Séminaire Lotharingien de Combinatoire **XX** (2026)
Article #30, 12 pp.
Proceedings of the 38th Conference on Formal Power Series and Algebraic Combinatorics (Seattle)

h^*-vectors for matroids

Christopher Eur*[1] and Alex Fink†[2] and Matt Larson‡[3]

[1] *Department of Mathematical Sciences, Carnegie Mellon University. Pittsburgh, PA. USA*
[2] *School of Mathematical Sciences, Queen Mary University of London. London. UK*
[3] *Department of Mathematics, Princeton University. Princeton, NJ. USA*

Abstract. The Hodge theory of matroids by Adiprasito–Huh–Katz resolved outstanding conjectures in matroid theory by establishing positivity properties for Chow rings of matroids. In algebraic geometry, a notion complementary to the Chow ring is the K-ring of a variety. We establish positivity properties for K-rings of matroids. Our results give a new proof of the nonnegativity of the omega invariant of a matroid, in support of Speyer's f-vector conjecture, and resolve the conjecture of Tohăneanu that higher order Orlik–Terao algebras are Cohen–Macaulay.

Keywords: matroids, h^*-vectors, tropical geometry, K-theory, Cohen–Macaulay rings

1 Introduction

Let $[n] = \{1, \ldots, n\}$ for a nonnegative integer n. For a d-dimensional lattice polytope P in $\mathbb{R}^{[n]}$ with the lattice $\mathbb{Z}^{[n]}$, it is a classical fact in Ehrhart theory that $k \mapsto |kP \cap \mathbb{Z}^{[n]}|$ is a polynomial in k of degree d, so one may define the *h^*-vector* $(h_0^*(P), \ldots, h_d^*(P))$ by

$$\sum_{k \geq 0} |kP \cap \mathbb{Z}^{[n]}| t^k = \frac{h_0^*(P) + h_1^*(P)t + \cdots + h_d^*(P)t^d}{(1-t)^{d+1}}.$$

Stanley [37] showed that the h^*-vector is nonnegative. Moreover, if P is *normal* (i.e. all lattice points in kP are sums of lattice points in P for every k), he showed that the h^*-vector is a Macaulay vector (also known as an M-vector or O-sequence). This means that the sequence $(h_0^*(P), \ldots, h_d^*(P))$ is nonnegative and satisfies certain explicit inequalities bounding how quickly it can grow; see [9, Theorem 4.2.10] for details.

*ceur@cmu.edu. Christopher Eur was partially supported by US NSF DMS-2246518.

†a.fink@qmul.ac.uk. Alex Fink received support from the Engineering and Physical Sciences Research Council [grant number EP/X001229/1], as well as the Institute for Advanced Study and the Fields Institute for Research in Mathematical Sciences.

‡mattlarson@princeton.edu. Matt Larson was supported by the Charles Simonyi Endowment and the Oswald Veblen Fund at the Institute for Advanced Study.

Let us explain Stanley's result in algebraic terms. Let $\Bbbk$ be a field, and let $\Bbbk^* = \Bbbk \setminus \{0\}$. For a vector space V, we denote the projective space of lines in V by $\mathbb{P}V$. A lattice polytope $P \subseteq \mathbb{R}^{[n]}$ defines a map

$$f_P \colon (\Bbbk^*)^{[n]} \to \mathbb{P}(\Bbbk^{P \cap \mathbb{Z}^{[n]}}) \quad \text{via} \quad (t_1, \ldots, t_n) \mapsto [t^m]_{m \in P \cap \mathbb{Z}^{[n]}},$$

where $t^m = t_1^{m_1} \cdots t_n^{m_n}$. Let Y_P be the Zariski closure of the image of f_P, and let $S(Y_P)_\bullet$ be its homogeneous coordinate ring. When P is normal, it is known that $\dim_\Bbbk S(Y_P)_k = |kP \cap \mathbb{Z}^{[n]}|$ for all $k \geq 0$ and that the subvariety $Y_P \subseteq \mathbb{P}(\Bbbk^{P \cap \mathbb{Z}^{[n]}})$ is *arithmetically Cohen–Macaulay* (i.e. $S(Y_P)_\bullet$ is Cohen–Macaulay). Now, we note the following result of Macaulay: for a finitely generated graded $\Bbbk$-algebra $S_\bullet = \bigoplus_{k \geq 0} S_k$ of Krull dimension $d+1$ with $S_0 = \Bbbk$, its *h^*-vector* $(h_0^*, \ldots, h_m^*)$ defined by $\sum_{k \geq 0} (\dim_\Bbbk S_k) t^k = \frac{h_0^* + h_1^* t + \cdots + h_m^* t^m}{(1-t)^{d+1}}$ is a Macaulay vector if $S_\bullet$ is Cohen–Macaulay and generated in degree 1. Lastly, it is easy to verify that if the function $k \mapsto \dim_\Bbbk S_k$ is a polynomial in k (necessarily of degree d, i.e. one less than the Krull dimension of $S_\bullet$), then $h_j^* = 0$ for all $j > d$.

One may restrict the domain of f_P to the intersection of $(\Bbbk^*)^{[n]}$ with an affine subspace of $\Bbbk^{[n]}$ and study the Zariski closure of the restricted image. Such objects have appeared in both combinatorics and algebraic geometry; see discussion after Theorem 1. One can ask whether the closure of the restricted image is arithmetically Cohen–Macaulay, and ask for a description of the combinatorics underlying the h^*-vector of its homogeneous coordinate ring. We affirmatively answer these questions in the context of *matroids* when P is a *generalized permutohedron*,[1] defined as follows.

A *generalized permutohedron*, introduced in [33], is a lattice polytope $P \subset \mathbb{R}^n$ satisfying the property that for every edge of P, there is a pair $\{i \neq j\} \subseteq [n]$ such that the edge is parallel to $\mathbf{e}_i - \mathbf{e}_j$. A *matroid* M of *rank* r on $[n]$ is a nonempty collection $\mathcal{B}$ of subsets of $[n]$ of cardinality r, called the set of *bases* of M, such that the polytope

$$P(\mathrm{M}) := \text{the convex hull of } \{\mathbf{e}_B : B \in \mathcal{B}\} \subset \mathbb{R}^{[n]}$$

is a generalized permutohedron, where $\mathbf{e}_S := \sum_{i \in S} \mathbf{e}_i \in \mathbb{R}^{[n]}$ denotes the sum of standard basis vectors indexed by a subset $S \subseteq [n]$. The polytope $P(\mathrm{M})$ is called the *base polytope* of M. We say that M is *loopless* if every $i \in [n]$ is contained in a basis of M. We point to [41, 32] as standard references on matroid theory and to [20] for the equivalence of the definition of matroids given here with theirs. The prototypical example of a matroid is given by a linear subspace $L \subseteq \Bbbk^{[n]}$ over a field $\Bbbk$, which defines a matroid of rank $r = \dim L$ with set of bases

$$\{B \subseteq [n] : \text{the composition of } L \hookrightarrow \Bbbk^{[n]} \text{ with the projection } \Bbbk^{[n]} \twoheadrightarrow \Bbbk^B \text{ is an isomorphism}\}.$$

[1] For the underlying geometry that provides a natural reason for restricting to generalized permutohedra rather than considering all polytopes, see Remark 7 as well as the discussion above Remark 5.

Matroids that arise in this way are called *realizable matroids*.

We will define h^*-vectors for a pair (M, P) of a loopless matroid M on $[n]$ and a generalized permutohedron $P \subseteq \mathbb{R}^{[n]}$, starting with the case of realizable matroids, and extending to all matroids via K-theory. Our main results state that the h^*-vectors are Macaulay vectors, leading to solutions for certain outstanding problems in matroid theory.

1.1 h^*-vectors for realizable matroids

Let $\mathbb{P}T := \{[t_1 : \ldots : t_n] \in \mathbb{P}(\Bbbk^{[n]}) : t_1 \cdots t_n \neq 0\}$ be the projective torus. To avoid trivialities, we assume that the realization $L \subseteq \Bbbk^{[n]}$ of a matroid satisfies $\mathbb{P}L \cap \mathbb{P}T \neq \emptyset$, or, equivalently, that the matroid of L is loopless. For a generalized permutohedron $P \subset \mathbb{R}^{[n]}$, define by $f_P(W_L)$ the Zariski closure of the image of the map

$$f_P \colon \mathbb{P}L \cap \mathbb{P}T \to \mathbb{P}(\Bbbk^{P \cap \mathbb{Z}^{[n]}}) \quad \text{via} \quad [t_1 : \ldots : t_n] \mapsto [t^m]_{m \in P \cap \mathbb{Z}^{[n]}}. \tag{1.1}$$

The notation $f_P(W_L)$ is explained in Remark 5. Our result for realizable matroids is the following.

Theorem 1. *The subvariety $f_P(W_L) \subseteq \mathbb{P}(\Bbbk^{P \cap \mathbb{Z}^n})$ is arithmetically Cohen–Macaulay. In particular, the h^*-vector of its homogeneous coordinate ring $S(f_P(W_L))_\bullet$ is a Macaulay vector.*

Theorem 3, to be stated later, will imply that the h^*-vector $(h_0^*, \ldots, h_m^*)$ of $S(f_P(W_L))_\bullet$ satisfies $h_j^* = 0$ for all $j > \dim f_P(W_L)$ and that the h^*-vector in fact depends only on the matroid that $L \subseteq \Bbbk^{[n]}$ realizes (having fixed the generalized permutohedron P). Special cases of this theorem recover results or resolve conjectures in the prior literature:

- When P is the convex hull of $\{\mathbf{e}_S : S \subset [n] \text{ and } |S| = n-1\}$, the homogeneous coordinate ring of $f_P(W_L)$ is known as the *Orlik–Terao algebra* [31, 38], and Theorem 1 recovers a main result of [34] that this algebra is Cohen–Macaulay.

- More generally, when P is the convex hull of $\{\mathbf{e}_S : S \subset [n] \text{ and } |S| = n-k\}$ for $1 \leq k \leq n-1$, the homogeneous coordinate ring of $f_P(W_L)$ is known as the *higher order Orlik–Terao algebra*, studied extensively in the commutative algebra literature [19, 40, 10]. In this case, Theorem 1 resolves and generalizes the conjecture of Tohăneanu that the higher order Orlik–Terao algebra for $k = 2$ is Cohen–Macaulay [39].

- When $P = -P(\mathrm{M})$, the convex hull of $\{-\mathbf{e}_B : B \text{ a basis of the matroid M of } L\}$, the variety $f_P(W_L)$ is known as *Kapranov's visible contours compactification*, appearing in the study of moduli of hyperplane arrangements [22, 21, 24]. This variety is the log-canonical model of a hyperplane arrangement complement associated to the data of $L \subseteq \Bbbk^{[n]}$. Theorem 1 states that this variety is arithmetically Cohen–Macaulay.

1.2 h^*-vectors for arbitrary matroids

As n grows, almost all matroids on $[n]$ are not realizable [30], yet geometric properties of realizable matroids sometimes persist for all matroids. A recent notable example is the Hodge theory of matroids [1], which established positivity properties for Chow rings of matroids. In algebraic geometry, the Chow ring of a variety has a counterpart known as the *K-ring*. Through K-theory, we define h^*-vectors for nonrealizable matroids in the absence of the subvariety $f_P(W_L)$.

Building upon [5], the authors of [27] defined the *matroid K-ring* $K(\mathrm{M})$ of a loopless matroid M, equipped with a map $\chi(\mathrm{M}, -)\colon K(\mathrm{M}) \to \mathbb{Z}$. We will explain the underlying geometry in Section 2. For now, key features of $\chi(\mathrm{M}, -)$ are as follows:

- every generalized permutohedron $P \subseteq \mathbb{R}^{[n]}$ defines a multiplicatively invertible element $\mathcal{L}_P$ in $K(\mathrm{M})$ such that $\mathcal{L}_P^k = \mathcal{L}_{kP}$ for $k \geq 0$,
- the function $k \mapsto \chi(\mathrm{M}, \mathcal{L}_P^k)$ is a polynomial in k, and
- when M is the *Boolean matroid* on $[n]$ (i.e. the ground set $[n]$ is a basis of M), we have $\chi(\mathrm{M}, \mathcal{L}_P) = |P \cap \mathbb{Z}^{[n]}|$.

We prove the following positivity property for K-rings of matroids, which was conjectured in [15, Conjecture 4.8].

Theorem 2. *Let* M *be a loopless matroid on* $[n]$*, and let* $P \subset \mathbb{R}^{[n]}$ *be a generalized permutohedron. Let* d *be the degree of the polynomial* $k \mapsto \chi(\mathrm{M}, \mathcal{L}_P^k)$*. Then, the* h^**-vector* $h^*(\mathrm{M}, \mathcal{L}_P)$ *defined by*

$$\sum_{k \geq 0} \chi(\mathrm{M}, \mathcal{L}_P^k) t^k = \frac{h_0^*(\mathrm{M}, \mathcal{L}_P) + h_1^*(\mathrm{M}, \mathcal{L}_P) t + \cdots + h_d^*(\mathrm{M}, \mathcal{L}_P) t^d}{(1-t)^{d+1}}$$

is a Macaulay vector.

An explicit combinatorial formula for the degree d of the polynomial $k \mapsto \chi(\mathrm{M}, \mathcal{L}_P^k)$ is given in Theorem 10. If $P \subset \mathbb{R}^{[n]}$ has the maximal dimension $(n-1)$, then $d = \mathrm{rank}(\mathrm{M}) - 1$. We also show that the h^*-vector $h^*(\mathrm{M}, \mathcal{L}_P)$ defined above via $\chi(\mathrm{M}, -)$ agrees with the realizable case in the following way.

Theorem 3. *If* M *admits a realization* $L \subseteq \mathbb{k}^{[n]}$*, then* $\chi(\mathrm{M}, \mathcal{L}_P^k)$ *is equal to the dimension* $\dim_{\mathbb{k}} S(f_P(W_L))_k$ *of the k-th graded component of the homogeneous coordinate ring of* $f_P(W_L)$*. In particular, the* h^**-vector of* $S(f_P(W_L))_\bullet$ *is equal to* $h^*(\mathrm{M}, \mathcal{L}_P)$*.*

We remark that the proof of this theorem is not a straightforward consequence of the definition of the map $\chi(\mathrm{M}, -)$; it involves showing certain cohomology vanishing results for $f_P(W_L)$, as explained in Section 2.

A consequence of Theorem 2 is a new proof of the nonnegativity of the omega invariant of a matroid, in support of the 20-year-old f-vector conjecture of Speyer [35], as we now explain. If P is a generalized permutohedron, then the negated polytope $-P = \{-p : p \in P\}$ is as well. Define the *omega invariant* $\omega(\mathrm{M}) \in \mathbb{Z}$ of a loopless matroid M of rank r by

$$\omega(\mathrm{M}) = (-1)^{r-n+\dim P(\mathrm{M})}\chi(\mathrm{M}, \mathcal{L}^{-1}_{-P(\mathrm{M})}).$$

The ω invariant is the leading coefficient of Speyer's invariant $g_{\mathrm{M}}(t) \in \mathbb{Z}[t]$, which was defined in [36, 17] in an attempt to bound the complexity of polyhedral subdivisions of the base polytope $P(\mathrm{M})$ of a matroid M into base polytopes of matroids. Such subdivisions had arisen in the study of Grassmannians and moduli spaces of hyperplane arrangements [26, 22, 21]. Speyer conjectured an upper bound on the number of faces of each dimension of such a subdivision, the titular f-vector. He reduced the conjecture to showing that the coefficients of $g_{\mathrm{M}}(t)$ were of predictable sign, and he proved this in [36] for matroids realizable over fields of characteristic zero. Our result shows the nonnegativity of the omega invariant, which was first proved in the recent preprint [6].

Corollary 4. *For any loopless matroid* M*, one has* $\omega(\mathrm{M}) \geq 0$.

Proof of Corollary 4. Because $P(\mathrm{M})$ and $-P(\mathrm{M})$ induce the same partition of $[n]$ in the sense explained in Section 4, Corollary 11 implies that the polynomial $k \mapsto \chi(\mathrm{M}, \mathcal{L}^k_{-P(\mathrm{M})})$ has degree $d := r - n + \dim P(\mathrm{M})$. Then $(-1)^d\chi(\mathrm{M}, \mathcal{L}^{-1}_{-P(\mathrm{M})}) = h^*_d(\mathrm{M}, \mathcal{L}_{-P(\mathrm{M})})$, which is nonnegative by Theorem 1. □

While our work and [6] both contain Corollary 4 as a highlighted application, the techniques of [6] cannot be used to prove Theorems 1, 2, and 3 here, nor can those theorems be used to deduce the theorem [6, Theorem D] that the authors of [6] use to prove Corollary 4.

This is an extended abstract of the paper [14]. In Section 2, we recast the theorems from the introduction into algebro-geometric terms. In Section 3, we sketch the main ideas of the proof. In Section 4, we state a formula for the degree of the polynomial $\chi(\mathrm{M}, \mathcal{L}^k_P)$.

The study of inequalities on h^*-vectors of normal polytopes continues to be active, with many outstanding questions and partial results; see [16] for a survey. It would be interesting to pursue similar questions for the h^*-vector $h^*(\mathrm{M}, \mathcal{L}_P)$ of a pair (M, P) of a matroid and a generalized permutohedron that we discuss here. A `Sage` program to compute these h^*-vectors in special cases can be found here, whereas a `Macaulay2` program can be found here.

2 Underlying geometry: wonderful varieties

Our main theorems are deduced by studying a geometric model of realizable matroids known as the *wonderful variety* W_L of $L \subseteq \Bbbk^{[n]}$, introduced by de Concini and Procesi [11]. It is a smooth subvariety of a toric variety, as we now explain. We point to [12] as a standard reference on toric geometry and adopt its conventions.

The *permutohedral variety (of dimension $n-1$)*, denoted $X_{[n]}$, is a smooth projective toric variety whose fan is the normal fan of the *permutohedron*

$$\text{the convex hull of } \{(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^{[n]} : \sigma \text{ a permutation of } [n]\}.$$

$X_{[n]}$ contains the projective torus $\mathbb{P}T$ as its open dense torus. As before, we assume that $\mathbb{P}L \cap \mathbb{P}T \neq \emptyset$, or, equivalently, that the matroid of $L \subseteq \Bbbk^{[n]}$ is loopless. The *wonderful variety* W_L is the closure of $\mathbb{P}L \cap \mathbb{P}T$ inside $X_{[n]}$. It is smooth and can be described also as an iterated blow-up of $\mathbb{P}L$ [11, Proposition 1.6].

Let $K(W_L)$ be the Grothendieck ring of vector bundles on W_L, also known as the K-ring. It is equipped with the sheaf Euler characteristic map $\chi(W_L, -)\colon K(W_L) \to \mathbb{Z}$ defined by $\chi(W_L, \mathcal{E}) = \sum_{k\geq 0}(-1)^k \dim_{\Bbbk} H^k(W_L; \mathcal{E})$ for a vector bundle $\mathcal{E}$ on W_L. The K-ring $K(\mathrm{M})$ of a matroid M introduced in [27] is modeled after $K(W_L)$; when $L \subseteq \Bbbk^{[n]}$ realizes a matroid M, there is an identification of $K(\mathrm{M})$ with $K(W_L)$ and $\chi(\mathrm{M}, -) = \chi(W_L, -)$. Moreover, the ring $K(\mathrm{M})$ is equipped with a surjection $K(X_{[n]}) \to K(\mathrm{M})$ which, when L realizes M, agrees with the pullback map along the inclusion $W_L \hookrightarrow X_{[n]}$.

"K-theoretic positivity" generally refers to nonnegativity of χ in the presence of ample or nef line bundles; see [28] for a reference. In our case, a natural collection of nef line bundles is supplied by nef line bundles on the ambient toric variety $X_{[n]}$. Applying to $X_{[n]}$ the standard dictionary between toric nef divisors and polytopes [12, Chapter 6], one obtains a correspondence between nef toric divisor classes on $X_{[n]}$ and generalized permutohedra $P \subseteq \mathbb{R}^{[n]}$ (see [5, Section 2.7] and [2] for details of this correspondence). Let $\mathcal{L}_P$ denote the corresponding nef line bundle on $X_{[n]}$, and also write $\mathcal{L}_P$ for the restriction of the line bundle to W_L. Nef line bundles on projective toric varieties are basepoint-free, and the complete linear system of $\mathcal{L}_P$ induces a map $f_P\colon X_{[n]} \to \mathbb{P}(\Bbbk^{P\cap\mathbb{Z}^n})$.

Remark 5. The map f_P here is the extension to $X_{[n]}$ of the map f_P in the introduction where it was defined only on the open dense torus $\mathbb{P}T$ of $X_{[n]}$. In particular, as $\mathbb{P}L \cap \mathbb{P}T$ is by definition a dense open subvariety of W_L, its image $f_P(W_L)$ is the closure of the image of $\mathbb{P}L \cap \mathbb{P}T$ under f_P.

Our main result is the following, which implies Theorems 1 and 3.

Theorem 6. *Let P be a generalized permutohedron in $\mathbb{R}^n$. Then:*

(1) $H^i(W_L, \mathcal{L}_P^{\otimes k}) = 0$ *unless either:* (i) $k \geq 0$ *and* $i = 0$, *or* (ii) $k < 0$ *and* $i = \dim f_P(W_L)$.

(2) *For any $k \geq 0$, the restriction map $H^0(X_{[n]}, \mathcal{L}_P^{\otimes k}) \to H^0(W_L, \mathcal{L}_P^{\otimes k})$ is surjective.*

(3) *We have $Rf_{P*}\mathcal{O}_{W_L} = \mathcal{O}_{f_P(W_L)}$.*

Proof of Theorems 1 & 3 from Theorem 6. The projection formula and (3) imply that the natural map $H^i(f_P(W_L), \mathcal{O}(k)) \to H^i(W_L, \mathcal{L}_P^{\otimes k})$ is an isomorphism for all i and k. In particular, the section ring $\bigoplus_{k\geq 0} H^0(W_L, \mathcal{L}_P^{\otimes k})$ and the section ring $\bigoplus_{k\geq 0} H^0(f_P(W_L), \mathcal{O}(k))$ of $\mathcal{O}(1)$ on $\mathbb{P}(\Bbbk^{P\cap\mathbb{Z}^n})$ are naturally isomorphic. Because a generalized permutohedron P is a normal lattice polytope [41, Theorem 18.6.3], the homogeneous coordinate ring of $\mathbb{P}(\Bbbk^{P\cap\mathbb{Z}^n})$ surjects onto the section ring $\bigoplus_{k\geq 0} H^0(X_{[n]}, \mathcal{L}_P^{\otimes k})$, and therefore (2) implies that the section ring $\bigoplus_{k\geq 0} H^0(W_L, \mathcal{L}_P^{\otimes k})$ is the homogeneous coordinate ring of the subvariety $f_P(W_L)$ in $\mathbb{P}(\Bbbk^{P\cap\mathbb{Z}^n})$. The theorems now follow from the cohomology vanishing (1): part (i) of (1) implies that $\chi(\mathrm{M}, \mathcal{L}_P^k) = \chi(W_L, \mathcal{L}_P^k) = \dim_\Bbbk H^0(W_L, \mathcal{L}_P^k)$ for all $k \geq 0$, proving Theorem 3, and parts (i) and (ii) of (1) together imply that $f_P(W_L)$ is arithmetically Cohen–Macaulay (see, e.g., [15, Proposition 4.5]), proving Theorem 1. □

Examples where the conclusion of Theorem 6 fails for other classes of nef or ample line bundles on wonderful varieties can be found in [14, Section 5]. In this sense, Theorem 6 appears to be sharp.

Remark 7. The conclusion of Theorem 1 can fail for normal lattice polytopes P that are not generalized permutohedra. For example, when $L = \{x + y + z = 0\} \subset \Bbbk^{[3]}$ and P is the convex hull of $\{(4,0,0), (3,1,0), (1,2,1), (0,3,1)\}$, the image $f_P(W_L)$ is a rational quartic curve in $\mathbb{P}^3$, whose homogeneous coordinate ring has depth 1 (and hence neither normal nor Cohen–Macaulay). Moreover, when P is not a generalized permutohedron, the Hilbert function of $S_\bullet(f_P(W_L))$ may not be determined by the matroid of L and P. For example, let P be the convex hull of $\{\mathbf{e}_{156}, \mathbf{e}_{678}, \mathbf{e}_{468}, \mathbf{e}_{256}, \mathbf{e}_{278}, \mathbf{e}_{137}, \mathbf{e}_{127}, \mathbf{e}_{459}, \mathbf{e}_{145}\}$ in $\mathbb{R}^{[9]}$, and let L_1 and L_2 respectively be the row spans of the matrices

$$\begin{bmatrix} 0 & 3 & 21 & 84 & 240 & 0 & 0 & 12 & 60 \\ 1 & 4 & 10 & 19 & 32 & 20 & 7 & 13 & 23 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & -1 & 1 & -7 & 0 & 0 & 3 & -3 \\ 0 & 0 & 1 & -2 & 5 & 1 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 & -1 & -3 & 1 & -1 & 0 \end{bmatrix}.$$

The matroid of L_1 is equal to the matroid of L_2. Then, the Hilbert function of $S_\bullet(f_P(W_{L_1}))$ is $(1, 9, 27, 54, \dots)$, while that of $S_\bullet(f_P(W_{L_2}))$ is $(1, 8, 27, 54, \dots)$. Both coordinate rings are neither normal nor Cohen–Macaulay.

3 Sketch of the proof

We now sketch the proof of Theorems 6 and 2 in the case where $\mathcal{L}_P$ is *ample*, i.e. P has the same normal fan as that of the permutohedron. While restricting to the ample case avoids nontrivial difficulties that arise in the general case, the proof for the ample case nonetheless contains many of the main ideas.

The main objective is to degenerate W_L to a reduced and Cohen–Macaulay union of toric boundary strata in $X_{[n]}$. Then, Theorem 6 would follow from known arguments using upper semicontinuity and (toric) Frobenius splitting techniques [18, 8]. Furthermore, if the degeneration is combinatorially determined by the matroid that L realizes, one can synthetically define the "degeneration" for an arbitrary, not necessarily realizable, matroid by assigning to each matroid M a certain reduced union of toric boundary strata in $X_{[n]}$. Then, Theorem 2 would follow if such a synthetic degeneration is Cohen–Macaulay and its sheaf Euler characteristic map χ agrees with $\chi(\mathrm{M}, -)$.

Step 1.

The torus $T = (\Bbbk^*)^{[n]}$ acts on $X_{[n]}$. For $w \in \mathbb{Z}^{[n]}$, which defines a one-parameter subgroup $\lambda^w\colon \Bbbk^* \to T$ by $t \mapsto (t^{w_1}, \dots, t^{w_n})$, one considers the flat limit $\lim_{t\to 0} \lambda^w(t) \cdot W_L$, denoted $\mathrm{ind}_w(W_L)$, where "ind" stands for tropical <u>in</u>itial <u>d</u>egeneration. When w is sufficiently general, $\mathrm{ind}_w(W_L)$ is a degeneration of W_L into a (possibly nonreduced) union of toric boundary strata of $X_{[n]}$.

Step 2.

In [23, Theorem 10.1], Katz provides a combinatorial description of the strata which occur in a tropical initial degeneration in terms of tropical intersection theory. In our case, one obtains a combinatorial description for $\mathrm{ind}_w(W_L)$ in terms of the *Bergman fan* [3] of the matroid M that L realizes. This allows one to assign to each M a reduced union of toric boundary strata, denoted $\mathrm{ind}_w(\mathrm{M})$, such that if M has a realization L then $\mathrm{ind}_w(\mathrm{M})$ equals $\mathrm{ind}_w(W_L)$ at least as sub*sets* of $X_{[n]}$.

Step 3.

One notes that W_L is a multiplicity-free irreducible subvariety in the sense of [7] when $X_{[n]}$ is embedded in a product of projective lines, by using [29] or [4]. This allows one to show that $\mathrm{ind}_w(W_L)$ is reduced and Cohen–Macaulay, and completes the proof of Theorem 6 when P is ample.

Step 4.

For any line bundle $\mathcal{L}$ on $X_{[n]}$, one has $\chi(W_L, \mathcal{L}) = \chi(\mathrm{ind}_w(W_L), \mathcal{L})$ because χ is locally constant along flat degenerations. One wishes to show further that $\chi(\mathrm{M}, \mathcal{L}) = \chi(\mathrm{ind}_w(\mathrm{M}), \mathcal{L})$. One can deduce this by using Knutson's formula [25] for computing the

K-class of the structure sheaf of $\mathrm{ind}_w(\mathrm{M})$ as a subscheme of $X_{[n]}$, and by using valuativity for matroid base polytopes [13]. The previous (Step 3) is an essential input; one needs that $\mathrm{ind}_w(\mathrm{M})$ equals $\mathrm{ind}_w(W_L)$ as sub*schemes* of $X_{[n]}$, not just as subsets.

Step 5.

One now wishes to deduce that $\mathrm{ind}_w(\mathrm{M})$ is also Cohen–Macaulay, from which Theorem 2 would follow when P is ample. However, the argument in (Step 3) via [7] does not generalize easily to the case of nonrealizable matroids. Overcoming this is one of the most delicate parts of our work. The key notion we introduce is *kindred subschemes*, defined as follows. A set is *independent* in a matroid M if it is a subset of a basis of M.

Definition 8. *A closed subscheme X of $(\mathbb{P}^1)^\ell$ is said to be* kindred *if it is empty, or if there is a matroid* M *on $[\ell]$ such that the multigraded Hilbert polynomial $\chi(X, \mathcal{O}(a_1, \ldots, a_\ell))$ is equal to*

$$\sum_{I \text{ independent in } \mathrm{M}} \prod_{i \in I} a_i.$$

It follows from (Step 4) and a formula [27, Corollary 7.5] for $\chi(\mathrm{M}, \mathcal{L})$ for special $\mathcal{L} \in K(\mathrm{M})$ that $\mathrm{ind}_w(\mathrm{M})$ is kindred. The key properties of kindred subschemes that we establish are the following.

Theorem 9. *Let $X \subseteq (\mathbb{P}^1)^{[\ell]}$ be a kindred subscheme. Then, the following hold:*

(1) X is reduced and Cohen–Macaulay.

(2) For any coordinate projection $p\colon (\mathbb{P}^1)^\ell \to (\mathbb{P}^1)^m$, the image $p(X)$ is a kindred subscheme of $(\mathbb{P}^1)^m$ and $Rp_*\mathcal{O}_X = \mathcal{O}_{p(X)}$.

Part (1) of the theorem completes (Step 5), and thereby Theorem 2 is proved when P is ample. Part (2) of the theorem is a crucial input for extending the steps laid out here to the general case where P is an arbitrary, not necessarily ample, generalized permutohedron.

4 A formula for the degree

We now state a combinatorial formula for the degree of the polynomial $k \mapsto \chi(\mathrm{M}, \mathcal{L}_P^k)$, which is an essential part of the definition of the h^*-vector $h^*(\mathrm{M}, \mathcal{L}_P)$. The formula will imply that the degree only depends on the lineality space of the normal fan of P.

A generalized permutohedron P in $\mathbb{R}^{[n]}$ induces a partition $[n] = S_1 \sqcup \cdots \sqcup S_\ell$, where the sets are the equivalence classes induced by the equivalence relation generated by setting $i \sim j$ if there is an edge of direction $\mathbf{e}_i - \mathbf{e}_j$ in P. The parts of the partition induced by the base polytope $P(\mathrm{M})$ of a matroid M are usually called the *connected components* of M.

Theorem 10. *Let* M *be a loopless matroid on* $[n]$*, and let* P *be a generalized permutohedron with induced partition* $[n] = S_1 \sqcup \cdots \sqcup S_\ell$*. Then the degree of the polynomial* $k \mapsto \chi(\mathrm{M}, \mathcal{L}_P^k)$ *is the minimum over partitions* $[n] = T_1 \sqcup \cdots \sqcup T_k$ *coarsening* $[n] = S_1 \sqcup \cdots \sqcup S_\ell$ *of the quantity* $\sum_{i=1}^k (\mathrm{rk}_\mathrm{M}(T_i) - 1)$.

Our proof of Theorem 10 uses properties of the Chow ring of a loopless matroid M in an essential way, namely the validity of Poincaré duality and Hodge–Riemann relations in degree 0 [1]. A generalized permutohedron defines an element $c_1(\mathcal{L}_P)$ in the Chow ring of a loopless matroid M, and we prove the theorem by relating the degree to the *numerical dimension* of $c_1(\mathcal{L}_P)$, i.e. the largest d such that $c_1(\mathcal{L}_P)^d \neq 0$ in the Chow ring of M. It may be interesting to find a proof of the theorem that does not utilize Chow rings of matroids. We record a case in which the formula of Theorem 10 is easy to evaluate.

Corollary 11. *If* M *is a matroid of rank* r *on* $[n]$ *and* P *a generalized permutohedron such that the induced partition* $S_1 \sqcup \cdots \sqcup S_\ell$ *of* P *coarsens the induced partition of* $P(\mathrm{M})$ *(into connected components), then the degree of the polynomial* $k \mapsto \chi(\mathrm{M}, \mathcal{L}^k)$ *equals* $r - \ell = r - n + \dim P$.

Proof. The rank function rk_M is additive on unions of connected components [32, Fact 4.2.13]. Any partition $T_1 \sqcup \cdots \sqcup T_k$ coarsening the induced partition of P also coarsens the partition into connected components of M, and therefore

$$\sum_{i=1}^{k} (\mathrm{rk}_\mathrm{M}(T_i) - 1) = \mathrm{rk}_\mathrm{M}([n]) - k = r - k$$

which is minimized by maximizing k, i.e., taking $T_\bullet$ and $S_\bullet$ identical with $k = \ell$. □

Acknowledgements

We thank Jenia Tevelev, Ronnie Cheng, Louis Esser, and Michel Brion for helpful conversations. We thank the Mathematischen Forschungsinstitut Oberwolfach for hosting the 2025 Toric Geometry workshop where this work began.

References

[1] K. Adiprasito, J. Huh, and E. Katz. "Hodge theory for combinatorial geometries". *Ann. of Math. (2)* **188**.2 (2018), pp. 381–452. DOI.

[2] F. Ardila, F. Castillo, C. Eur, and A. Postnikov. "Coxeter submodular functions and deformations of Coxeter permutahedra". *Adv. Math.* **365** (2020), p. 107039. DOI.

[3] F. Ardila and C. J. Klivans. "The Bergman complex of a matroid and phylogenetic trees". *J. Combin. Theory Ser. B* **96**.1 (2006), pp. 38–49. DOI.

[4] S. Backman, C. Eur, and C. Simpson. "Simplicial generation of Chow rings of matroids". *J. Eur. Math. Soc. (JEMS)* **26**.11 (2024), pp. 4491–4535. DOI.

[5] A. Berget, C. Eur, H. Spink, and D. Tseng. "Tautological classes of matroids". *Invent. Math.* **233**.2 (2023), pp. 951–1039. DOI.

[6] A. Berget and A. Fink. "The external activity complex of a pair of matroids" (2024). arXiv: 2412.11759v2.

[7] M. Brion. "Multiplicity-free subvarieties of flag varieties". *Commutative algebra (Grenoble/Lyon, 2001)*. Vol. 331. Contemp. Math. Amer. Math. Soc., Providence, RI, 2003, pp. 13–23. DOI.

[8] M. Brion and S. Kumar. *Frobenius splitting methods in geometry and representation theory*. Vol. 231. Progress in Mathematics. Birkhäuser Boston Inc., Boston MA, 2005, pp. x+250.

[9] W. Bruns and J. Herzog. *Cohen-Macaulay rings*. Vol. 39. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1993, pp. xii+403.

[10] R. Burity. "On a-fold products ideals of hyperplane arrangements". *Bull. Sci. Math.* **189** (2023), Paper No. 103359, 20. DOI.

[11] C. de Concini and C. Procesi. "Wonderful models of subspace arrangements". *Selecta Math. (N.S.)* **1**.3 (1995), pp. 459–494. DOI.

[12] D. A. Cox, J. B. Little, and H. K. Schenck. *Toric varieties*. Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011, pp. xxiv+841. DOI.

[13] H. Derksen and A. Fink. "Valuative invariants for polymatroids". *Adv. Math.* **225**.4 (2010), pp. 1840–1892. DOI.

[14] C. Eur, A. Fink, and M. Larson. "Vanishing theorems for combinatorial geometries" (2025). arXiv:2510.05207.

[15] C. Eur and M. Larson. "K-theoretic positivity for matroids". *Alg. Geom. (to appear)* (2023). arXiv:2311.11996v2. Link.

[16] L. Ferroni and A. Higashitani. "Examples and counterexamples in Ehrhart theory". *EMS Surv. Math. Sci.* (2024). DOI.

[17] A. Fink and D. E. Speyer. "K-classes for matroids and equivariant localization". *Duke Math. J.* **161**.14 (2012), pp. 2699–2723. DOI.

[18] O. Fujino. "Multiplication maps and vanishing theorems for toric varieties". *Math. Z.* **257**.3 (2007), pp. 631–641. DOI.

[19] M. Garrousian, A. Simis, and Ş. O. Tohăneanu. "A blowup algebra for hyperplane arrangements". *Algebra Number Theory* **12**.6 (2018), pp. 1401–1429. DOI.

[20] I. M. Gelfand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova. "Combinatorial geometries, convex polyhedra, and Schubert cells". *Adv. in Math.* **63**.3 (1987), pp. 301–316.

[21] P. Hacking, S. Keel, and J. Tevelev. "Compactification of the moduli space of hyperplane arrangements". *J. Algebraic Geom.* **15**.4 (2006), pp. 657–680. DOI.

[22] M. M. Kapranov. "Chow quotients of Grassmannians. I". *I. M. Gelfand Seminar*. Vol. 16. Adv. Soviet Math. Amer. Math. Soc., Providence, RI, 1993, pp. 29–110.

[23] E. Katz. "A tropical toolkit". *Expo. Math.* **27**.1 (2009), pp. 1–36. DOI.

[24] S. Keel and J. Tevelev. "Geometry of Chow quotients of Grassmannians". *Duke Math. J.* **134**.2 (2006), pp. 259–311. DOI.

[25] A. Knutson. "Frobenius splitting and Möbius inversion" (2009). arXiv:0902.1930v1.

[26] L. Lafforgue. *Chirurgie des grassmanniennes*. Vol. 19. CRM Monograph Series. American Mathematical Society, Providence, RI, 2003, pp. xx+170. DOI.

[27] M. Larson, S. Li, S. Payne, and N. Proudfoot. "K-rings of wonderful varieties and matroids". *Adv. Math.* **441** (2024), Paper No. 109554, 43. DOI.

[28] R. Lazarsfeld. *Positivity in algebraic geometry. I*. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004, pp. xviii+387. DOI.

[29] B. Li. "Images of rational maps of projective spaces". *Int. Math. Res. Not. IMRN* 13 (2018), pp. 4190–4228. DOI.

[30] P. Nelson. "Almost all matroids are nonrepresentable". *Bull. Lond. Math. Soc.* **50**.2 (2018), pp. 245–248. DOI.

[31] P. Orlik and H. Terao. "Commutative algebras for arrangements". *Nagoya Math. J.* **134** (1994), pp. 65–73. DOI.

[32] J. Oxley. *Matroid theory*. 2nd ed. Vol. 21. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2011, pp. xiv+684.

[33] A. Postnikov. "Permutohedra, associahedra, and beyond". *Int. Math. Res. Not. IMRN* 6 (2009), pp. 1026–1106. DOI.

[34] N. Proudfoot and D. E. Speyer. "A broken circuit ring". *Beiträge Algebra Geom.* **47**.1 (2006), pp. 161–166.

[35] D. E. Speyer. "Tropical geometry". PhD thesis. University of California, Berkeley, 2005.

[36] D. E. Speyer. "A matroid invariant via the K-theory of the Grassmannian". *Adv. Math.* **221**.3 (2009), pp. 882–913. DOI.

[37] R. P. Stanley. "A monotonicity property of h-vectors and h^*-vectors". *European J. Combin.* **14**.3 (1993), pp. 251–258. DOI.

[38] H. Terao. "Algebras generated by reciprocals of linear forms". *J. Algebra* **250**.2 (2002), pp. 549–558. DOI.

[39] Ş. O. Tohăneanu. "On ideals generated by a-fold products of linear forms". *J. Commut. Algebra* **13**.4 (2021), pp. 549–570. DOI.

[40] Ş. O. Tohăneanu and Y. Xie. "On the Geramita-Harbourne-Migliore conjecture". *Trans. Amer. Math. Soc.* **374**.6 (2021), pp. 4059–4073. DOI.

[41] D. J. A. Welsh. *Matroid theory*. L. M. S. Monographs, No. 8. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976, pp. xi+433. Link.