for subgraphs, orientations and fourientations

In this section we state, illustrate and prove our main result, which is a set of enumerative identities between classes of subgraphs, orientations, and fourientations.

3.1 Main result

Our main result is stated in terms of the cardinality of classes of fourientations with a given solid edge set S (they interpolate between subgraphs and orientations).

Theorem 3.1. *For any graph $G = (V, E)$ and sets $A, B \subseteq V^2$, the number $|\mathrm{Four}_{G,S}(A, B)|$ of (A, B)-valid fourientations with solid edge set S is independent of S.*

By comparing the case $S = \varnothing$ to the case $S = E$ in Theorem 3.1, we get:

Corollary 3.2. *Let $G = (V, E)$ be a graph. For any sets $A, B \subseteq V^2$ the number of (A, B)-valid orientations is equal to the number of (A, B)-valid subgraphs.*

Let us illustrate Theorem 3.1 and Corollary 3.2 by considering various connectivity constraints A, B.

Example 3.3. Let $G = (V, E)$ be a graph and let $u \in U \subseteq V$. By Corollary 3.2 with $A = \{(u, v) \mid v \in V \setminus U\}$ and $B = \{(u, v) \mid v \in U\}$, the number of orientations such that for every vertex in U can reach u and every vertex in $V \setminus U$ cannot reach u is equal to the number of subgraphs such that the connected component containing u is U. For $U = V$, we get that the number of orientations such that every vertex can reach u is equal to the number of connected subgraphs. Applying Theorem 3.1 to this case shows that the number of fourientations such that every vertex can reach u is $2^{|E|}$ times the number of connected subgraphs.

Example 3.4. Let $G = (V, E)$, and let $X, Y \subseteq V$ be such that $X \uplus Y = V$. Let $A = \varnothing$ and $B = \{(x, y) \mid x \in X,\ y \in Y\}$. Applying Corollary 3.2 gives that the number of orientations of G such that each vertex $y \in Y$ can reach at least one vertex in X and every vertex $x \in X$ can be reached by at least one vertex in Y is equal to the number of subgraphs of G in which each connected component contains at least one vertex in X and one vertex in Y.

Let us reframe Theorem 3.1 in terms of cyclic and acyclic edges. Let $G = (V, E)$ be a graph, and let $\overleftrightarrow{G} = (V, \overleftrightarrow{E})$ be the corresponding digraph. For an orientation α of G, we define $\mathrm{Acy}(\alpha) \subseteq \overleftrightarrow{E}$ and $\mathrm{Cyc}(\alpha) \subseteq \overleftrightarrow{E}$ as the sets of acyclic arcs and cyclic arcs respectively. We extend this definitions to fourientations: for an fourientation ϕ of a graph G, we define $\mathrm{Acy}(\phi) \subseteq \overleftrightarrow{E}$ and $\mathrm{Cyc}(\phi) \subseteq \overleftrightarrow{E}$ as the sets of acyclic 1-way edges and cyclic 1-way edges respectively. The following is a consequence of Theorem 3.1.

Corollary 3.5. *Let $G = (V, E)$ be a graph, and let $\overleftrightarrow{G} = (V, \overleftrightarrow{E})$ be the corresponding digraph. Let $(y_a)_{a \in \overleftrightarrow{E}}$ and $(z_a)_{a \in \overleftrightarrow{E}}$ be variables. Then*

$$\sum_{\alpha \text{ orientation of } G} \prod_{a \in Acy(\alpha)} (1 + y_a) \prod_{b \in Cyc(\alpha)} (1 + z_b) = \sum_{\phi \text{ fourientation of } G} \prod_{a \in Acy(\phi)} y_a \prod_{b \in Cyc(\phi)} z_b. \tag{3.1}$$

Proof. For every subsets $A, B \subseteq \overleftrightarrow{E}$ we can compare the coefficient of $\prod_{a \in A} y_a \prod_{b \in B} z_b$ in the left-hand side and right-hand side of (3.1). Denoting by $G \setminus A \setminus B$ the subgraph of G obtained by deleting the edges in E corresponding to the arcs in $A \cup B$, one gets

$$\left[\prod_{a \in A} y_a \prod_{b \in B} z_b\right] LHS = \#(A, B)\text{-valid orientations of } G \setminus A \setminus B$$

and

$$\left[\prod_{a \in A} y_a \prod_{b \in B} z_b\right] RHS = \#(A, B)\text{-valid subgraphs of } G \setminus A \setminus B.$$

Since these quantities are equal for all sets A, B by Theorem 3.1 (see Remark 2.2), we conclude that (3.1) holds. □

A fourientation is called *acyclic* (resp. *totally cyclic*) if all its 1-way edges are acyclic (resp. cyclic). Note that by setting $y_a = 1$ and $z_a = 0$ for all $a \in \overleftrightarrow{E}$ in (3.1) one gets

$$\#\text{ acyclic fourientations of } G = \sum_{\alpha \text{ orientation of } G} 2^{|\mathrm{Acy}(\alpha)|}, \tag{3.2}$$

while by setting $y_a = 0$ and $z_a = 1$ for all $a \in \overleftrightarrow{E}$ one gets

$$\#\text{ totally cyclic fourientations of } G = \sum_{\alpha \text{ orientation of } G} 2^{|\mathrm{Cyc}(\alpha)|}. \tag{3.3}$$

From (3.3), one can derive the following corollary first obtained in [1].

Corollary 3.6 ([1]). *For all $n \in \mathbb{N}$, we let t_n be the number of totally cyclic orientations of the complete graph K_n (strongly connected tournaments on n vertices), and we let s_n be the number of strongly connected fourientations of K_n (strongly connected simple digraphs on n vertices). These numbers are related by the following equation*

$$\frac{1}{1 - \sum_{n=1}^{\infty} 2^{\binom{n}{2}} t_n \frac{x^n}{n!}} = \exp\left(\sum_{n=1}^{\infty} s_n \frac{x^n}{n!}\right). \tag{3.4}$$

We omit the derivation of Corollary 3.6, and simply mention the main idea: the left-hand side of (3.4) can be interpreted as the generating function of oriented complete graphs with a weight of 2 per cyclic edge, while the right-hand side can be interpreted as the generating function of totally cyclic fouriented complete graphs. These are shown to be equal by applying (3.3) to complete graphs.

3.2 Proof of Theorem 3.1

In this subsection we prove Theorem 3.1. We fix a graph $G = (V, E)$ and the sets $A, B \subseteq V^2$ throughout. It suffices to show that for every set $S \subset V$ and edge $e \in E \setminus S$,

$$|\mathrm{Four}_{G,S}(A, B)| = |\mathrm{Four}_{G,S\cup e}(A, B)|. \tag{3.5}$$

We now fix S and $e \in E \setminus S$, and proceed to prove (3.5).

Let $\Omega = \{\mathring{e}, \overrightarrow{e}, \overleftarrow{e}, \overleftrightarrow{e}\}$ be the set of possible configurations of e, and let $(\Omega, \preceq)$ be the poset given by $\mathring{e} \prec \overrightarrow{e} \prec \overleftrightarrow{e}$ and $\mathring{e} \prec \overleftarrow{e} \prec \overleftrightarrow{e}$. For a fourientation ϕ of $G \setminus e$ and a configuration $c \in \Omega$ we denote by $\phi \cup c$ the fourientation of G obtained by giving configuration c to e. We then define the function $f_\phi : \Omega \to \{0,1\}$, by:

$$f_\phi(c) = \begin{cases} 1 & \text{if } \phi \cup c \text{ is } (A,B)\text{-valid}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$|\mathrm{Four}_{G,S}(A,B)| = \sum_\phi f_\phi(\overrightarrow{e}) + f_\phi(\overleftarrow{e}) \ \text{ and } \ |\mathrm{Four}_{G,S\cup e}(A,B)| = \sum_\phi f_\phi(\mathring{e}) + f_\phi(\overleftrightarrow{e}),$$

where the sums are over the fourientations ϕ of $G \setminus e$ with solid edge set S. Thus, in order to prove (3.5) it suffices to prove that for every fourientation ϕ of $G \setminus e$,

$$f_\phi(\overrightarrow{e}) + f_\phi(\overleftarrow{e}) = f_\phi(\mathring{e}) + f_\phi(\overleftrightarrow{e}). \tag{3.6}$$

We now fix an orientation ϕ of $G \setminus e$, and proceed to prove (3.6). For a configuration $c \in \Omega$, we denote by $D(c)$ the directed graph $\vec{G}(A,B;\phi \cup c)$. Recall that a fourientation is (A,B)-valid if it satisfies Conditions (a) and (b). We say that $c \in \Omega$ is

- *acyclic-valid* if $\phi \cup c$ satisfies (a), that is, every arc in A is acyclic in $D(c)$,
- *cyclic-valid* if $\phi \cup c$ satisfies (b), that is, every arc in B is cyclic in $D(c)$.

We omit the proof of the following easy lemmas.

Lemma 3.7. *Let $c \preceq d$ in the poset $(\Omega, \preceq)$. If c is cyclic-valid, then d is cyclic-valid. If d is acyclic-valid, then c is acyclic-valid.*

Lemma 3.8. *Let $c \in \{\overleftarrow{e}, \overrightarrow{e}\}$ such that $\phi \cup c$ is (A,B)-valid. If c is cyclic in $D(c)$, then $\phi \cup \overleftrightarrow{e}$ is (A,B)-valid. If c is acyclic in $D(c)$, then $\phi \cup \mathring{e}$ is (A,B)-valid.*

We will now prove (3.6), starting with the following:

Claim 1. $f_\phi(\overleftrightarrow{e}) + f_\phi(\mathring{e}) = 2$ if and only if $f_\phi(\overrightarrow{e}) + f_\phi(\overleftarrow{e}) = 2$.

Suppose that $f_\phi(\overleftrightarrow{e}) + f_\phi(\mathring{e}) = 2$. Since $\overleftrightarrow{e}$ is acyclic-valid, Lemma 3.7 implies that $\overrightarrow{e}$ and $\overleftarrow{e}$ are both acyclic-valid. Since $\mathring{e}$ is cyclic-valid, Lemma 3.7 implies that $\overrightarrow{e}$ and $\overleftarrow{e}$ are both cyclic-valid. Thus, $f_\phi(\overrightarrow{e}) + f_\phi(\overleftarrow{e}) = 2$.

Now suppose that $f_\phi(\overrightarrow{e}) + f_\phi(\overleftarrow{e}) = 2$. By Lemma 3.7, $\overleftrightarrow{e}$ is cyclic-valid and $\mathring{e}$ is acyclic-valid. We need to show that $\overleftrightarrow{e}$ is acyclic-valid and $\mathring{e}$ is cyclic-valid. If $\overleftrightarrow{e}$ is not acyclic-valid, then there exists an arc $a = (u,v) \in A$ such that there is a directed path from v to u in $D(\overleftrightarrow{e})$. But then there is a directed path from v to u in either $D(\overrightarrow{e})$ or $D(\overleftarrow{e})$, which is impossible since $\overrightarrow{e}$ and $\overleftarrow{e}$ are acyclic-valid. It remains to show that $\mathring{e}$ is cyclic-valid, that is, for every arc $b = (u,v) \in B$ we must show that there is a directed path from v to u in $D(\overset{\circ}{e})$. If there is a directed path from v to u not containing e, then we are done. Otherwise there exists a directed path in $D(\overrightarrow{e})$ containing $\overrightarrow{e}$ and a directed path in $D(\overleftarrow{e})$ containing $\overleftarrow{e}$. But then, as shown in Figure 2(a), there is a directed path from v to u avoiding e, which contradicts our assumption. This completes the proof of Claim 1.

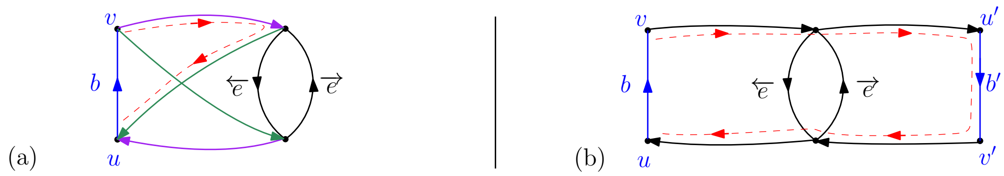

Figure 2: (a) The case where there are directed paths from v to u containing $\overrightarrow{e}$, and one containing $\overleftarrow{e}$. The dashed path avoids e. (b) The case where there is a path from v to u in $D(\overleftarrow{e})$, and a path from v' to u' in $D(\overrightarrow{e})$. The dashed path avoids e.

Claim 2. $f_\phi(\overleftrightarrow{e}) + f_\phi(\overset{\circ}{e}) > 0$ if and only if $f_\phi(\overrightarrow{e}) + f_\phi(\overleftarrow{e}) > 0$.

Suppose that $f_\phi(\overleftarrow{e}) + f_\phi(\overrightarrow{e}) > 0$. Since any 1-way arc must be either cyclic or acyclic, Lemma 3.8 implies $f_\phi(\overset{\circ}{e}) + f_\phi(\overleftrightarrow{e}) > 0$.

Suppose conversely that $f_\phi(\overset{\circ}{e}) + f_\phi(\overleftrightarrow{e}) > 0$. Consider first the situation where $\phi \cup \overleftrightarrow{e}$ is (A, B)-valid. By Lemma 3.7, both $\overleftarrow{e}$ and $\overrightarrow{e}$ are acyclic-valid. We need to prove that one of them is cyclic-valid. If this is not the case, then there are arcs $b = (u, v)$ and $b' = (u', v')$ in B such that there is no path from v to u in $D(\overrightarrow{e})$, and no path from v' to u' in $D(\overleftarrow{e})$. Since there are such paths in $D(\overleftrightarrow{e})$, we conclude that there is a path from v to u using $\overleftarrow{e}$ in $D(\overleftarrow{e})$, and a path from v' to u' using $\overrightarrow{e}$ in $D(\overrightarrow{e})$. But then, as shown in Figure 2(b), there is a directed path from v to u in $D(\overrightarrow{e})$ avoiding e entirely, which is a contradiction.

Consider now the situation where $\phi \cup \overset{\circ}{e}$ is (A, B)-valid. By Lemma 3.7, both $\overleftarrow{e}$ and $\overrightarrow{e}$ are cyclic-valid. We need to prove that one of them is acyclic-valid. If this is not the case, then there exist arcs $a, a' \in A$ such that there is a directed cycle containing a and $\overrightarrow{e}$ in $D(\overrightarrow{e})$ and a directed cycle containing a' and $\overleftarrow{e}$ in $D(\overleftarrow{e})$. But then, there is a directed cycle containing a and a' avoiding e entirely, which contradicts the acyclic-validity of $\overset{\circ}{e}$. This completes the proof of Claim 2.

The above claims imply (3.6), which completes the proof of Theorem 3.1.

4 Equivalence classes under cycle-cocycle reversal

In this section we study equivalence classes of (A, B)-valid orientations up to cycle or cocycle reversals. We fix the graph $G = (V, E)$ and the sets $A, B \subseteq V^2$ throughout.

By *reversing a directed cycle* (resp. *cocycle*) of a fourientation ϕ of G we mean reversing all the 1-way edges on that directed cycle (resp. cocycle), and leaving the 2-way edges (resp. 0-way edges) unchanged. We call (A,B)*-cocycle* of ϕ a cocycle of $\vec{G}(A,B;\phi)$ which does not contain any arc from $A \cup B$.

Definition 4.1. We say that two fourientations ϕ and ϕ' of G are *cycle equivalent* (resp. *cocycle equivalent*) if ϕ can be obtained from ϕ' by repeatedly reversing some directed cycles (resp. (A,B)-cocycles). We say that two fourientations ϕ and ϕ' are *cycle-cocycle equivalent* if ϕ can be obtained from ϕ' by repeatedly reversing some directed cycles and/or (A,B)-cocycles.

The following result is easy to check.

Lemma 4.2. *If ϕ and ϕ' are cycle-cocycle equivalent fourientations of G, then ϕ is (A,B)-valid if and only if ϕ' is (A,B)-valid.*

By Lemma 4.2 it makes sense to consider equivalence classes of (A,B)-valid fourientations.

Definition 4.3.

(i) An (A,B)*-valid cycle class* is a cycle equivalence class of (A,B)-valid fourientations which do not contain any cycle made entirely of 2-way edges.

(ii) An (A,B)*-valid cocycle class* is a cocycle equivalence class of (A,B)-valid fourientations which do not contain any (A,B)-cocycle made entirely of 0-way edges.

(iii) An (A,B)*-valid cycle-cocycle class* is a cycle-cocycle equivalence class of (A,B)-valid fourientations which does not contain any cycle made entirely of 2-way edges nor any (A,B)-cocycle made entirely of 0-way edges.

For a set $S \subseteq E$ of edges, we denote by $\mathrm{Four}_{G,S}(A,B)/\mathrm{cyc}$ (resp. $\mathrm{Four}_{G,S}(A,B)/\mathrm{coc}$, $\mathrm{Four}_{G,S}(A,B)/\mathrm{cc}$) the set of (A,B)-valid cycle (resp. cocycle, cycle-cocycle) classes.

Our main result is the following.

Theorem 4.4. *Let $G=(V,E)$ be a graph and let $A,B \subseteq V^2$.*

(i) $|\mathit{Four}_{G,S}(A,B)/\mathit{cyc}\,|$ *is independent of S.*

(ii) $|\mathit{Four}_{G,S}(A,B)/\mathit{coc}\,|$ *is independent of S.*

(iii) $|\mathit{Four}_{G,S}(A,B)/\mathit{cc}\,|$ *is independent of S.*

Note that $\mathrm{Four}_{G,E}(A,B)/\mathrm{cyc}$ is the set of *forests* of G (subgraphs with no cycle). For a subgraph F of G, let us denote by $F \cup \underline{A} \cup \underline{B}$ the graph obtained from F by adding the set of edges $\{\{u,v\} \mid (u,v) \in A \cup B\}$. We say that F is (A,B)*-connected* if the connected components of $F \cup \underline{A} \cup \underline{B}$ and $G \cup \underline{A} \cup \underline{B}$ are equal. Then $\mathrm{Four}_{G,E}(A,B)/\mathrm{coc}$ is the set of (A,B)-connected subgraphs of G, while $\mathrm{Four}_{G,E}(A,B)/\mathrm{cc}$ is the set of (A,B)-connected forests of G. Comparing the cases of $S=\emptyset$ and $S=E$ in Theorem 4.4 we obtain the following corollary:

Corollary 4.5. *Let $G = (V, E)$ be a graph and let $A, B \subseteq V^2$.*

(i) The number of cycle equivalence classes of (A, B)-valid orientations is equal to the number of (A, B)-valid forests.

(ii) The number of cocycle equivalence classes of (A, B)-valid orientations is equal to the number of (A, B)-valid (A, B)-connected subgraphs.

(iii) The number of cycle-cocycle equivalence classes of (A, B)-valid orientations is equal to the number of (A, B)-valid (A, B)-connected forests.

Example 4.6. Applying Corollary 4.5(i) with the sets $A = \emptyset, B = \{(u, v)\}$ for some vertices $u, v \in V$, we get that the cycle equivalence classes of orientations of G such that v cannot reach u is equal to the number of forests of G such that u and v are in separate trees.

The results in Corollary 4.5 for the case $A = B = \emptyset$ were first established by Gioan [3]. Our proof technique is entirely different. We will only prove statement (i) of Theorem 4.4 in this extended abstract.

It suffices to show that for every set $S \subset E$ and edge $e \in E \setminus S$ one has

$$|\mathrm{Four}_{G,S}(A, B)/\mathrm{cyc}| = |\mathrm{Four}_{G,S\cup\{e\}}(A, B)/\mathrm{cyc}|. \tag{4.1}$$

We examine the relationship between the equivalence classes through the lens of a graph $\mathcal{G}_S(A, B)$ on the set of equivalence classes $\mathcal{V} := \mathrm{Four}_{G,S}(A, B)/\mathrm{cyc}$. Let us say that two equivalence classes $\overline{\phi}$ and $\overline{\psi}$ in $\mathcal{V}$ are *e-adjacent* if there are fourientation $\phi' \in \overline{\phi}$ and $\psi' \in \overline{\psi}$ such that ϕ' and ψ' coincide except that they have opposite orientations of e. Let $\mathcal{G}_S(A, B)$ be the graph (without multiple edges) with vertex set $\mathcal{V}$, where two classes are connected by an edge if and only if they are e-adjacent. See Figure 3.

We make a few observations and definitions. Observe that in each class $\overline{\phi} \in \mathcal{V}$ the edge e is either always cyclic or always acyclic. We refer to the class ϕ as either *e-cyclic* or *e-acyclic* accordingly. For a class $\overline{\phi} \in \mathcal{V}$ we denote by $\overline{\phi}_{\overrightarrow{e}}$ (resp. $\overline{\phi}_{\overleftarrow{e}}$) the subset of fourientations in $\overline{\phi}$ containing $\overrightarrow{e}$ (resp. $\overleftarrow{e}$). We call *$\mathcal{V}$-block* a non-empty subset of the form $\overline{\phi}_{\overrightarrow{e}}$ or $\overline{\phi}_{\overleftarrow{e}}$ for $\overline{\phi} \in \mathcal{V}$.

Lemma 4.7. *Let $\overline{\phi} \in \mathcal{V}$ be a class of fourientations. If $\overline{\phi}$ is e-acyclic (resp. e-cyclic), then it has degree at most 1 (resp. 2) in $\mathcal{G}_S(A, B)$.*

Proof. First note that the e-acyclic classes have a single $\mathcal{V}$-block while the e-cyclic classes have two $\mathcal{V}$-blocks.

Suppose that ϕ and ϕ' are in the same block of a class $\overline{\phi} \in \mathcal{V}$, and let ψ and ψ' be the fourientations obtained from ϕ and ϕ' by reversing the orientation of e. It can be shown that ϕ can be obtained from ϕ' by reversing some directed cycles not containing e. Hence the same holds for ψ and ψ', which implies that they are cycle equivalent; and if they are (A,B)-valid then they belong to the same $\mathcal{V}$-block. Hence the number of classes e-adjacent to the class $\overline{\phi}$ is at most its number of blocks. □

Lemma 4.8. *Each connected component of $\mathcal{G}_S(A,B)$ is a either a path, or a vertex $\overline{\phi}$ with a loop (which happens if e is the unique 1-way edge in a directed cycle of the fourientation ϕ).*

Proof. By Lemma 4.7, the graph $\mathcal{G}_S(A,B)$ must be a collection of paths or cycles. We need to show that a cycle of classes of length greater than 1 is not possible. Consider a set $\mathcal{U}$ of classes in $\mathcal{V}$ which are connected in the graph $\mathcal{G}_S(A,B)$. The set $T \subseteq S$ of 2-way edges is the same for every fourientation of every class in $\mathcal{U}$. Let (u,v) be the endpoints of $\overrightarrow{e}$ in the contracted graph G/T. If $u = v$ (which occurs when $\overrightarrow{e}$ is the only 1-way edge in a directed cycle) then it is easy to see that $\mathcal{U}$ is just one class with a loop in $\mathcal{G}_S(A,B)$. Assume now that $u \neq v$. For a class $\overline{\phi} \in \mathcal{U}$, the outdegree of u in the contracted fourientations ϕ'/T is the same for every fourientation ϕ' in the class (since it is unchanged by a cycle reversal). However this outdegree will change by 1 when going from a class $\overline{\phi} \in \mathcal{U}$ to an e-adjacent class (it increases by 1 when going from a class $\overline{\phi}$ to an e-adjacent class $\overline{\psi}$ obtained by replacing $\overleftarrow{e}$ by $\overrightarrow{e}$). In fact the outdegree of u will change monotonously along a simple path of $\mathcal{G}_S(A,B)$, hence a cycle of length greater than 1 is impossible. □

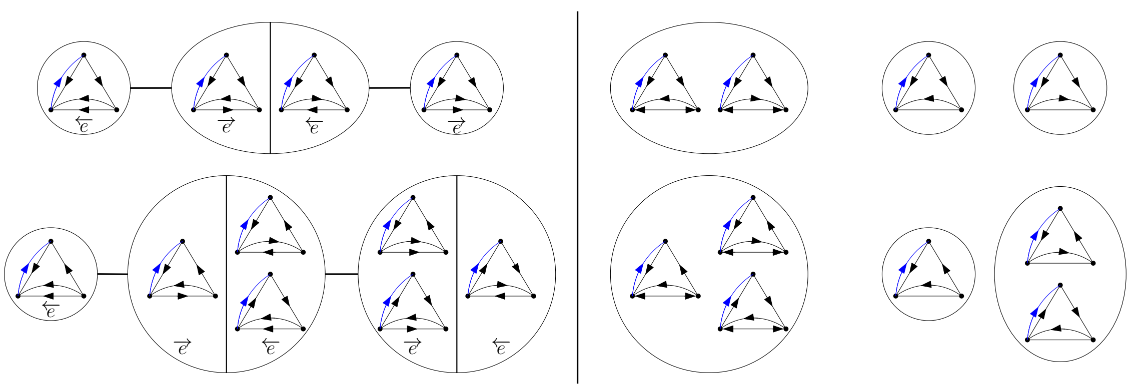

Figure 3: Left: The graph $\mathcal{G}_S(A,B)$ with 6 classes. Right: the resulting 6 (A,B)-valid classes of $\text{Four}_{G,S\cup\{e\}}$ after replacement with 2-way and 0-way e. The classes with configuration $\overleftrightarrow{e}$ correspond to the (A,B)-valid paths of $\mathcal{G}_S(A,B)$, while the classes with configuration $\mathring{e}$ correspond to the edges of $\mathcal{G}_S(A,B)$.

We will now prove (4.1). Let $\mathcal{V}' = \text{Four}_{G,S\cup\{e\}}(A,B)/\text{cyc}$. We need to prove $|\mathcal{V}| = |\mathcal{V}'|$. This will be done by describing how the cycle classes change (merge and split) if the 1-way arc $\overrightarrow{e}$ or $\overleftarrow{e}$ is replaced with a solid edge $\overleftrightarrow{e}$ or $\mathring{e}$. This is illustrated in Figure 3.

In short, we prove that each connected component of $\mathcal{G}_S(A,B)$ and each non-loop edge of $\mathcal{G}_S(A,B)$ give raise to one class in $\mathcal{V}'$. This readily implies $|\mathcal{V}| = |\mathcal{V}'|$. We now sketch the proof of the above statement.

Given a class $\overline{\phi} \in \mathcal{V}'$ where e is oriented as $\overset{\circ}{e}$, the set of fourientations obtained from from those in $\overline{\phi}$ by replacing $\overset{\circ}{e}$ by $\overrightarrow{e}$ (resp. $\overleftarrow{e}$) are all cyclic equivalent. By (3.6), at least one of these choices ($\overrightarrow{e}$ or $\overleftarrow{e}$) gives some (A, B)-valid fourientations, hence a $\mathcal{V}$-block. Given a class $\overline{\phi} \in \mathcal{V}'$ where e is oriented as $\overleftrightarrow{e}$, the set of (A, B)-valid fourientations obtained by replacing $\overleftrightarrow{e}$ by $\overrightarrow{e}$ or $\overleftarrow{e}$ form a (non-empty) union of $\mathcal{V}$-blocks all in the same connected component of $\mathcal{G}_S(A, B)$.

Conversely, for a connected component γ of $\mathcal{G}_S(A, B)$, we consider the effect of replacing the 1-way arc e with either $\overleftrightarrow{e}$ or $\overset{\circ}{e}$. When replacing e by $\overleftrightarrow{e}$, all the fourientations in γ become cycle equivalent. When replacing e by $\overset{\circ}{e}$, the fourientations in adjacent $\mathcal{V}$-blocks of γ become identified (they merge in $\mathcal{V}'$), but the fourientations in the same class of $\mathcal{V}$ having different orientations of e become cycle inequivalent (they split in $\mathcal{V}'$).

We now count the classes of $\mathcal{V}'$ coming from γ. If γ is a path with some edges, (3.6) implies that each edge of γ gives a (A, B)-valid class of $\mathcal{V}'$ with configuration $\overset{\circ}{e}$, and we also get one class of $\mathcal{V}'$ with configuration $\overleftrightarrow{e}$. If γ is an isolated vertex $\overline{\phi}$ without a loop, then by (3.6) exactly one choice among $\overleftrightarrow{e}$ or $\overset{\circ}{e}$ is (A, B)-valid and gives a class of $\mathcal{V}'$. Finally if γ is an isolated vertex $\overline{\phi}$ with a loop, then the edge e is the only 1-way edge in a directed cycle of ϕ, and since an entire cycle of 2-way edges is disallowed for (A, B)-valid classes, we may only replace e with the configuration $\overset{\circ}{e}$, resulting in a single class in $\mathcal{V}'$. This proves $|\mathcal{V}| = |\mathcal{V}'|$, hence Theorem 4.4(i).

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References

[1] K. Archer, I. M. Gessel, C. Graves, and X. Liang. "Counting acyclic and strong digraphs by descents". *Discrete Math.* **343**.11 (2020), 112041, 14 pp. DOI.

[2] S. Backman and S. Hopkins. "Fourientations and the Tutte polynomial". *Res. Math. Sci.* **4** (2017), Paper No. 18, 57 pp. DOI.

[3] E. Gioan. "Enumerating degree sequences in digraphs and a cycle-cocycle reversing system". *European J. Combin.* **28**.4 (2007), pp. 1351–1366. DOI.

[4] W. T. Tutte. *Graph Theory*. Cambridge Mathematical Library. Cambridge University Press, 2001.