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Quadratic exchange equations for Coxeter matroids

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Abstract. Tropicalisation (with trivial coefficients) is a process that turns a polynomial equation into a combinatorial predicate on subsets of the set of variables. We show that for each minuscule representation of a simple reductive group, there is a set of quadratic equations cutting out the orbit of the highest weight vector whose tropicalisation characterises the set of Coxeter matroids for that representation which satisfy the strong exchange property. In this extended abstract we focus on type B, with all other types being handled in [4].

Keywords: Coxeter matroids, strong exchange, minuscule weights, tropicalisation

1 Introduction

A linear space of dimension r over $\mathbb{C}^n$ represents a matroid of rank r on the ground set $\{1, \ldots, n\}$, and these linear spaces are parametrised by $\mathbb{C}$-valued points of the Grassmannian $\mathrm{Gr}(r, n)$. This relationship between the Grassmannian and matroids is used to great profit at the overlap of algebraic geometry and combinatorics, for example in tropical geometry. We will be tropicalising equations over $\mathbb{C}$ in this paper. To forestall confusion, let us say right away that when we do so we are only recording their monomial support, i.e. we are tropicalising with respect to the trivial valuation.

The parameter space for tropical linear spaces, known as the Dressian, strictly contains the tropicalisation of the Grassmannian. In the trivial valuation setting, tropical linear spaces are simply matroids, and the previous containment is the statement that the set of matroids strictly contains the set of realisable matroids. However, the ideal of

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the Grassmannian – over $\mathbb{Z}$, and therefore over $\mathbb{C}$ – contains a privileged set of quadrics, the quadratic (Grassmann–) Plücker relations, and tropicalising only these quadrics cuts out exactly the set of matroids (and more generally the Dressian).

Many have studied how well this happy state of affairs replicates for other quotients $\mathbb{G}/\mathbb{P}$ of a reductive group by a parabolic subgroup. In these studies, the first choice to be made is an embedding of $\mathbb{G}/\mathbb{P}$ in a projective space with a prescribed coordinate basis, which is required to define the tropicalisation. Our objective in this project is to do so uniformly in $\mathbb{G}$. Representation theory gives a uniform procedure to define a projective embedding of $\mathbb{G}/\mathbb{P}$ in a projectivised $\mathbb{G}$-representation $\operatorname{proj}(V_\lambda)$, in which it is cut out by quadrics. As for the coordinate basis, for Grassmannians we can concisely express the choice as the basis of homogeneous linear forms in the weight grading for $\mathbb{G}$; since the weight spaces are one-dimensional, this uniquely specifies our basis up to scalars. The same approach works for $\mathbb{G}/\mathbb{P}$ when V_λ is minuscule (we also call $\mathbb{P}$ minuscule in this case), i.e. all the weights of V_λ are in a single Weyl group orbit. This implies that all weight spaces are one-dimensional. So it is the minuscule case we study here. It is for convenience in the representation theory that we have fixed our coefficient field to be $\mathbb{C}$.

On the other side of the connection, Coxeter matroids [1] are to $\mathbb{G}/\mathbb{P}$ as matroids are to the Grassmannian. A Coxeter matroid is any subpolytope of the orbit polytope for $\mathbb{P}$ of the Weyl group $W(\mathbb{G})$ whose edges are parallel to roots of $\mathbb{G}$ (see Section 2.2 for a careful definition). With this, we have every piece but one of our main theorem:

Theorem 1. *Let $\mathbb{G}$ be a simply connected complex Lie group and $\mathbb{P}$ a minuscule parabolic subgroup. There exists a set Q spanning the quadrics cutting out $\mathbb{G}/\mathbb{P} \subset \operatorname{proj}(V_\lambda)$ such that a set is the vertex set of a strong Coxeter matroid if and only if it satisfies the tropicalisations of Q.*

The one word yet to be defined is "strong". We say that a Coxeter matroid P is strong if for any two vertices v, w of P, either $\operatorname{conv}\{v, w\}$ is parallel to a root or it is the diagonal of a parallelogram all four of whose vertices are vertices of P. (Such Coxeter matroids are also said to satisfy the strong exchange property. For more details see Section 2.2 below.) The condition of being strong will emerge from the structure of tropicalised quadrics homogeneous in the weight grading. In particular, if P is realisable in the sense that there is a point $x \in \mathbb{G}/\mathbb{P}$ so that a vertex v of the orbit polytope is present in P if and only if the weight-v coordinate of x is nonzero, then P satisfies every tropical equation of $\mathbb{G}/\mathbb{P}$, so P is strong. All (ordinary) matroids are strong, but not, for example, all Δ-matroids (Section 2.3).

Borovik, Gelfand and White recognised the importance of minuscule representations in Open Problem 6.16.1 of [1]. Our theorem can be read as morally confirming the "if" direction of their expectation in that open problem, strengthened by the connection to equations of $\mathbb{G}/\mathbb{P}$.

Despite the statement of Theorem 1 being type-agnostic, the proof is an analysis of each individual type in the classification of minuscule representations. In this extended abstract we will focus entirely on type B, leaving the other types for our paper [4], outside type A (usual Grassmannians) which is well known. We sketch the structure of this extended abstract. Section 2 is dedicated to background on Coxeter matroids. In Section 3 we characterise strong Coxeter matroids in terms of the distribution of pairs of antipodal vertices on faces of the orbit polytope for type B. In Section 4 we introduce the quadrics we need in type B. We show the agreement of their tropicalisation with the characterisation that we obtained in Section 3, proving Theorem 11. The calculations for the quadrics and all proofs can be found in [4].

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2 Coxeter matroids

In this section we fix notation for Coxeter matroids. For a more thorough introduction, the interested reader is invited to look at [1, 5, 6, 7, 8].

2.1 Coxeter groups

2.1.1 Coxeter groups, root systems and Weyl groups

Let $\mathcal{V}$ be an n-dimensional Euclidean real vector space with inner product $(\cdot,\cdot)$. For a given hyperplane $H \subset \mathcal{V}$, we let s_H denote the reflection which fixes H pointwise and sends a normal vector of H to its opposite. A Coxeter group W is a finite group generated by a set of reflections in the orthogonal group $O(\mathcal{V})$. A reflecting hyperplane for W is a hyperplane H such that $s_H \in W$. The set of reflecting hyperplanes of W forms the Coxeter arrangement of W.

For a given W we consider a root system $\Phi \subset \mathcal{V}$ and let Φ^+ and Φ^- denote the set of positive roots and negative roots respectively. We let $\Pi \subseteq \Phi^+$ denote the set of simple roots and $S \subseteq W$ consist of the reflections with a simple root as a normal vector, named the simple reflections. Then S generates W. We will require our root systems be crystallographic: they satisfy $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$ for all α, $\beta \in \Phi$. For ease of notation we let $\langle \alpha, \beta \rangle := \frac{2(\alpha,\beta)}{(\beta,\beta)}$. A Weyl group is a Coxeter group with a crystallographic root system.

The Dynkin diagram of a Weyl group encodes the above data as a graph where the vertex set is the set of simple roots and the number of edges between two roots is given by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$. In addition, we draw an arrow pointing to the smaller of the two roots if they have different lengths. The following is a list of all Dynkin diagrams for irreducible Weyl groups.

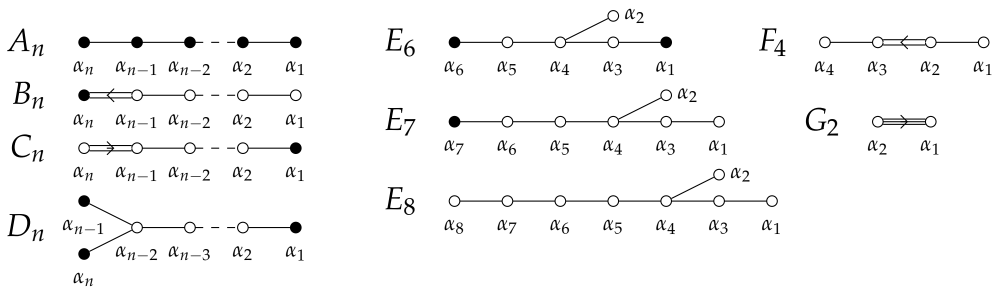


Figure 1: The Dynkin diagrams for all irreducible Weyl groups. The simple roots associated to minuscule fundamental weights are highlighted in black.

2.1.2 Weights and representations

For a given crystallographic root system Φ with a fixed choice of simple roots Π, let $\mathfrak{g}$ be its associated complex semisimple Lie algebra and let $\mathfrak{h}$ denote a Cartan subalgabra of $\mathfrak{g}$. A representation of $\mathfrak{g}$ is a pair $V_\pi := (\pi, V)$ where V is a complex vector space and $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism. We say V_π is finite-dimensional if V is finite-dimensional, and irreducible if there are no non-zero $\mathfrak{g}$-invariant proper subspaces.

For V_π and $\lambda \in \mathbb{R}\Phi$ set $V(\lambda) = \{v \in V : \pi(h)v = \langle \lambda, h\rangle\, v \text{ for all } h \in \mathfrak{h}\}$. Whenever $V(\lambda) \neq 0$ we call λ a weight of V_π and $V(\lambda)$ a weight space. We let $\Lambda = \{\lambda \in \mathbb{R}\Phi : \langle \lambda, h\rangle \in \mathbb{Z} \text{ for all } h \in \mathfrak{h}\}$ denote the set of all weights. Let $\{\lambda_i\}_{i\in[n]}$ denote the dual basis of the vectors $\left\{\frac{2\alpha_i}{(\alpha_i,\alpha_i)}\right\}_{i\in[n]}$. The λ_i are known as the fundamental weights. The set $\Lambda^+ = \{\lambda \in \Lambda : \lambda = \sum c_i\lambda_i,\ c_i \geq 0\}$ is known as the set of dominant weights.

It is well-known that a representation V of $\mathfrak{g}$ can be decomposed such that $V = \bigoplus V(\lambda)$ where each $V(\lambda)$ is a weight space associated to a weight λ. For every irreducible finite dimensional representation V_π there is a unique maximal dominant (integral) weight in the root poset which we call the highest weight of V_π. When we use the notations V_λ for an irreducible finite-dimensional representation of $\mathfrak{g}$ we will assume that λ is the unique highest weight of V_λ unless otherwise specified.

2.1.3 Parabolic subgroups and minuscule representations

For a given subset $J \subseteq S = \{s_1, \ldots, s_n\}$, we let $P_J = \langle J\rangle$ be the standard parabolic subgroup of W generated by J and $W^J = W/P_J$ denote the standard parabolic cosets. The maximal parabolic subgroups are precisely the standard parabolic subgroups of W where $J = S \setminus \{s_i\}$ for some $i \in [n]$. The maximal parabolic subgroups are an important part of our study due to their relationship with minuscule representations.

Given a representation V_λ of $\mathfrak{g}$, let $J = \{s \in S : s(\lambda) = \lambda\}$. A fundamental weight λ_i is said to be minuscule if $\langle \lambda_i, \alpha\rangle \leq 1$ for all $\alpha \in \Phi^+$. Equivalently, and crucially,

λ_i is minuscule if and only if the weights of V_{λ_i} are in one Weyl group orbit. If λ_i is minuscule, we say that the representation V_{λ_i} is minuscule. Similarly, we say that the maximal parabolic subgroup P_J for $J = S \setminus \{s_i\}$ is minuscule. In Figure 1, the simple roots associated to minuscule fundamental weights are highlighted in black. Note that Lie algebras of type E_8, F_4 and G_2 do not have any minuscule weights.

2.2 Coxeter matroids

2.2.1 Coxeter matroids from polytopes

For a given standard parabolic subgroup P_J, we choose a point $\omega_J \in V$ such that ω_J is in the intersection of all hyperplanes of the Coxeter arrangement restricted to the parabolic subgroup and is in the negative half-space of the rest. The choice of ω_J allows us to define an injective map from cosets W^J to the vector space $\mathcal{V}$ under which $wP_J \mapsto w \cdot \omega_J$. Thus, we will write $A \cdot \omega_J$ for the point associated to the coset $A \in W^J$.

Given a subset $M \subseteq W^J$, we let $P(M)$ be the polytope constructed by taking the convex hull of $A \cdot \omega_J$ for all $A \in M \subseteq W^J$. When $M = W^J$, we call $P(M) = \mathrm{conv}\{W \cdot \omega_J\}$ the ambient polytope of W^J. We say that M is a Coxeter matroid if and only if every edge of $P(M)$ is parallel to a root in Φ.

2.2.2 Strong exchange property

The strong exchange property for Coxeter matroids is the following property that a Coxeter matroid $M \subseteq W^J$ may have:

$$\begin{gathered}\text{for any distinct cosets } A, B \text{ in } M \subseteq W^J \\ \text{there is a reflecting hyperplane } H \text{ separating } A \text{ and } B \text{ such that } s_H A, s_H B \in M\,. \end{gathered} \tag{2.1}$$

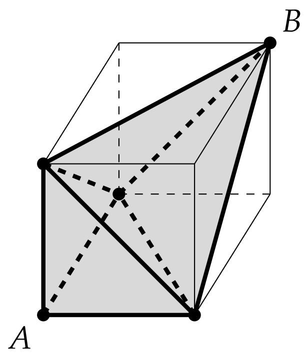
We say a Coxeter matroid is strong if it satisfies the strong exchange property. In fact the strong exchange property is a sufficient condition for an arbitrary subset $M \subseteq W^J$ to form a Coxeter matroid.

All ordinary matroids are strong [3], but not every Coxeter matroid satisfies the strong exchange property in general. Borovik, Gelfand and White ask the following in Open Problem 6.16.1 of [1] as a potential characterisation of strong Coxeter matroids.

Question 2. Let W be a Coxeter group and let P be a parabolic subgroup of W. Is it true that every matroid for W and P satisfies the strong exchange property if and only if P is a minuscule weight parabolic subgroup?

The next example answers this question in the negative, showing that there exists a matroid M for W and P such that P is a minuscule weight parabolic subgroup and M does not satisfy the strong exchange property.

Example 3. Let W be the type B_n Coxeter group generated by $\{s_1, s_2, s_3\}$ and let $J = \{s_1, s_2\}$. Then P_J is a minuscule weight parabolic. Fixing $\omega_J = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, the ambient polytope $\operatorname{conv}\{W \cdot \omega_J\}$ is the 3-cube $[-\frac{1}{2}, \frac{1}{2}]^3$ with vertices associated to W^J. Suppose $M \subseteq W^J$ has polytope as drawn here:



As every edge is parallel to a root, this subset of cosets gives us a Coxeter matroid by definition. Then taking A and B to be the cosets whose vertices are shown in the figure, one can verify that no such hyperplane H exists such that both $s_H A$ and $s_H B$ are in M.

2.3 Δ-matroids

In this section we define Δ-matroids, first as set systems then as type B_n Coxeter matroids. A set system on a finite set E is a set of subsets of E.

Definition 4 ([2, Definition 6])**.** A Δ-matroid is a set system M on $[n]$ such that the following symmetric exchange property for Δ-matroids is satisfied:

$$\begin{gathered}\text{For all } A, B \in M,\ a \in A \triangle B, \text{ there exists } b \in A \triangle B \\ \text{such that } A \triangle \{a, b\} \in M\,.\end{gathered} \tag{2.2}$$

Note that a and b might be equal.

We consider the following strong exchange property for Δ-matroids:

$$\begin{gathered}\text{For all } A, B \in M,\ a \in A \triangle B, \text{ there exists } b \in A \triangle B \\ \text{such that } A \triangle \{a, b\}\,,\ B \triangle \{a, b\} \in M\,.\end{gathered} \tag{2.3}$$

We emphasise that (2.3) again allows a, b to be equal. This is a specialisation of the strong exchange property (2.1) for Coxeter matroids to type B_n.

We now describe how all Δ-matroids arise as Coxeter matroids of type B_n. The root system of B_n has the following choices of simple roots:

$$\Pi(B_n) = \{\alpha_i := e_i - e_{i+1} :\ 1 \le i \le n-1\} \cup \{\alpha_n := e_n\}\,.$$

For $W = W(B_n)$, the only minuscule parabolic subgroup is P_J where $J = S \setminus \{s_n\}$. By concretely choosing $\omega_J = (-\frac{1}{2}, \ldots, -\frac{1}{2})$, we see that the ambient polytope is the cube $[-\frac{1}{2}, \frac{1}{2}]^n$. As with ordinary matroids, we can identify set systems with subpolytopes of this cube by labelling its vertices by sets. For a set $I \subseteq [n]$, define $\varepsilon_I \in \mathcal{V} = \mathbb{R}^n$ to be the indicator vector of I translated by $(-\frac{1}{2}, \ldots, -\frac{1}{2})$. Then

$$Q_n := \{\varepsilon_I : I \subseteq [n]\} = W(B_n) \cdot \omega_{S\setminus\{s_n\}}, \quad (\varepsilon_I)_i = \begin{cases} -\frac{1}{2} & i \notin I \\ \frac{1}{2} & i \in I\,. \end{cases} \tag{2.4}$$

The ambient polytope is the cube obtained as the convex hull of these points. With this dictionary established, it follows that the strong exchange property for Δ-matroids is a specialisation of the strong exchange property for Coxeter matroids to type B_n for the parabolic subgroup that excludes s_n.

Example 5. Let M be the following set system on $[3]$:

$$M = \{\varnothing, 1, 2, 3, 123\}\,.$$

These sets correspond to vertices of the polytope from Example 3 under the identification (2.4). It follows that M is a Δ-matroid but does not satisfy the strong exchange property. Taking $A = \varnothing$ and $B = 123$, we see there is no choice of a and b such that both $A \triangle \{a, b\}$ and $B \triangle \{a, b\}$ are contained in M, as at least one must have cardinality two.

3 Characterising strong Coxeter matroids

In this section we give a combinatorial characterisation of strong minuscule Coxeter matroids M in terms of tropical equations for type B. Recall we handle the other types in [4].

3.1 Preliminaries

3.1.1 Tropical equations

We define tropical equations here in a bare-bones way. For context and a detailed treatment of tropical algebra, see [11].

Let $\mathbb{B} = (\{0,1\}, \oplus, \odot)$ be the Boolean semifield, where 0 is the zero, 1 the multiplicative identity, and $1 \oplus 1 = 1$. Let X be a finite set of indeterminates. Then $\mathbb{B}[X]$ is the monoid semiring with coefficients in $\mathbb{B}$ of the free commutative monoid $\mathbb{N}^X = \mathrm{Hom}_{\mathrm{Set}}(X, \mathbb{N})$ on the generators X. Explicitly, for $a \in \mathbb{N}^X$, let $X^{\odot a}$ denote the formal monomial $\bigodot_{x\in X} \underbrace{x \odot \cdots \odot x}_{a(x)\text{ times}}$. Then every element of $\mathbb{B}[X]$ is of the form

$$\bigoplus_{a\in A} X^{\odot a}$$

for some finite subset $A \subset \mathbb{N}^X$. Addition in $\mathbb{B}[X]$ corresponds to union of the sets A, and multiplication to Minkowski sum. The tropicalisation map trop : $\mathbb{C}[X] \to \mathbb{B}[X]$ is defined by $\operatorname{trop}(\sum_{a\in A} z_a X^a) = \bigoplus_{a\in A} X^{\odot a}$ where $z_a \in \mathbb{C}$ is nonzero for all $a \in A$.

When we call an element $f \in \mathbb{B}[X]$ a tropical equation, we are thinking of it as a condition that may or may not be satisfied by a tuple $p \in \mathbb{B}^X$. The tropical equation $f = \bigoplus_{a\in A} X^{\odot a}$ is satisfied by p if the number of $a \in A$ such that $p^{\odot a} = 1$ – i.e. such that $p(x) = 1$ for all x with $a(x) > 0$ – is not exactly 1.

The tropical equations arising in this section will have $X = \{x_S : S \subseteq [n]\}$. Set systems M on $[n]$ are in bijection with tuples $\nu_M \in \mathbb{B}^X$ by the rule that $\nu_M(x_S) = 1$ if and only if $S \in M$. In this way we may make sense of saying that a set system M satisfies a tropical equation $f \in \mathbb{B}[X]$: we mean that ν_M satisfies f.

3.2 Cubes

This section deals with Δ-matroids which can be viewed as subpolytopes of cubes. The corresponding formulations in tropical equations are Theorem 8.

3.2.1 Δ-matroids (Type B)

Strong exchange equations (Type B) The type B strong exchange equations on $[n]$ are the equations in the collection $\mathcal{F}_n^{(B)} := \left\{ f_{I,J}^{(B)} : I, J \subseteq [n], |I \triangle J| \geq 3 \right\}$ where

$$f_{I,J}^{(B)} := \bigoplus_{i\in I\triangle J} x_{I\triangle i} \odot x_{J\triangle i} \qquad \text{if } |I \triangle J| \equiv 0 \mod 2 \text{ and} \tag{3.1}$$

$$f_{I,J}^{(B)} := x_I \odot x_J \oplus \bigoplus_{i\in I\triangle J} x_{I\triangle i} \odot x_{J\triangle i} \qquad \text{if } |I \triangle J| \equiv 1 \mod 2\,. \tag{3.2}$$

Remark 6. While not necessary for any proofs, we give some geometric intuition for these equations in terms of antipodes of cubes. Recall from (2.4) the notation $\varepsilon_I \in \mathbb{R}^n$ for the vertex of the cube Q_n indexed by the subset $I \subseteq [n]$. Given two sets $I, J \subseteq [n]$, we can consider the subcube of Q_n

$$\square_{I,J} := \{\varepsilon_K : I \cap J \subseteq K \subseteq I \cup J\} \subseteq Q_n\,.$$

This notation has the property that ε_I is an antipode of ε_J within $\square_{I,J}$: by an abuse of notation we will call the pair $(\varepsilon_I, \varepsilon_J)$ an antipode. Moreover, for each $S \subseteq I \triangle J$ we also have that $(\varepsilon_{I\triangle S}, \varepsilon_{J\triangle S})$ is an antipode of $\square_{I,J}$. In particular, the antipodes that neighbour $(\varepsilon_I, \varepsilon_J)$ along an antipodal pair of edges are of the form $(\varepsilon_{I\triangle i}, \varepsilon_{J\triangle i})$ for $i \in I \triangle J$.

With this viewpoint, the equation (3.1) fixes some antipode $(\varepsilon_I, \varepsilon_J)$ and is satisfied by ν_M if and only if there is not exactly one neighbouring antipode present in the polytope $P(M)$. The equation (3.2) similarly fixes some antipode $(\varepsilon_I, \varepsilon_J)$ and is satisfied by ν_M if and only if there is not exactly one of $(\varepsilon_I, \varepsilon_J)$ or its neighbouring antipodes present in M.

Example 7. To better understand these equations, we work through them for $n = 3$ and $n = 4$. For ease of notation we will let $i_1 \dots i_k$ denote $\{i_1, \dots, i_k\}$, e.g. 12 denotes $\{1, 2\}$.

Let $n = 3$, where the list of subsets of [3] is $\varnothing$, 1, 2, 3, 12, 13, 23, 123. The requirement that $|I \triangle J| \geq 3$ means that $\mathcal{F}_3^{(B)}$ contains no equation in which $|I \triangle J| \equiv 0 \mod 2$. There are potentially four equations with $|I \triangle J| = 3 \equiv 1 \mod 2$. Two of these are

$$\begin{aligned} f^{(B)}_{\varnothing,123} &= x_\varnothing \odot x_{123} \oplus \bigoplus_{i \in 123} x_{\varnothing \triangle i} \odot x_{123 \triangle i} = x_\varnothing \odot x_{123} \oplus x_1 \odot x_{23} \oplus x_2 \odot x_{13} \oplus x_3 \odot x_{12} \\ f^{(B)}_{1,23} &= x_1 \odot x_{23} \oplus \bigoplus_{i \in 123} x_{1 \triangle i} \odot x_{23 \triangle i} = x_1 \odot x_{23} \oplus x_\varnothing \odot x_{123} \oplus x_{13} \odot x_2 \oplus x_{12} \odot x_3 \end{aligned} \tag{3.3}$$

We see that they are equal elements of $\mathbb{B}[X]$, giving us a sum of antipodes of the 3-cube. The other two equations are equal to this same sum as well. So in fact $\left|\mathcal{F}_3^{(B)}\right| = 1$.

When $n = 4$, the set $\mathcal{F}_4^{(B)}$ contains the S_4-orbit of the equation in (3.3) of size 4, and three additional S_4-orbits of sizes 1, 1, and 4 respectively:

$$\begin{aligned} f^{(B)}_{\varnothing,1234} &= \bigoplus_{i \in 1234} x_{\varnothing \triangle i} \odot x_{1234 \triangle i} = x_1 \odot x_{234} \oplus x_2 \odot x_{134} \oplus x_3 \odot x_{124} \oplus x_4 \odot x_{123} = f^{(B)}_{12,34} \\ f^{(B)}_{1,234} &= \bigoplus_{i \in 1234} x_{1 \triangle i} \odot x_{234 \triangle i} = x_\varnothing \odot x_{1234} \oplus x_{12} \odot x_{34} \oplus x_{13} \odot x_{24} \oplus x_{14} \odot x_{23} \\ f^{(B)}_{1,1234} &= x_1 \odot x_{1234} \oplus \bigoplus_{i \in 234} x_{1 \triangle i} \odot x_{1234 \triangle i} = x_1 \odot x_{1234} \oplus x_{12} \odot x_{134} \oplus x_{13} \odot x_{124} \oplus x_{14} \odot x_{123} \end{aligned}$$

Altogether we have $\left|\mathcal{F}_4^{(B)}\right| = 10$.

It turns out that the type B strong exchange equations precisely characterise strong Δ-matroids.

Theorem 8. *Let M be a set system on $[n]$. Then M is a strong Δ-matroid if and only if ν_M satisfies the type B strong exchange equations $\mathcal{F}_n^{(B)}$.*

This gives us a novel combinatorial way to view the strong exchange property of a type B Coxeter matroid. In the following section, we see that these same equations are tropical versions of equations coming from the Lichtenstein embeddings.

4 Equations from homogenous spaces

In this section, we show that the strong exchange equations defined in Section 3 arise as tropicalisations of equations of the Lichtenstein embedding, the inclusion of the homogenous space $\mathbb{G}/\mathbb{P}_\alpha$ into the projective space $\mathrm{proj}(V_\lambda)$ (Section 4.1.1). For the minuscule case of type B, we will derive a set of quadratic embedding equations whose tropicalisations match the previous section and which span the degree 2 graded component of the Lichtenstein embedding equations. Recall that proofs and the miniscule cases of all other types can be found in [4].

4.1 Preliminaries

4.1.1 Minuscule varieties

We begin with some preliminaries on the Lie theoretic approach to Grassmannians and their generalisations, minuscule varieties, in particular how we obtain equations cutting out these miniscule varieties. A general reference for this section is [5].

Let W be a Weyl group with root system Φ and a particular (fixed) choice of simple roots Π. Fix $\alpha \in \Pi$ and let $P_{S\setminus\{s_\alpha\}}$ be the parabolic subgroup of W associated to $\Pi \setminus \{\alpha\}$. A Coxeter matroid is a subset of $W^{\Pi\setminus\alpha} = W/P_{S\setminus\{s_\alpha\}}$ whose associated polytope $P(M)$ has edges parallel to roots (see Section 2.2.1). Let $\mathbb{G}$ be the simply connected complex Lie group with root system Φ and Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathbb{G})$. Let $\{\lambda_i\}$ be the set of fundamental weights in bijection with Π. We fix λ to be the fundamental weight that pairs with the omitted simple root α and let V_λ be the irreducible representation of $\mathbb{G}$ with highest weight λ and highest weight vector v_λ. The homogeneous space we study in this section is $\mathbb{G}/\mathbb{P}_\alpha$ where $\mathbb{P}_\alpha$ is the stabiliser of $v_\lambda \in \mathrm{proj}(V_\lambda)$. The subgroup $\mathbb{P}_\alpha$ is a maximal parabolic of $\mathbb{G}$ and $\mathbb{G}/\mathbb{P}_\alpha$ embeds into the projective space $\mathrm{proj}(V_\lambda)$ as the $\mathbb{G}$-orbit of v_λ.

The equations that cut out $\mathbb{G}/\mathbb{P}_\alpha$ in $\mathrm{proj}(V_\lambda)$ are all quadrics, hence we can view them as linear forms on $S^2(V_\lambda)$. Moreover, they can be calculated by the action of the Casimir operator Ω [10].

Rather than working directly with the Casimir operator, consider the decomposition of $S^2(V_\lambda) = \bigoplus V_\mu$ into irreducible $\mathbb{G}$-modules. Fixing a basis for each V_μ, gives a basis $\{b_i\}$ of $S^2(V_\lambda)$ with associated dual basis $\{b_i^\vee\}$ of $S^2(V_\lambda^\vee)$. We consider the dual basis as a basis for the space of quadrics on V_λ. The following lemma states that this choice of basis gives a simple description of the quadratic equations cutting out $\mathbb{G}/\mathbb{P}$ inside $\mathrm{proj}(V_\lambda)$.

Lemma 9. *Let $S^2(V_\lambda) = \bigoplus V_\mu$ be a decomposition into irreducible $\mathbb{G}$-modules and let the set $\{b_1, \dots, b_k\}$ a basis for $S^2(V_\lambda)$ such that $b_i \in V_\mu$ for some weight μ. Then the system of quadrics cutting out $\mathbb{G}/\mathbb{P}$ inside* $\mathrm{proj}(V_\lambda)$ *is the subspace*

$$\mathrm{Span}\left\{b_i^\vee : b_i \notin V_{2\lambda}\right\} \subset S^2(V_\lambda^\vee), \tag{4.1}$$

where $\{b_1^\vee, \dots, b_k^\vee\}$ is the dual basis for $S^2(V_\lambda^\vee)$.

We call the equations (4.1) from Lemma 9 the Lichtenstein embedding equations. While these are elegantly phrased in this basis, they do not yield explicit polynomial equations in terms of variables on V_λ. To do obtain explicit equations, we restrict our analysis to minuscule representations for the following reason.

Recall we can decompose V_λ as a direct sum of weight spaces $V_\lambda = \bigoplus_{\mu\in\Lambda} V_\lambda(\mu)$. If V_λ is minuscule, each weight space is one-dimensional. Fixing some vector $v_\mu \in V_\lambda(\mu)$ for each $\mu \in \Lambda$ gives us a canonical basis $\{v_\mu : \mu \in \Lambda\}$ for V_λ up to scaling. Likewise, we get a canonical dual basis $\{x_\mu : \mu \in \Lambda\}$ for $V_\lambda^\vee$. In this sense, the equations (4.1) can be viewed not just as elements of $S^2(V_\lambda)^\vee \cong S^2(V_\lambda^\vee)$, but as quadrics in variables x_μ indexed by the weights of V_λ. This allows us to write the Lichtenstein embedding equations from Lemma 9 as explicit polynomial equations using only the weight space decomposition.

4.2 Cubes

4.2.1 Type B embedding equations

We begin with type B_n, where the only minuscule fundamental weight is λ_n corresponding to the simple root α_n. This means we have simply connected complex Lie group $\mathbb{G} = \mathrm{Spin}(2n+1)$ with corresponding Lie algebra $\mathfrak{so}_{2n+1}$. The fundamental representation of $\mathfrak{so}_{2n+1}$ corresponding to λ_n is the spin representation $\mathcal{S}$. It is 2^n-dimensional whose weights are $(\pm\frac{1}{2}, \dots, \pm\frac{1}{2})$. Hence, if $\mathbb{P} = \mathbb{P}_{\alpha_n}$ is the maximal parabolic corresponding to the last fundamental weight we have that $\mathbb{G}/\mathbb{P} \subset \mathrm{proj}(\mathcal{S})$.

Recall that we define the embedding equations cutting out $\mathbb{G}/\mathbb{P} \subset \mathrm{proj}(\mathcal{S})$ in terms of variables indexed by the weights of $\mathcal{S}$. The weights of $\mathcal{S}$ are exactly $Q_n = \{\varepsilon_I : I \subseteq [n]\}$ as defined in (2.4) and hence in 1-1 correspondence with subsets of $[n]$ or, equivalently, the vertices of the n-cube. Hence, we write $X = \{x_I : I \subseteq [n]\}$ where x_I is the variable corresponding to weight ε_I.

For any two sets $A, B \subseteq [n]$, define $\sigma'_{A,B}$ to be the permutation which stably sorts the concatenation $A \circ B$, where A and B are written in increasing order, into weakly increasing order overall. That is, for each $i \in A \cap B$, the order of the is remains the same. We write $|\sigma'_{A,B}|$ for the length of this permutation.

Definition 10. The type B quadratic embedding equations are the collection of equations

$$\mathcal{E}_n^{(B)} := \left\{ F_{I,J}^{(B)} : I, J \subseteq [n], |I \triangle J| \geq 3 \right\} \quad \text{where}$$

$$F_{I,J}^{(B)} := \sum_{i\in I\triangle J} (-1)^{|\sigma'_{i,I}|+|\sigma'_{i,J}|} x_{I\triangle i} x_{J\triangle i} \qquad \text{if } |I \triangle J| \equiv 0 \mod 2 \tag{4.2}$$

$$F_{I,J}^{(B)} := -x_I x_J + \sum_{i\in I\triangle J} (-1)^{|\sigma'_{i,I}|+|\sigma'_{i,J}|} x_{I\triangle i} x_{J\triangle i} \qquad \text{if } |I \triangle J| \equiv 1 \mod 2 \tag{4.3}$$

The tropicalisation of the type B quadratic embedding equations $\mathcal{E}_n^{(B)}$ is clearly equal to the type B strong exchange equations $\mathcal{F}_n^{(B)}$. Moreover, we motivate calling them the quadratic embedding equations via the following theorem.

Theorem 11. *Let $\mathbb{G} = \mathrm{Spin}(2n+1)$ and $\mathbb{P}$ be a the maximal parabolic corresponding to the last fundamental weight. The type B quadratic embedding equations $\mathcal{E}_n^{(B)}$ are a spanning set for the equations of the embedding $\mathbb{G}/\mathbb{P} \subset \mathrm{proj}(\mathcal{S})$ and tropicalise to the type B strong exchange equations $\mathcal{F}_n^{(B)}$.*

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