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On the finiteness of the group associated with weighted walks in multidimensional orthants

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Abstract. In the study of walks with small steps confined to multidimensional orthants, a certain group of transformations plays a central role. In particular, several techniques to potentially compute the generating function, including the orbit sum method, can only be applied when this group is finite. In this extended abstract, we present three new results concerning this group. First, in two dimensions, we provide a complete characterization of the weight parameters that yield a finite group. In higher dimensions, we show that whenever the group is finite, it must necessarily be isomorphic to a simpler reflection group. Finally, in three dimensions, we give a full classification of the parameters leading to a finite group that also satisfies an additional Weyl property.

Résumé. Dans l'étude des marches restreintes à des orthants multidimensionnels, il s'avère qu'un certain groupe de transformations joue un rôle crucial, en particulier au travers de son éventuelle finitude. En effet, si ce groupe est fini, plusieurs méthodes peuvent être appliquées (comme la somme sur l'orbite) pour calculer la fonction génératrice d'intérêt. Dans cette note nous présentons trois nouveaux résultats sur ce groupe. Tout d'abord, en dimension deux, nous donnons une caractérisation complète des poids conduisant à un groupe fini. En dimension supérieure, nous montrons que si le groupe est fini, il est nécessairement isomorphe à un groupe de réflexions plus simple. Enfin, en dimension trois, nous proposons une classification complète des paramètres menant à un groupe fini possédant en outre une propriété de type Weyl.

Keywords: Walks in orthants, group of the walk, reflection groups; power series

1 Introduction

A lattice walk is a sequence of points $P_0, P_1, \dots, P_n$ of $\mathbb{Z}^d$, $d \geqslant 1$. The points P_0 and P_n are its starting and end points, respectively, the consecutive differences $P_{i+1} - P_i$ its steps, and n is its length. Given a set $\mathcal{S} \subset \mathbb{Z}^d$, called the step-set, a set $C \subset \mathbb{R}^d$

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called the domain (which in this paper will systematically be the cone $\mathbb{R}_+^d$, called the d-dimensional orthant), and elements P and Q of C, we are interested in the number $e_C(P, Q; n)$ of (possibly weighted) walks of length n that start at $P = P_0$, have all their steps in $\mathcal{S}$, have all their points in C, and end at $Q = P_n$. Normalising the weights (which are assumed to be non-negative) with the condition that they sum to one, we obtain transition probabilities, and the number $e_C(P, Q; n)$ can be interpreted as the probability that a random walk starting at P will reach the point Q at time n while remaining in the domain C.

In the last twenty years, there has been a dense research activity in the mathematical community on the enumerative aspects of walks confined to cones, in particular to the d-dimensional orthant. To summarise, three main questions have attracted most attention: the first is to determine, if possible, a closed-form formula for the number of walks $e_C(P, Q; n)$. Of course, such an explicit formula is not expected to exist in general, and in most cases can be explained by bijections with other combinatorial objects. The second question concerns the asymptotic behaviour, e.g. of the number of walks $e_C(P, Q; n)$, in the regime where the length $n \to \infty$. The last question focuses on the complexity of generating functions associated with these models, such as the series

$$\sum_{n \geqslant 0} e_C(P, Q; n) t^n. \tag{1.1}$$

One then asks whether the series satisfies any algebraic or differential equation. Answering this question allows us to classify the models according to the complexity of their generating function. The three questions are by no means independent: for example the possible asymptotic forms of the numbers $e_C(P, Q; n)$ depend heavily on the complexity of the associated generating function.

To present the main results of this paper, we recall a tool that has played a crucial role in obtaining several previous results in the literature: the *group* of the walk model. This is a group that was first introduced in two dimensions, in a combinatorial context, in [3], following the idea of Fayolle, Iasnogorodski and Malyshev in [6]. This group will be properly defined later in the paper, but can be presented informally as a symmetry group of involutions defined by the step-set $\mathcal{S}$ of the model. The main application is that, if the group is finite, its action on a functional equation naturally associated with the model can lead to explicit expressions for the generating functions (1.1) as well as information on the asymptotics and algebraic complexity. In principle, this method works in two dimensions [6, 3] and, in higher dimensions, may apply in a few favorable cases [2, 11, 12, 4]. However, there is no known criterion to determine whether the group is finite (even in two dimensions). This is precisely the question we address in this paper.

2 Main results

We work under the small step-set assumption, that is, $\mathcal{S} \subset \{-1,0,1\}^d$. Define the inventory of the model $\mathcal{S}$ as follows:

$$\chi_{\mathcal{S}}(x_1,\ldots,x_d) = \sum_{(i_1,\ldots,i_d)\in\mathcal{S}} w(i_1,\ldots,i_d)x_1^{i_1}\cdots x_d^{i_d}, \tag{2.1}$$

where $w(i_1,\ldots,i_d) > 0$ is the weight of the step $(i_1,\ldots,i_d) \in \mathcal{S}$. We will normalize the non-negative weights in such a way that $\chi_{\mathcal{S}}(1,\ldots,1) = 1$, so that they are also transition probabilities. Most of the time we shall assume that the step-set satisfies the following irreducibility assumption:

(H1) The step-set $\mathcal{S}$ is not included in any half-space $\{y \in \mathbb{R}^d : \langle x,y\rangle \geqslant 0\}$ with $x \in \mathbb{R}^d \setminus \{0\}$, $\langle\cdot,\cdot\rangle$ denoting the classical Euclidean inner product.

A consequence of (H1) is that the model is truly d-dimensional and is not directed towards a half-space, unlike, for example, the singular walks considered in [3].

2.1 The combinatorial group

This group was first introduced in the context of two-dimensional walks [6, 3] and turns out to be very useful. Let $\chi_{\mathcal{S}}$ be the inventory (2.1), and define A_j, B_j, C_j for $j = 1,\ldots,d$ as follows

$$\begin{aligned}\chi_{\mathcal{S}}(x_1,\ldots,x_d) &= x_1A_1(x_2,\ldots,x_d) + B_1(x_2,\ldots,x_d) + \overline{x_1}C_1(x_2,\ldots,x_d)\\ &= x_2A_2(x_1,x_3,\ldots,x_d) + B_2(x_1,x_3,\ldots,x_d) + \overline{x_2}C_2(x_1,x_3,\ldots,x_d)\\ &= \cdots\\ &= x_dA_d(x_1,\ldots,x_{d-1}) + B_d(x_1,\ldots,x_{d-1}) + \overline{x_d}C_d(x_1,\ldots,x_{d-1}),\end{aligned}$$

where $\overline{x_i} = \frac{1}{x_i}$. Under the assumption (H1), the functions $A_1,\ldots,A_d$ and $C_1,\ldots,C_d$ are all non-zero. The group of $\mathcal{S}$ is defined as the group

$$G = \langle\varphi_1,\ldots,\varphi_d\rangle \tag{2.2}$$

of birational transformations of the variables $[x_1,\ldots,x_d]$ generated by the involutions:

$$\begin{cases}\varphi_1([x_1,\ldots,x_d]) &= \left[\overline{x_1}\frac{C_1(x_2,\ldots,x_d)}{A_1(x_2,\ldots,x_d)}, x_2,\ldots,x_d\right],\\ \varphi_2([x_1,\ldots,x_d]) &= \left[x_1,\overline{x_2}\frac{C_2(x_1,x_3,\ldots,x_d)}{A_2(x_1,x_3,\ldots,x_d)}, x_3,\ldots,x_d\right],\\ & \cdots\\ \varphi_d([x_1,\ldots,x_d]) &= \left[x_1,\ldots,x_{d-1},\overline{x_d}\frac{C_d(x_1,\ldots,x_{d-1})}{A_d(x_1,\ldots,x_{d-1})}\right].\end{cases} \tag{2.3}$$

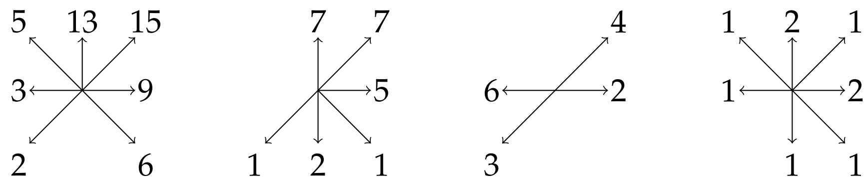

Figure 1: Examples of finite group models taken from [10]. From left to right, a model with a group of order 4, 6, 8 and 10. They should be normalised to satisfy the property that $\sum_{(i,j)\in\mathcal{S}} w(i,j) = 1$. The paper [10] actually contains infinite families of finite group examples. For example, the third example defines a group model of order 8 if and only if the associated weights $w(i,j)$ satisfy $w(1,0)w(-1,0) = w(1,1)w(-1,-1) \neq 0$.

The generators of the group G therefore satisfy the relations $\varphi_1^2 = \cdots = \varphi_d^2 = \mathrm{Id}$, plus possible other relations, depending on the model. Note that if $\mu, \alpha_1, \ldots, \alpha_d > 0$, then the model defined by the inventory $(x_1, \ldots, x_d) \mapsto \mu\chi_{\mathcal{S}}(\alpha_1 x_1, \ldots, \alpha_d x_d)$ shares the same combinatorial group as the model $\mathcal{S}$. Such transformations are referred to as central weightings.

2.2 A new complete classification in two dimensions

In the unweighted case (that is, when all jumps in the step-set have the same weight), the models were classified in [3] according to the finiteness of the group. Among the 79 non-equivalent models, only 23 have a finite group. For example, unweighting the first and third models of Figure 1 yields models with groups of order 4 and 8, respectively.

In the general weighted setting, the paper [10] characterizes the weights that produce models with groups of order 4, 6, and 8. For example, they find two families of models with a group of order eight: the first one (family 4a in [10]) is defined by $w(1,1) = w(0,1) = w(0,-1) = w(-1,-1) = 0$ and $w(1,-1)w(-1,1) = w(1,0)w(-1,0) \neq 0$, the second one is our third example on Figure 1. Furthermore, the authors identify three isolated models with a group of order 10, namely the one on the right of Figure 1, together with its vertical reflection and its reflection through the origin. Kauers and Yatchak leave open the possibility of additional finite-group models. Essentially, we prove that these are indeed all the finite-group models.

Theorem 1. *In two dimensions, the only possible orders of G are* 4, 6, 8 *and* 10.

In fact, we prove more: first, we recover all the families identified in [10], and we show that the only models with a group of order 10 are the rightmost model in Figure 1, its two symmetric versions, and all their central weightings.

See [6, 5] for recent progress on the classification of a related group over an underlying Riemann surface.

2.3 A reflection group

Let $\mathcal{S}$ be a step-set satisfying (H1), and let $\chi_{\mathcal{S}}$ be its inventory (2.1). The system of equations

$$\frac{\partial \chi_{\mathcal{S}}}{\partial x_1} = \cdots = \frac{\partial \chi_{\mathcal{S}}}{\partial x_d} = 0 \tag{2.4}$$

admits a unique solution in $(0,\infty)^d$, denoted by $\boldsymbol{x_0}$. The point $\boldsymbol{x_0}$ has the following interpretation: if we consider the central weighting of the model $\mathcal{S}$ with the weights $w(i_1,\dots,i_d)\frac{\boldsymbol{x_0}^{(i_1,\dots,i_d)}}{\chi_{\mathcal{S}}(\boldsymbol{x_0})}$ using the multi-index notation, we get a zero drift model. Define now the covariance matrix

$$\Delta = \left(a_{ij}\right)_{1\leqslant i,j\leqslant d}, \quad \text{with } a_{ij} = \frac{\frac{\partial^2 \chi_{\mathcal{S}}}{\partial x_i \partial x_j}(\boldsymbol{x_0})}{\sqrt{\frac{\partial^2 \chi_{\mathcal{S}}}{\partial x_i^2}(\boldsymbol{x_0}) \cdot \frac{\partial^2 \chi_{\mathcal{S}}}{\partial x_j^2}(\boldsymbol{x_0})}}. \tag{2.5}$$

Let $\Delta^{-\frac{1}{2}}$ denote the inverse of the symmetric, positive definite square root of the covariance matrix Δ. This construction is designed to satisfy the following property: the walk with steps $\Delta^{-\frac{1}{2}}\mathcal{S}$ (together with the above reweighting) has zero drift and identity covariance matrix. In other words, the matrix Δ is canonical in the sense that it allows one to transform the random walk into a new walk lying in the domain of attraction of a universal Brownian motion with zero drift and identity covariance matrix. Consider the d-dimensional unbounded polytope

$$T = \Delta^{-\frac{1}{2}}\mathbb{R}_+^d. \tag{2.6}$$

We denote the canonical basis of $\mathbb{R}^d$ by $(e_i)_{1\leqslant i\leqslant d}$, so the orthant $\mathbb{R}_+^d$ is bounded by the hyperplanes

$$G_i = \operatorname{Span}\{e_1,\dots,e_{i-1},e_{i+1},\dots,e_d\}, \quad i\in\{1,\dots,d\}, \tag{2.7}$$

and thus T in (2.6) is bounded by the hyperplanes $H_i = \Delta^{-\frac{1}{2}}G_i$. Finally, we introduce the reflection group

$$H = \langle r_1,\dots,r_d\rangle, \tag{2.8}$$

where r_i is the orthogonal reflection in side H_i of T.

In [8, Cor. 6], the authors prove the existence of a surjective morphism that sends the set of generators $(\varphi_1,\dots,\varphi_d)$ of G in (2.3) to the set of generators $(r_1,\dots,r_d)$ of H. In this extended abstract, we present a strong refinement of that previous result:

Theorem 2. *If G is finite, then G is isomorphic to H.*

In particular, this implies that G is a reflection group, which is convenient as reflection groups are well understood.

2.4 Classification of finite groups in three dimensions and beyond

Theorems 1 and 2 are not only interesting in their own right, but can also be combined to completely determine, in arbitrary dimension, the cases where G is finite and the cone T in (2.6) is a *Weyl chamber* of the reflection group H. In these cases we will say that G has the Weyl property. Focusing on these models is natural, as we believe that they are precisely the cases where the reflection principle and orbit-sum methods can be used to directly compute the number of walks; see [7, 11, 12, 4]. We don't state a precise general result, but illustrate how this can be done in a specific case by classifying all irreducible groups in three dimensions that satisfy the Weyl property.

Theorem 3. *In $d = 3$ dimensions, the models that give rise to finite groups with the Weyl property can be completely classified. See the end of Section 5 for a concrete example.*

The basic idea is that when G has the Weyl property, its generators form a Coxeter system, meaning that the only relations between the generators are of the form $(\varphi_i\varphi_j)^{m_{ij}} = 1$. As a consequence we can reduce the problem to two dimensions and apply Theorem 1. More generally this idea allows one to classify all models in arbitrary dimension for which the group is finite and the generators form a Coxeter system. We note that whenever G is finite, Theorem 2 implies that G is a Coxeter group, but the generators φ_i do not always form a Coxeter system.

3 Proof outline for Theorem 1

While the full proof would exceed the space available in this short note, we can sketch it as follows:

1. We first establish a criterion for the finiteness of the group $\langle \varphi_1, \varphi_2 \rangle$ introduced in (2.2). To each model we associate a function $r : (0,1) \to (0,\infty)$, which is real-analytic in t and expressed as the ratio of two elliptic integrals. The criterion can be stated as follows: the group is finite if and only if $r(t) = \frac{p}{q}$ is a rational constant; in that case, the order of the group is $2q$.

2. We show that $\lim_{t\to 0} r(t) = r_0$, where

$$r_0 \in \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{3}{4}, \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, \frac{4}{7}, \frac{5}{7}, \frac{5}{8} \right\}. \tag{3.1}$$

   As a consequence, the only possible orders of the group are $4, 6, 8, 10, 14$, and 16. Surprisingly, 12 does not appear as a possible order.

3. Using, on the one hand, that groups of orders 4, 6, and 8 are already classified, and on the other hand, that the order of the group is at most 12 (see [9, Rem. 5.1]), the only remaining case is that of a group of order 10. In this case we conclude by analysing the behaviour of $r(t)$ around 0 that the only cases where the group has order 10 are those found in [10].

We now give some more details on each item above. We start with 1. We use a detour via complex analysis. As it turns out, the group $\langle \varphi_1, \varphi_2 \rangle$ in (2.2) cannot only be viewed as a group of birational transformations as in (2.3), but also as a group of automorphisms of the Riemann surface

$$\{(x, y) \in (\mathbb{C} \cup \infty)^2 : 1 - t\chi_{\mathcal{S}}(x, y) = 0\}.$$

For $t \in (0, 1)$, and under hypothesis (H1), this surface has genus 1; it is therefore homeomorphic to a torus $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$, where the fundamental periods ω_1, ω_2 of the lattice can be computed explicitly in terms of t as follows: $\omega_1 = i\alpha K(k')$, $\omega_2 = \alpha K(k)$, where k is the elliptic modulus, which is an algebraic function of t, $\alpha > 0$ is a non-essential quantity (also algebraically dependent on t), and K denotes the complete elliptic integral of the first kind; see, e.g., [1, Sec. 6.3]. It is then shown that the group takes the form $\langle \omega \mapsto -\omega, \omega \mapsto -\omega + \omega_3 \rangle$, with $\omega_3 = \alpha F(w, k)$, where w is another algebraic function of t and $F(w, k)$ denotes the incomplete elliptic integral of the first kind. The function $r(t)$ is equal to the ratio $\frac{\omega_3}{\omega_2} = \frac{F(w,k)}{K(k)}$, and the group is finite if and only if $r(t)$ is a fixed rational number.

For point 2, we proceed as follows: On the one hand, given the support of the step-set $\{(i, j) : w(i, j) \neq 0\}$, we can directly write w^2 and k^2 as series in $\mathbb{C}[[\sqrt{t}]]$ with coefficients depending on the (non-zero) weights $w(i, j)$. On the other hand, we can write w^2 and k^2 as series in the elliptic nome $q = \exp(i\pi\tau) = \exp(i\pi \frac{K(k)}{K'(k)})$ and $q^{r(t)}$, by inverting the relation $r = \frac{F(w,k)}{K(k)}$ using Jacobi elliptic functions: $w = \operatorname{sn}(rK(k), k) = -i\operatorname{sc}(rK(k), \sqrt{1 - k^2})$, where the first equality is the classical inversion formula for $F(w, k)$, while the second is by the Jacobi transformation. Writing the Jacobi elliptic function sc in terms of Jacobi theta functions yields the following expression for w^2:

$$\begin{aligned}
w^2 = -\frac{\theta_3(q)^2\theta_1(\frac{r}{2}\tau, q)^2}{\theta_4(q)^2\theta_2(\frac{r}{2}\tau, q)^2} &= \frac{(1 + 2q + 2q^4 + \cdots)^2(1 - q^r - q^{2-r} + q^{2+2r} + \cdots)^2}{(1 - 2q - 2q^4 + \cdots)^2(1 + q^r + q^{2-r} + q^{2+2r} + \cdots)^2}, \\
k^2 = \frac{\theta_4(q)^4}{\theta_3(q)^4} &= \frac{(1 - 2q - 2q^4 + \cdots)^4}{(1 + 2q + 2q^4 + \cdots)^4}.
\end{aligned}$$

Using these series in q, we have $\frac{\log(1-w^2)}{\log(1-k^2)} \to r_0$ as $q \to 0$, or equivalently as $t \to 0$. Then using the series in t, we can identify this limit, and show that it only depends on the support of the step-set.

Finally, for point 3, we consider the situation where the group has order 10. By symmetry arguments, we only need to consider the case where $r(t) = \frac{2}{5}$ and $w(1,1) = 0$, while $w(1,-1), w(-1,-1), w(-1,1) = 1$ and $w(1,0), w(0,1) \neq 0$. Then from the series in q, we have

$$\frac{\left(385\, w^6 - 1415\, w^4 + 1835\, w^2 - 869\right)}{\left(w^2-1\right)^3} + 32\frac{\left(k^2-1\right)\left(8\, w^2 - 13\right)}{\left(w^2-1\right)^6} - 256\frac{\left(k^2-1\right)^2}{\left(w^2-1\right)^8} = o(1),$$

as $q \to 0$. On the other hand, writing this expression directly as a series in t, the coefficient for t^{-3}, t^{-2}, t^{-1} and t^0 are all rational functions of the square-roots of the unknown weights $w(1,0)$, $w(0,1)$, $w(-1,0)$, $w(0,-1)$. The only case where these coefficients are all 0 is when $w(1,0) = w(0,1) = 1$ and $w(-1,0) = w(0,-1) = 2$. This corresponds, after a central reflection, to the rightmost model in Figure 1.

4 Proof of Theorem 2

It was shown in [8] that there is a surjective morphism $\phi : G \to H$, implying that whenever G is finite, H is also finite, and more precisely

$$H \equiv \frac{G}{\ker \phi}.$$

Theorem 2 is then equivalent to the statement that $\ker \phi$ is trivial whenever G is finite. Actually we prove the stronger statement that $\ker \phi$ is torsion-free, that is every non-identity element of $\ker \phi$ has infinite order.

We start by defining the transformation. From the definition (2.4) of x_0, one can show that x_0 is fixed by each generator φ_j of G, and hence by every element $g \in G$. This allows us to define the morphism

$$J : \left| \begin{array}{ccc} G & \to & GL_d(\mathbb{R}) \\ g & \mapsto & \mathrm{Jac}_{x_0}\, g \end{array} \right. \tag{4.1}$$

In [8], it is shown that $\mathrm{Im}\, J$ and H are isomorphic, which ensures the existence of a surjective morphism $\phi : G \to H$ for which $\ker J$ and $\ker \phi$ are isomorphic. Hence it suffices to show that $\ker J$ is torsion free. This statement follows directly from the following lemma:

Lemma 1. *Let $\Omega \subset \mathbb{R}^d$ be a connected open set and $f : \Omega \to \Omega$ be a $\mathcal{C}^2$-mapping such that*

- *there exists a positive integer n such that $f^n = \mathrm{Id}$ (implying that f is a $\mathcal{C}^2$-diffeomorphism);*
- *f admits a fixed point $X \in \Omega$ with $df_X = \mathrm{Id}$.*

Then $f = \mathrm{Id}$.

Indeed, let $g \in \ker J$. Then g has the fixed point x_0 at which dg_{x_0} is the identity. If additionally the order n of g is finite, then g satisfies the assumptions of Lemma 1 with $\Omega = (0,\infty)^d$ and $X = \boldsymbol{x_0}$ which proves that $g = \mathrm{Id}$. Hence, the only element of $\ker J$ of finite order is the identity, so $\ker J$ is torsion free, as required. It remains to prove Lemma 1. Unlike Sections 3 and 5, we can present the complete proof here.

Proof. Let

$$F := \{x \in \Omega | f(x) = x \text{ and } df_x = \mathrm{Id}\}.$$

From the assumptions, we have $F \neq \emptyset$. Moreover, F is clearly a closed set. The proof will be complete once we show that F is also open, which, by the connectedness of Ω, will imply that $F = \Omega$. Hence it suffices to show that for $x \in F$ there is some $\gamma > 0$ satisfying $B(x,\gamma) \subset F$. For the rest of the proof we fix $x \in F$. The idea is to show that for z in an open ball centered at x, we have $f^n(z) - z \approx n(f(z) - z)$; then, since the left-hand side is 0 by the first assumption of Lemma 1, we must have $f(z) = z$.

To make this approximation precise, we fix $\alpha > 0$ such that $B(x,\alpha) \subset \Omega$. We use the Taylor-Lagrange theorem, together with the fact that f^k is of class $\mathcal{C}^2$, which implies that there exists $C_0 > 0$ such that for all $y, z \in B(x,\alpha)$ and all $k \in \{0,\dots,n-1\}$ (where we recall that n is the order of f in the diffeomorphism group of Ω),

$$\|f^k(y) - f^k(z) - d(f^k)_z(y-z)\| \leqslant C_0\|y-z\|^2.$$

In particular, if $f(z), z \in B(x,\alpha)$ and $v = f(z) - z$ then

$$\|f^{k+1}(z) - f^k(z) - d(f^k)_z(v)\| \leqslant C_0\|v\|^2. \tag{4.2}$$

Now let $\varepsilon \in (0,\alpha)$ be sufficiently small that $\varepsilon(1+2C_0) < 1$. Let $k \in \{0,\dots,n-1\}$. Since f^k is differentiable at x, $df_x = \mathrm{Id}$, and x is a fixed point of f, it follows that $d(f^k)_x = \mathrm{Id}$. Since f is of class $\mathcal{C}^1$, there exists $\delta_k > 0$ such that $B(x,\delta_k) \subset \Omega$ and such that for all $z \in B(x,\delta_k)$,

$$\|d(f^k)_z - \mathrm{Id}\,\| \leqslant \varepsilon. \tag{4.3}$$

Finally, by continuity of f^k, we can choose $\gamma_k \in (0,\delta_k)$ such that

$$f^k(B(x,\gamma_k)) \subset B(x, \min(\varepsilon,\delta_k)).$$

Set $\gamma = \min_k(\gamma_k,\varepsilon)$. We will show that $B(x,\gamma) \subset F$. Combining (4.2) and (4.3) applied at $v = f(z) - z$ yields

$$\|f^{k+1}(z) - f^k(z) - v\| \leqslant C_0\|v\|^2 + \varepsilon\|v\|. \tag{4.4}$$

By summing (4.4) from $k=0$ to $k=n-1$, we obtain that, since $f^n = \mathrm{Id}$,

$$n\|v\| = \|f^n(z) - z - nv\| \leqslant \sum_{k=0}^{n-1} \|f^{k+1}(z) - f^k(z) - v\| \leqslant n(C_0\|v\|^2 + \varepsilon\|v\|). \tag{4.5}$$

Now observe that, by definition of γ, we have $f(z), z \in B(x,\varepsilon)$, which implies that $\|v\| \leqslant 2\varepsilon$, and thus the right-hand side of (4.5) is at most $(2C_0\varepsilon+\varepsilon)n\|v\|$. But our choice of ε implies $(2C_0\varepsilon+\varepsilon) < 1$, which makes (4.5) possible only if $v=0$, that is, $f(z)=z$. Hence $f(z)=z$ for all $z \in B(x,\gamma)$. This implies that $df_z = \mathrm{Id}$ for $z \in B(x,\gamma)$, and therefore $B(x,\gamma) \subset F$. This completes the proof of Lemma 1. □

5 Proof outline for Theorem 3

We assume in this section that $d=3$ and thus, $\chi_{\mathcal{S}}$ can be written as $\chi_{\mathcal{S}}(x,y,z) = \sum_{(i,j,k)\in\{-1,0,1\}^3} w(i,j,k)x^iy^jz^k$, where (x,y,z) are the standard coordinates on $\mathbb{R}^3$. Denote by m_{ij} the order of $\varphi_i\varphi_j$ in G. In this section, the main tool we use is the following result:

Proposition 1. *The group G is a finite group with the Weyl property if and only if*

1. *up to a permutation of the coordinates (x,y,z), the triple (m_{12}, m_{13}, m_{23}) belongs to the following list:*
$$\{(2,2,k),(3,2,3),(3,2,4),(3,2,5) | k \in \mathbb{N}^*\}; \tag{5.1}$$
2. *for all $i \neq j$, the coefficient a_{ij} of Δ in (2.5) is equal to $-\cos\left(\frac{\pi}{m_{ij}}\right)$.*

The above result is a direct consequence of the results in [8] combined with Theorem 2 and the classification of finite Coxeter groups. Indeed, if conditions 1 and 2 both hold, then the classification of finite Coxeter groups (see [8, Sec. 6.2]), combined with [8, Prop. 4] imply that the group H has the presentation

$$\left\{r_1^2, r_2^2, r_3^2, (r_1r_2)^{m_{12}}, (r_1r_3)^{m_{13}}, (r_2r_3)^{m_{23}}\right\},$$

where r_1, r_2, r_3 are the generators of H defined in Section 2.3, and that T in (2.6) is a Weyl chamber of the group. Then condition 1 implies that G is finite and therefore isomorphic to H by Theorem 2. In the other direction, if G is finite and has the Weyl property then by Theorem 2, H is isomorphic to G and H is the reflection group associated to a Weyl chamber T, so it is a finite Coxeter system. Moreover, the order of r_ir_j in H is m_{ij}, which is the order of $\varphi_i\varphi_j$ in G. Then condition 1 follows from the classification of finite Coxeter groups while condition 2 follows from [8, Prop. 4].

Note that if (m_{12}, m_{13}, m_{23}) belongs to the list (5.1), then G is respectively isomorphic to $\frac{\mathbb{Z}}{2\mathbb{Z}} \times D_{2k}$, A_3, B_3, or H_3.

The next step is to use the classification of two-dimensional finite group models (according to Theorem 1 and Kauers and Yatchak's paper [10]) to determine when conditions 1 and 2 of Proposition 1 hold. To do this, we first prescribe the value of the triple (m_{12}, m_{13}, m_{23}) (and thus $a_{ij} = -\cos\left(\frac{\pi}{m_{ij}}\right)$). Considering z to be fixed, we define

$$\chi_z(x,y) := \chi_{\mathcal{S}}(x,y,z) - (w(0,0,1)z + w(0,0,-1)\overline{z}). \tag{5.2}$$

Note that removing the constant term $w(0,0,1)z + w(0,0,-1)\bar{z}$ in (5.2) does not affect the nature of the two-dimensional model in the variables x, y. Hence, for any $z > 0$, the inventory $\chi_z(x,y)$ corresponds to a two-dimensional model with group of order $2m_{1,2}$, and coefficients

$$w^{xy}(i,j) := w(i,j,1)z + w(i,j,0) + w(i,j,-1)\bar{z}.$$

Moreover, the value a_{12} satisfies condition 2, which immediately allows us to eliminate the possibility that $m_{12} = 5$. We denote by G_z the group associated with the model χ_z. Analogously, we define χ_x, χ_y, G_x, and G_y. With this notation, and since in two dimensions the only finite combinatorial groups are D_4, D_6, D_8, and D_{10}, Proposition 1 can be restated as follows:

Proposition 2. *The group G is a finite group with the Weyl property if and only if*

1. *up to a permutation of the coordinates (x,y,z), the triple of groups (G_z, G_y, G_x) appears in the following list:*
$$\{(D_4, D_4, D_{2k}), (D_6, D_4, D_6), (D_6, D_4, D_8) : 2 \leqslant k \leqslant 4\};$$
2. $a_{12} = -\cos\left(\frac{2\pi}{|G_z|}\right)$, $a_{13} = -\cos\left(\frac{2\pi}{|G_y|}\right)$ *and* $a_{23} = -\cos\left(\frac{2\pi}{|G_x|}\right)$.

So constructing a finite group G with the Weyl property is equivalent to prescribing G_x, G_y and G_z, which gives rise to equations on the coefficients $w^{xy}(i,j)$, $w^{xz}(i,j)$ and $w^{yz}(i,j)$ (from the classification in two dimensions, see Section 2.2) which in turn lead to equations on the weights $w(i,j,k)$. Note that, to fulfil the second condition in Proposition 2, the classification in two dimensions must only take into account the models whose covariance matrix has non-positive off-diagonal entries.

Let us conclude this section by giving the classification for the Coxeter groups A_3 and B_3. For any inventory $\chi_{\mathcal{S}}(x,y,z)$ corresponding to a finite group G with the Weyl property, inventories defined by permuting x, y, z, as well as the inventories $a\chi_{\mathcal{S}}(bx, cy, dz)$ and $\chi_{\mathcal{S}}(\bar{x}, \bar{y}, \bar{z})$ also have these properties. Up to these transformations, we show that there are exactly two families of models for which $(m_{12}, m_{13}, m_{23}) = (3,2,3)$, so $G = A_3$. The first is represented by

$$\chi_{\mathcal{S}}(x,y,z) = \frac{1}{yz} + \frac{2}{y} + \frac{z}{y} + \frac{x}{z} + x + \frac{y}{xz} + \frac{y}{x} + \frac{c}{z},$$

for $c \geqslant 0$, while the second is represented by

$$\chi_{\mathcal{S}}(x,y,z) = a\left(x + \frac{y}{x} + \frac{z}{y} + \frac{1}{z}\right) + b\left(\frac{1}{x} + \frac{x}{y} + \frac{y}{z} + z\right) + c\left(y + \frac{1}{y} + \frac{xz}{y} + \frac{y}{xz} + \frac{z}{x} + \frac{x}{z}\right),$$

where $a, b, c \geqslant 0$ and a, b, c are not all 0. We have also fully classified the models with $(m_{12}, m_{13}, m_{23}) = (3,2,4)$, so the group $G = B_3$. In this case, all models are related by the symmetries described above to one of two unweighted models:

$$\chi_{\mathcal{S}}(x,y,z) = \frac{y}{z} + \frac{z}{y} + \frac{x}{y} + \frac{y}{x} + \frac{1}{x} + x \text{ and } \chi_{\mathcal{S}}(x,y,z) = \frac{x}{z} + \frac{z}{x} + \frac{y}{z} + \frac{z}{y} + \frac{y}{xz} + \frac{xz}{y} + z + \frac{1}{z}.$$

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References

[1] O. Bernardi, M. Bousquet-Mélou, and K. Raschel. "Counting quadrant walks via Tutte's invariant method". *Comb. Theory* **1** (2021). Id/No 3, p. 77. DOI.

[2] A. Bostan, M. Bousquet-Mélou, M. Kauers, and S. Melczer. "On 3-dimensional lattice walks confined to the positive octant". *Ann. Comb.* **20**.4 (2016), pp. 661–704. DOI.

[3] M. Bousquet-Mélou and M. Mishna. "Walks with small steps in the quarter plane". *Algorithmic probability and combinatorics*. Providence, RI: American Mathematical Society (AMS), 2010, pp. 1–39.

[4] M. Buchacher, S. Hofmanninger, and M. Kauers. "Walks with small steps in the 4D-orthant". *Ann. Comb.* **25**.1 (2021), pp. 153–166. DOI.

[5] V. Dragović and M. Radnović. "Finite Groups of Random Walks in the Quarter Plane and Periodic 4-bar Links". 2026. arXiv:2512.21976.

[6] G. Fayolle, R. Iasnogorodski, and V. Malyshev. *Random walks in the quarter plane.* 2nd edition. Vol. 40. Probab. Theory Stoch. Model. Cham: Springer, 2017. DOI.

[7] T. Feierl. "Asymptotics for the number of walks in a Weyl chamber of type B". *Random Struct. Algorithms* **45**.2 (2014), pp. 261–305. DOI.

[8] L. Gohier, E. Humbert, and K. Raschel. "Enumeration of walks in multidimensional orthants and reflection groups". *Indiana Univ. Math. J.* (to appear). Link.

[9] C. Hardouin and M. F. Singer. "On differentially algebraic generating series for walks in the quarter plane". *Sel. Math., New Ser.* **27**.5 (2021). Id/No 89, p. 49. DOI.

[10] M. Kauers and R. Yatchak. "Walks in the quarter plane with multiple steps". *Proceedings of FPSAC 2015, Daejeon, South Korea*. Discrete Mathematics & Theoretical Computer Science (DMTCS), 2015, pp. 25–36.

[11] S. Melczer and M. Mishna. "Asymptotic lattice path enumeration using diagonals". *Algorithmica* **75**.4 (2016), pp. 782–811. DOI.

[12] R. Yatchak. "Automated positive part extraction for lattice path generating functions in the octant". *Extended abstracts of EuroComb 2017, Vienna, Austria*. Amsterdam: Elsevier, 2017, pp. 1061–1067. DOI.