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Extended Cyclic Sieving Phenomena

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Abstract. The Cyclic Sieving Phenomenon (CSP) of Reiner-Stanton-White is a remarkable framework in which the symmetries of a cyclic action on a set X of combinatorial objects are described in terms of root-of-unity evaluations of some q-version of the cardinality of X.

We generalize the CSP, by replacing the set X with some vector space that carries a combinatorial but not necessarily permutation action of a cyclic group (for instance the action on the homology of a complex induced by some cyclic action on its vertices). In this extended Cyclic Sieving Phenomenon (eCSP), the polynomial encodes the traces of the cyclic action. We prove eCSP for a variety of examples, including the positive q-Catalan numbers, the q-Kirkman numbers resolving an open problem of Armstrong-Reiner-Rhoades since 2012, and more.

Keywords: cyclic sieving, poset topology, W-noncrossing partitions, cluster complexes

1 Introduction

Some particularly popular results in combinatorics, involve evaluating polynomials $f(q)$ with positive integer coefficients at the number -1. Surprisingly, the answer is often meaningful: a positive integer that counts something. This happens in general reciprocity theories (like Ehrhart theory, which includes the set-multiset duality, the weak and strict order polynomials of posets, and more) and also in symmetry counting.

Stembridge's $q = -1$ phenomenon in symmetry counting occurs when $f(q)$ is a statistic on a set of combinatorial objects, and the evaluation $f(-1)$ counts those objects fixed by a certain involution. It originated in Stembridge's seminal work [19] which unified the study of symmetries in plane partitions and his later work [18] giving a representation theoretic interpretation to it.

Ten years later, amidst a cold Minnesotan winter, Dennis Reiner, Dennis Stanton, and Dennis White, generalized [15] this representation theoretic approach to what has become known as the Cyclic Sieving Phenomenon (CSP). The CSP occurs when a polynomial $f(q)$ carries orbital information for the action of a whole cyclic group $C_n = \langle c\rangle$, of order n, on a set X of combinatorial objects (as opposed to the action of an involution, i.e. with $n = 2$, in the $q = -1$ phenomenon). More precisely, if $\zeta = e^{2\pi i/n}$, we say that

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the triple $(X, f(q), C_n)$ *exhibits the cyclic sieving phenomenon* if for all integers d, we have that

$$\#\{x \in X \mid c^d \cdot x = x\} = f(\zeta^d), \tag{1.1}$$

where $\#A$ denotes the cardinality of the set A.

The statement of (1.1) resembles at first glance a magic trick: one plugs in a root of unity in a polynomial, and (tada!) the result is an integer and it counts something. On the other hand, interpolation could construct polynomials in $\mathbb{C}[q]$ with prescribed values, and with a little more effort we can show that under the conditions of (1.1) the coefficients can be forced to live in $\mathbb{Z}_{\geq 0}$ [15]. However, what is truely surprising is that often the polynomials that appear in CSP triples $(X, f(q), C_n)$ are natural combinatorial, geometric, representation-theoretic q-analogues of the cardinalities of the sets X. Because of this, CSPs have become a sort of collectors item in the community: everyone tries to prove at least one for their favorite combinatorial objects and q-versions.

After their introduction in [15], there has been a plethora of works on CSPs, including the development of general methods as well as various directions of generalizations: the cyclic group C_n is replaced by multicyclic, dihedral or more abstract groups. A common characteristic of these generalizations is that the group always acts by *permuting* the elements of the set X so that the left hand side of (1.1) still makes sense.

The purpose of this abstract is to change perspective on these generalizations and consider non-permutation actions of groups. Indeed, many natural cyclic actions in combinatorics, especially those induced on algebraic invariants of spaces are not permutation actions. We develop a notion of extended cyclic sieving for such actions and prove a collection of theorems for many popular combinatorial spaces.

Extended Cyclic Sieving Phenomena (eCSP)

The representation-theoretic approach to CSPs involves the study of (the linearized space) $\mathbb{C} \cdot X$ as a C_n-module. Write $(C_n)_j$ for the irreducible representation of the cyclic group C_n in which the generator c is sent to $e^{2\pi i j/n}$. If a polynomial $f(q)$ is given by $f(q) = a_m q^m + \cdots + a_1 q + a_0$, and we have a module isomorphism

$$\mathbb{C} \cdot X \simeq_{C_n} \bigoplus_{j=0}^{m} a_j \cdot (C_n)_j, \tag{1.2}$$

(where $a_j \cdot (C_n)_j$ denotes the representation $(C_n)_j$ taken with multiplicity a_j) then, *after computing traces in* (1.2), the triple $(X, f(q), C_n)$ exhibits the CSP [15, Prop. 2.1].

Note that j may be greater than n and then many of the $(C_n)_j$ will be isomorphic. There is often a graded vector space $A(X) \simeq_{C_n} \mathbb{C} \cdot X$ associated to the set X (for instance the homology of some manifold that has a cellular decomposition indexed by the objects of X) whose j-th degree components are naturally isomorphic to $a_j \cdot (C_n)_j$ and thus give a way to realize (1.2). In that case, the polynomial $f(q)$ is just the Hilbert series of $A(X)$.

The interpretation of CSPs in (1.2) does not require that the action of C_n is a permutation action. Replacing $\mathbb{C} \cdot X$ with an arbitrary C_n-module V, there will always be (infinitely many) decompositions as in (1.2). This takes us to the following definition.

Definition 1.1. Let $C_n = \langle c \rangle$ be a cyclic group of order n, V a vector space with a C_n-action, and $f(q) \in \mathbb{Z}_{\geq 0}[q]$. We say that the triple $(V, f(q), C_n)$ exhibits an *extended Cyclic Sieving Phenomenon (eCSP)* if for all integers d, we have (with $\zeta = e^{2\pi i/n}$)

$$\mathrm{Tr}_V(c^d) = f(\zeta^d).$$

Of course, if V is the linearization of a set X permuted by C_n, Definition 1.1 above recovers that original CSP. However, there are many natural examples of combinatorial vector spaces that carry cyclic actions that are not permutation actions (a frequent example is induced homology actions). Our thesis in this abstract is that such actions are often encoded in the same remarkable and suprising way as in the usual CSPs: evaluating a combinatorially/geometrically/algebraically meaningful polynomial at roots of unity gives as the traces of the representation.

Remark 1.2 (Exotic Cyclic Sieving Phenomena)**.** There are many other ways to generalize CSPs and we want to focus here on a particular direction. Instead of interpreting the trace in Defn. 1.1 **directly** as a polynomial evaluation, we may choose to apply additional operations to the number $f(\zeta^d)$ (like taking its absolute value). Without restricting what operations we may consider, we will refer to such examples as *exotic* CSPs (with the notation eCSP encompassing both versions).

One successful such operation is the field norm for the extension $\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q}$. Indeed all our evaluations $f(\zeta^d)$ live in this extension so their field norm is a rational number. We prove in Section 4 an exotic CSP for a natural induced cyclic action on the multilinear component of the free Lie algebra (equivalently on the top homology of the set partition lattice).

Remark 1.3 (Extended Dihedral Sieving Phenomena)**.** All existing generalizations of the CSP (with groups other that C_n, as in [14]) have a natural *extended* version as well. One just drops the requirement that the group action is a permutation action and considers the anologue of Definition 1.1. In Section 3, we describe how our framework can resolve some erroneous conjectures regarding the q,t-Kirkman numbers and dihedral sieving phenomena on cluster complexes.

Remark 1.4 (Some eCSPs are regular CSPs)**.** If the C_n-representation V is a permutation representation, meaning that it is equal to a sum of coset representations of C_n (i.e. $V \cong_{C_n} \bigoplus_{d|n} a_d \cdot \uparrow_{C_d}^{C_n} \mathrm{triv}$ with $a_d \in \mathbb{Z}_{\geq 0}$), then we can choose a basis of V that is permuted by C_n and interpret the eCSP as a regular CSP.

Of course, something like this would be particularly interesting if the basis and the C_n permutation action on it have nice combinatorial models.

Homology Actions, Lefschetz numbers, and fixed subcomplexes

A family of natural candidates for eCSP are induced cyclic actions on homology groups of simplicial complexes Δ. If g is a simplicial map on Δ, then the *Lefschetz number* $\Lambda(g)$ is defined via

$$\Lambda(g) := \sum_{i\geq 0}(-1)^i \operatorname{Tr}_{\tilde{H}_i(\Delta)}(g^*), \tag{1.3}$$

where g^* is the induced action of g on the reduced homology groups $\tilde{H}_i(\Delta, \mathbb{C})$. A particularly nice case here is when Δ is pure and shellable; then, if $\dim(\Delta) = n$, the (geometric realization of the) complex Δ is homotopy equivalent to a wedge of n-spheres and

$$\Lambda(g) = (-1)^n \cdot \operatorname{Tr}_{\tilde{H}_n(\Delta)}(g^*). \tag{1.4}$$

A special case of the Hopf trace formula allows us to compute such Lefschetz numbers combinatorially [6]. We have that

$$\Lambda(g) = \sum_{i\geq 0}(-1)^i \Big(\phi_i^+(g) - \phi_i^-(g)\Big), \tag{1.5}$$

where $\phi_i^+(g)$ and $\phi_i^-(g)$ are the numbers of i-dimensional faces of Δ that g fixes setwise and respects their orientation, and respectively fixes setwise but reverses orientation.

When the simplicial action g further has the property that if it fixes a face $F \in \Delta$, it fixes it pointwise (Delucchi and D'Alì [8] call these actions *proper*), then the Lefschetz number equals the reduced Euler characteristic of the fixed subcomplex Δ^g [6]. When we have both a pure, shellable complex Δ and a proper action by g, we get

$$\Lambda(g) = (-1)^n \cdot \operatorname{Tr}_{\tilde{H}_n(\Delta)}(g^*) = \tilde{\chi}(\Delta^g) = \sum_{i\geq 0}(-1)^i \phi^+(g), \tag{1.6}$$

where $\tilde{\chi}$ denotes the reduced Euler characteristic and n is the dimension of Δ.

Yet another specialization arises if the simplicial complex Δ is the *order complex* of a poset P. When P has bottom and top elements $\hat{0}$ and $\hat{1}$, we define its order complex $\mathcal{O}(P)$ as the simplicial complex whose facets are the maximal chains of $P \setminus \{\hat{0}, \hat{1}\}$. In that case, if g acts on P respecting its order —this implies in particular that g acts properly on $\mathcal{O}(P)$— and P itself is shellable, then equation (1.6) becomes [20, §4.6]

$$\Lambda(g) = (-1)^n \cdot \operatorname{Tr}_{\tilde{H}_n(\mathcal{O}(P))}(g^*) = \tilde{\chi}(\mathcal{O}(P)^g) = \tilde{\chi}(\mathcal{O}(P^g)) = \mu(P^g), \tag{1.7}$$

where μ is the Möbius function of the fixed subposet P^g.

After this discussion, homology spaces of simplicial complexes (especially top homologies of shellable complexes perhaps arising as order complexes of posets) are good contenders for eCSP: equations (1.4), (1.5), (1.6), (1.7) allow us to compute the left hand side of the eCSP Defn. 1.1 by combinatorial (enumerative or poset-theoretic) methods. Then, it could be easy to check whether a conjectural polynomial $f(q)$ satisfies the eCSP.

Moreover, in such examples the evaluation of the right hand side $f(\zeta^d)$ will not only encode a trace calculation, but in fact an invariant of a fixed space —the reduced Euler characteristic $\tilde{\chi}(\Delta^{c^d})$— in this way remaining in the spirit of the original CSP to somehow measure the fixed space X^{c^d}.

Remark 1.5. This construct of a polynomial encoding the trace of a group action on homology has been observed independently by Matthieu Josuat-Vergès who referred to it as a *homological* cyclic/dihedral sieving phenomenon [11].

2 An eCSP on the W-noncrossing-partitions lattice

Our first eCSP triple involves one of the central objects in Coxeter-Catalan combinatorics, the noncrossing partition lattice $NC(W)$.

Recall that in a finite Coxeter group W, with set of (all) reflections R, we may define a length function $\ell_R : W \to \mathbb{Z}_{\geq 0}$ via $\ell_R(w) = k$ when k is the smallest number such that $w = t_1 \cdots t_k$ with $t_i \in R$. This in turn defines the absolute reflection order $\leq_R$ on W via

$$u \leq_R v \quad \text{if and only if} \quad \ell_R(u) + \ell_R(u^{-1}v) = \ell_R(v), \tag{2.1}$$

which finally allows us to define the *noncrossing partition lattice* $NC(W)$ as the interval $[e, c]_{\leq_R}$ with e, c the identity and (any) Coxeter element c of W respectively.

The lattice $NC(W)$ generalizes the lattice NC_n of noncrossing partitions due to Kreweras (we have $NC(S_n) \simeq NC_n$), it is shellable, and has remarkable numerological properties. Its size and Möbius function are given via the W-Catalan (see [16]) and positive W-Catalan numbers (see [2, 4, 10]) respectively; that is, $\#NC(W) = \mathrm{Cat}(W)$ and $\mu\Big(NC(W)\Big) = (-1)^n \cdot \mathrm{Cat}^+(W)$, where

$$\mathrm{Cat}(W) = \prod_{i=1}^{n} \frac{h+1+e_i}{e_i+1} \qquad \text{and} \qquad \mathrm{Cat}^+(W) = \prod_{i=1}^{n} \frac{h-1+e_i}{1+e_i}.$$

Here h is the Coxeter number of W (the order of c), n its rank, and the $(e_i)_{i=1}^n$ are the *exponents* of W (i.e. $e_i = d_i - 1$ for the invariant degrees of W, or equivalently such that $\exp(2\pi i e_i/h)$ are the eigenvalues of $c \in W \leq \mathrm{GL}(V)$).

Shellability of $NC(W)$ [3] implies that the order complex $\Delta_W := \mathcal{O}\Big(NC(W)\Big)$ is a wedge of $(n-2)$-dimensional spheres and that we have $\dim \tilde{H}^{n-2}(\Delta_W; \mathbb{C}) = \mathrm{Cat}^+(W)$.

Bessis-Reiner [5] proved a CSP on the cyclic action on $NC(W)$ generated by conjugation with a Coxeter element $c \in W$, using a natural q-analogue of the numbers $\mathrm{Cat}(W)$. We define a similar version of the positive case recalling at the same time theirs:

$$\mathrm{Cat}(W;q) = \prod_{i=1}^{n} \frac{[h+1+e_i]_q}{[1+e_i]_q} \quad \text{while} \quad \mathrm{Cat}^+(W;q) := q^{n-1} \cdot \prod_{i=1}^{n} \frac{[h-1+e_i]_q}{[1+e_i]_q}. \tag{2.2}$$

Here as usual we write $[n]_q := (1-q^n)/(1-q)$ for the q-integer n.

Theorem 2.1. *Let W be an irreducible finite Coxeter group and let $V := H^{\mathrm{top}}\Big(\mathcal{O}\big(NC(W)\big);\mathbb{C}\Big)$ be the top dimensional homology space of the order complex of $NC(W)$. If $C_n := \langle c\rangle$ is a cyclic group generated by any Coxeter element $c \in W$, its action by conjugation on $NC(W)$ induces an action on V. Then, the triple $\big(V, \mathrm{Cat}^+(W;q), C_n\big)$ exhibits an eCSP.*

Sketch of proof. Since the space V comes from a shellable poset, we use equation (1.7) to compute the traces. Bessis-Reiner [5] showed that for any integer d the set of fixed elements $NC(W)^{c^d}$ equals the noncrossing partition lattice $NC(W')$ in the reflection group $W' := \mathrm{Cent}_W(c^d)$ with respect to its Coxeter element $c^d \in W'$. Here, W' may be a nonreal complex reflection group, but it will be well-generated and hence have well defined Coxeter elements.

Because the natural map $w \to V^w$ defines an injective, order preserving embedding of $NC(W)$ into the intersection lattice of the reflection arrangement $\mathcal{A}_W$, the Bessis-Reiner result further implies that as a poset $NC(W')$ is equal to the induced fixed subposet $NC(W)^{c^d}$. Now, the Möbius function of $NC(W')$ is given in terms of the exponents of W' which by Springer's theory of regular elements are the subset of the exponents e_i of W such that h/d divides $e_i + 1$. It is an easy check to show that these exactly agree with the evaluations $\mathrm{Cat}^+(W;q)\big|_{q=\zeta^d}$. □

Remark 2.2. Theorem 2.1 above holds also for well generated complex reflection groups. Even though these groups have no associated cluster complex, (hence there are no positive clusters to be enumerated by $\mathrm{Cat}^+(W)$), the lattices $NC(W)$ are still shellable [12], Springer's theory and the Bessis-Reiner result still apply and so does our proof.

3 q-Kirkman numbers and an eCSP on cluster complexes

Another popular Coxeter-Catalan object is the cluster complex $\Upsilon(W)$ associated to a finite real reflection group $W \leq \mathrm{GL}(V)$. It is a spherical simplicial complex with $\mathrm{Cat}(W)$-many facets, which is dual to the associahedron when $W = S_n$. Its f-vector $(f_0, f_1, \dots, f_n)$, where f_i counts the number of i-dimensional faces of $\Upsilon(W)$ and $n =$

rank(W), is naturally given [1] in terms of the noncrossing partition lattice $NC(W)$:

$$\sum_{i=0}^{n} f_i t^i = \sum_{w \in NC(W)} (1+t)^{\dim(V^w)}. \tag{3.1}$$

The parking character Park(W) associated to the irreducible reflection group W is given via $w \to (h+1)^{\dim(V^w)}$. Armstrong-Reiner-Rhoades [1] gave a representation theoretic interpretation of the numbers f_i as

$$f_i = \big\langle \wedge^{n-i}(V), \mathrm{Park}(W) \big\rangle. \tag{3.2}$$

The cluster complex $\Upsilon(W)$ has a natural action by a cyclic group $C_{h+2} = \langle g \rangle$ (of order $h+2$ for the Coxeter number h of W) that corresponds to rotation of triangulations when $W = S_n$. Moreover, the parking space Park(W) has a natural graded version given as the representation $\mathbb{C}[V]/(\Theta)$ where Θ is a system of parameters, of homogeneous degree $h+1$ that carries the reflection representation of W.

With this motivation, ARR [1] considered the natural q-versions of the numbers f_i above, which they called *q-Kirkman numbers*, given as

$$f_i(q) := \sum_{d \geq 0} \big\langle \wedge^{n-i}(V), \big(\mathbb{C}[V]/(\Theta)\big)_d \big\rangle, \tag{3.3}$$

and attempted to prove a CSP with the action of C_{h+2}. In fact, a CSP for the same action but with ad hoc polynomials had already been proven by Eu-Fu [9]. The representation-theoretic polynomials $f_i(q)$ were different from those of Eu-Fu and remarkably seem to exhibit the CSP only for the so called coincidental types $(A_n, B_n, I_2(m), H_3)$. Armstrong-Reiner-Rhoades asked for an explanation or resolution of this failure of the $f_i(q)$ to encode the C_n action.

The cluster complex $Y(W)$ is polytopal meaning in particular, that the link of every face is spherical. If some face $F \in \Upsilon(W)$ is fixed (setwise) by an element $g^d \in C_{h+2}$, then its link $lk(F)$ is also fixed. In particular, g^d must act on the top homology $H_{\mathrm{top}}(lk(F))$. Instead of considering the permutation action of C_{h+2} on the set of faces $F \in \Upsilon(W)$, we take its induced action on the following direct sum of 1-dimensional spaces

$$U_i := \bigoplus_{\substack{F \in \Upsilon(W) \\ \dim(F) = i}} H_{\mathrm{top}}(lk(F)). \tag{3.4}$$

Theorem 3.1. *For any irreducible finite Coxeter group W, with U_i and $f_i(q)$ as defined in (3.4) and (3.3), the triple $\Big(U_i, f_i(q), C_{h+2}\Big)$ exhibits an eCSP. In other words, the q-Kirkman numbers do not simply count fixed faces in $\Upsilon(W)$ but they weight each fixed face by ± 1 depending on the induced action of its stabilizer on its link.*

Remark 3.2. The motivation for the definition of the spaces U_i in Theorem 3.1 lies in the construction of the Kirkman numbers. Armstrong-Reiner-Rhoades connect (3.1) with (3.2) by showing that the representation theoretic definition models the f-to-h transformation formula between $NC(W)$ and $\Upsilon(W)$. In fact, this is misleading: when we consider the m-Fuss version of the parking character the m-Fuss Kirkman numbers do not count faces of the generalized cluster complex $\Upsilon(W,m)$. They instead model what Ceballos and Mühle [7] call the $f = h$ reciprocity: the m-Fuss-Kirkman numbers count faces $F \in \Upsilon(W,m)$ each weighted by $|\tilde{\chi}(lk(F))|$ (where $\tilde{\chi}$ denotes again the reduced Euler characteristic).

At this point, Theorem 3.1 should seem more natural: since the ungraded interpretation of the Kirkman numbers *should* keep track of links of faces, to get a CSP for their q-versions we must take into account the induced action on the links.

Sketch of Proof of Thm. 3.1. The proof of the Theorem is case-by-case. Building on the works of Eu-Fu [9] and the recent generalization by Pouillart [13] we can keep track of the parabolic types of the link of each (fixed) face, while Armstrong-Reiner-Rhoades [1] have given explicit formulas for the q-Kirkman numbers. The exceptional types are done via computer calculations. □

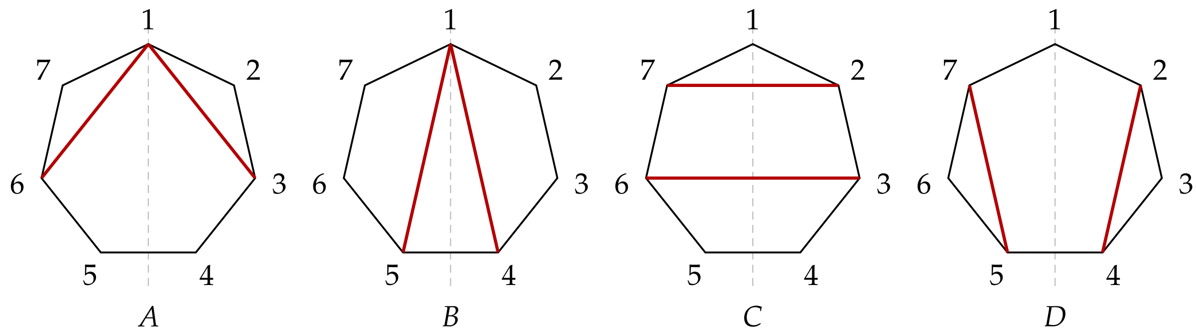
Remark 3.3. It turns out that in the coincidental types, the induced actions on links of faces of the same dimension have the same sign (it is unclear whether there is a deeper reason for this). On other types though, there are fixed faces of the same dimension whose links have different weights induced by the C_{h+2}-action. This is the reason why the Armstrong-Reiner-Rhoades CSP does not hold in general.

Remark 3.4. Theorem 3.1 conjecturally holds for the Fuss-generalized cluster complexes as well. This is work in progress with Cati Alexandru (for certain families of faces in type A), and with Matthieu Josuat-Vergès and Eric Sommers in arbitrary types.

Remark 3.5. This framework allows us to also correct an early conjecture of Rao-Suk [14] on dihedral sieving phenomena. Their conjecture 6.2 on the little q,t-Schröder numbers $\tilde{S}_{n,d}(q,t)$ implies that $\tilde{S}_{n,d}(1,-1)$ is equal, up to sign, with the number of dissections of an $(n+2)$-gon with $n-1-d$ (noncrossing) diagonals *that are invariant under reflection by a fixed axis of the polygon*. This is wrong already in $n = 7$: we have

$$\tilde{S}_{5,2}(1,-1) = \langle \nabla e_n, s_{(3,1,1)} \rangle|_{q=1,t=-1} = -2.$$

However, we have the following four suitable dissections:



The resolution of this seeming counterexample is again to consider the same polynomials but the spaces U_i with the (now dihedral) action described in the theorem. In the case above, the links of the four faces in the cluster complex are (topologically) a pentagon (A, D) or a square (B, C). It is easy to see when the reflection across the y-axis acts on them (the associated homology cycles) as multiplication by -1 (cases A, B, D) or multiplication by 1 (case C); the total contribution is now -2 as expected.

4 An exotic CSP for the set partition lattice

The lattice Π_n of set partitions of the set $[n] := \{1, 2, \ldots, n\}$, ordered by coarsening, has bottom element $\hat{0}$ the partition in n distinct singletons and top element $\hat{1}$ the partition with the single block $[n]$. It is a geometric lattice (Π_n is isomorphic to the intersection lattice of the braid arrangement Br_n) and hence shellable, and its order complex $\mathcal{O}(\Pi_n)$ is homotopy equivalent to a wedge of $(n-1)!$-many $(n-2)$-dimensional spheres; in particular, $\dim \tilde{H}_{n-2}(\mathcal{O}(\Pi_n); \mathbb{C}) = (n-1)!$.

The group S_n acts on $[n]$ by permuting its elements and this extends to an action on Π_n and induces an action on its top homology space $V = \tilde{H}_{n-2}(\mathcal{O}(\Pi_n); \mathbb{C})$. Stanley [17, §7], interpreting work of Hanlon, computed the induced S_n representation on V as

$$V \simeq_{S_n} \uparrow_{C_n}^{S_n} \xi, \tag{4.1}$$

where C_n is the subgroup of S_n generated by the long cycle $c := (12 \cdots n)$ and ξ is the irreducible character of C_n that sends c to $\zeta := e^{2\pi i/n}$.

The field extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ is a finite Galois extension, whose degree equals Euler's totient function $\phi(n)$, with $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \simeq \mathbb{Z}_n^\times$, and whose *field norm* is given via

$$N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\alpha) := \prod_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(\alpha). \tag{4.2}$$

By definition all elements of $\mathbb{Q}(\zeta)$ have a rational field norm, and if in particular $r \in \mathbb{Q}$, then $N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(r) = r^{\phi(n)}$. Because of this, we define a normalized function $||\cdot||$:

$$||\alpha|| := \sqrt[\phi(n)]{N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\alpha)}, \tag{4.3}$$

so that $||\cdot||$ agrees with the absolute value in $\mathbb{Q}$.

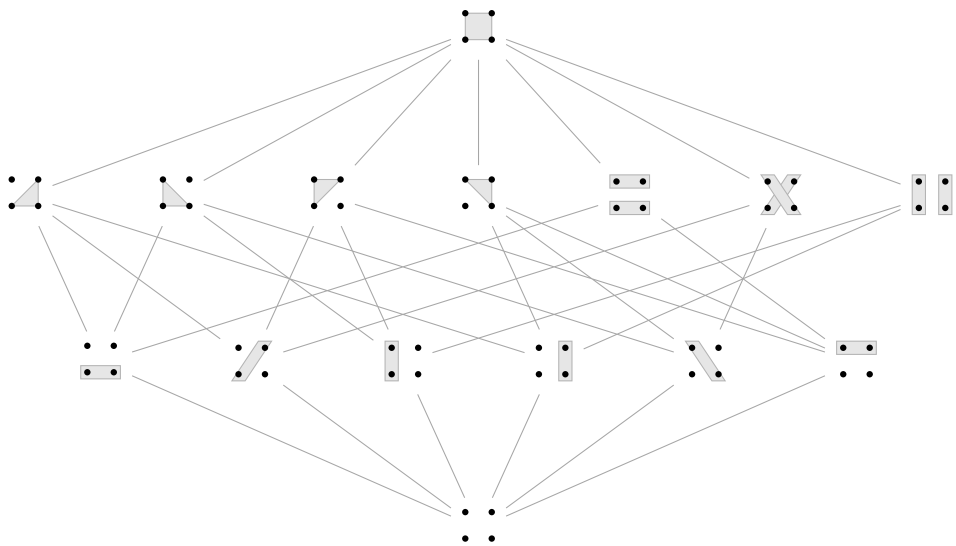


Figure 1: The partition lattice Π_4.

Theorem 4.1. *Let V be the top homology space for the set partition lattice Π_n and C_n the cyclic group acting on V defined above. Then, the triple*

$$\left(V, f(q) := \prod_{i=1}^{n-1} [i]_{q^i}, C_n\right)$$

exhibits up to sign an exotic cyclic sieving phenomenon with the normalization $||f(\zeta^d)||$ of (4.3).

Sketch of Proof. Stanley described[1] the character (4.1) explicitly: when $d|n$, we have

$$\mathrm{Tr}_V(c^d) = (-1)^{n+d}\mu(n/d)\cdot(d-1)!(n/d)^{d-1}, \tag{4.4}$$

where μ is the usual number-theoretic Möbius function. For other values of d, c^d is conjugate to $c^{(d,n)}$ and we may apply (4.4) again.

We will show that for any integer $d|n$, the normalized evaluation $||f(\zeta^d)||$ agrees up to sign with the right hand side of (4.4) (for other values of d, the same trick as above works). Note that in the product (with $\zeta = e^{2\pi i/n}$)

$$f(\zeta^d) = 1\cdot(1+\zeta^{2d})\cdot(1+\zeta^{3d}+\zeta^{6d})\cdots(1+\zeta^{d(n-1)}+\cdots+\zeta^{d(n-1)(n-2)}) = \prod_{i=1}^{n-1}\frac{1-\zeta^{di^2}}{1-\zeta^{di}}$$

[1]In fact, any other element $g \in S_n$ that is not conjugate to some c^d has zero trace in V.

it is possible that $1 = \zeta^{di^2}$ and simultaneously $1 \neq \zeta^{di}$ exactly when n/d has a square factor, i.e. when $\mu(n/d) = 0$.

Now, we are left with the case $\mu(n/d) \neq 0$ and we need to show that

$$||f(\zeta^d)|| = (d-1)! \cdot (n/d)^{d-1}.$$

When $i = k \cdot (n/d)$, it is clear that $||(1 + q^i + q^{2i} + \cdots + q^{i(i-1)})\Big|_{q=\zeta^d}|| = i = k \cdot (n/d)$. We will show that all remaining values of i contribute 1 to the norm. Indeed more generally, for any two integers m, s such that m is squarefree and does not divide s we have that

$$N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}\left(\frac{1-\zeta_m^{s^2}}{1-\zeta_m^{s}}\right) = 1,$$

where $\zeta_m = e^{2\pi i/m}$. If $r = m/(m,s)$ then using the transitivity of the field norm over towers, we can show that $N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(1-\zeta_m^s) = \Phi_r(1)^{\phi(m)/\phi(r)}$, where $\Phi_r(x)$ is the r-th cyclotomic polynomial. Because m is square-free, $(m,s) = (m,s^2)$ and the proof is complete because $N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(1-\zeta_m^s) = N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(1-\zeta_m^{s^2})$. $\square$

Example 4.2. Take $n = 12$ and $d = 4$, so that $\zeta_{12}^4 = \zeta_3 = e^{2\pi i/3}$. Any computer algebra system can easily calculate that $f(\zeta_3) = -81 + 81\sqrt{3} \cdot \mathrm{i}$. Here the normalization $|| \cdot ||$ agrees with the usual quadratic normalization, so that we have

$$||(-81 + 81\sqrt{3} \cdot \mathrm{i})|| = \sqrt{81^2 + 3 \cdot 81^2} = 162 = (4-1)! \cdot (12/4)^{4-1} = 3! \cdot 3^3,$$

exactly as expected.

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References

[1] D. Armstrong, V. Reiner, and B. Rhoades. "Parking spaces". *Adv. Math.* **269** (2015), pp. 647–706. DOI.

[2] C. A. Athanasiadis. "Generalized Catalan numbers, Weyl groups and arrangements of hyperplanes". *Bull. London Math. Soc.* **36**.3 (2004), pp. 294–302. DOI.

[3] C. A. Athanasiadis, T. Brady, and C. Watt. "Shellability of noncrossing partition lattices". *Proc. Amer. Math. Soc.* **135**.4 (2007), pp. 939–949. DOI.

[4] C. A. Athanasiadis and E. Tzanaki. "On the enumeration of positive cells in generalized cluster complexes and Catalan hyperplane arrangements". *J. Algebraic Combin.* **23**.4 (2006), pp. 355–375. DOI.

[5] D. Bessis and V. Reiner. "Cyclic sieving of noncrossing partitions for complex reflection groups". *Ann. Comb.* **15**.2 (2011), pp. 197–222. DOI.

[6] A. Björner. "Topological methods". *Handbook of combinatorics, Vol. 1, 2*. Elsevier Sci. B. V., Amsterdam, 1995, pp. 1819–1872.

[7] C. Ceballos and H. Mühle. "Revisiting generalizations of the Dehn-Sommerville relations". *Sém. Lothar. Combin.* **87** ([2022–2023]), Art. B87a, 25. DOI.

[8] A. D'Alì and E. Delucchi. "Equivariant Hilbert and Ehrhart series under translative group actions". 2025. arXiv:2312.14088. Link.

[9] S.-P. Eu and T.-S. Fu. "The cyclic sieving phenomenon for faces of generalized cluster complexes". *Adv. in Appl. Math.* **40**.3 (2008), pp. 350–376. DOI.

[10] S. Fomin and N. Reading. "Generalized cluster complexes and Coxeter combinatorics". *Int. Math. Res. Not.* 44 (2005), pp. 2709–2757. DOI.

[11] M. Josuat-Vergès. "Cluster parking functions II: dihedral symmetry". working paper or preprint. 2025. Link.

[12] H. Mühle. "EL-shellability and noncrossing partitions associated with well-generated complex reflection groups". *European J. Combin.* **43** (2015), pp. 249–278. DOI.

[13] L. Pouillart. "Cyclic sieving phenomena on parabolic classes of faces of the cluster complex". 2026. arXiv:2603.28242. Link.

[14] S. Rao and J. Suk. "Dihedral sieving phenomena". *Discrete Math.* **343**.6 (2020), pp. 111849, 12. DOI.

[15] V. Reiner, D. Stanton, and D. White. "The cyclic sieving phenomenon". *J. Combin. Theory Ser. A* **108**.1 (2004), pp. 17–50. DOI.

[16] V. Reiner. "Non-crossing partitions for classical reflection groups". *Discrete Math.* **177**.1-3 (1997), pp. 195–222. DOI.

[17] R. P. Stanley. "Some aspects of groups acting on finite posets". *J. Combin. Theory Ser. A* **32**.2 (1982), pp. 132–161. DOI.

[18] J. R. Stembridge. "On minuscule representations, plane partitions and involutions in complex Lie groups". *Duke Math. J.* **73**.2 (1994), pp. 469–490. DOI.

[19] J. R. Stembridge. "Some hidden relations involving the ten symmetry classes of plane partitions". *J. Combin. Theory Ser. A* **68**.2 (1994), pp. 372–409. DOI.

[20] M. L. Wachs. "Poset topology: tools and applications". *Geometric combinatorics*. Vol. 13. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, 2007, pp. 497–615. DOI.