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Flows on cyclic graphs, complete gentle algebras, and the Mutoperhedron

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Abstract. The flow cone of a directed acyclic graph admits a family of unimodular triangulations, as described by Danilov, Karzanov, and Koshevoy (DKK), that are of active interest. von Bell, Braun, Bruegge, Hanely, Peterson, Serhiyenko, and Yip recently discovered a correspondence between these triangulations for certain graphs and maximal cones of a g-vector fan of an associated gentle quiver. This correspondence has already been fruitful in uncovering lattice structures in the triangulations. In this extended abstract, we extend the correspondence to certain graphs with cycles. We also give a cyclic analogue of the results of Baldoni–Vergne and Postnikov–Stanley, expressing the volume of the flow polytope of acyclic graphs as the number of certain integer flows on that same graph. In addition, we introduce a new polytope, the *mutoperhedron*, whose f-vector is equal to that of the permutohedron but which has a different combinatorial type.

Keywords: Flows, locally gentle quivers, quiver representations, polytopes, fans.

1 Introduction

von Bell, Braun, Bruegge, Hanely, Peterson, Serhiyenko, and Yip [6] made a connection between flows on certain directed acyclic graphs and gentle algebras, building on earlier work of Danilov, Karzanov, and Koshevoy [9]. We strengthen and extend this link to include certain cyclic graphs and complete gentle algebras.

A ***balanced flow*** on a directed acyclic graph $H = (V, E)$ is an assignment of a nonnegative real number weight on each $e \in E$ such that, at any internal vertex, the sum of the weights of the incoming edges equals the sum of the weights of the outgoing edges.

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The collection of all balanced flows forms a cone $\mathcal{F}^+(H)$ and a specific slice of this cone called the ***flow polytope*** $\mathcal{F}_1(H)$, obtained by considering the flows that have the sum of the weights of the outgoing edges of sources equal to 1, is widely studied and has interesting properties. For instance, Baldoni–Vergne [5] and Postnikov–Stanley [14] showed that the volume of $\mathcal{F}_1(H)$ equals the number of certain integer flows on H.

In [9], Danilov, Karzanov, and Koshevoy defined a family of unimodular triangulations $\mathsf{DKK}(H,F)$ of these cones and polytopes where F is a ***framing*** of H, i.e., an ordering of incoming and outgoing edges at each vertex. The rays of $\mathcal{F}^+(H)$ are the indicator vectors of the paths ρ of H that go from a source to a sink, called ***routes***. Using the framing, they defined what it means for two routes to be compatible, and the maximal compatible sets of routes, called ***maximal cliques***, give the triangulation $\mathsf{DKK}(H,F)$.

Some routes appear in every maximal clique; in other words, they are compatible with any other route. These routes are said to be ***exceptional***. One can define the reduced triangulation $\mathsf{DKK}_{\mathrm{red}}(H,F)$ as the projection of $\mathsf{DKK}(H,F)$ via the quotient of $\mathcal{F}(H)$ by the vector space $E(H,F)$ spanned by the flows given by the exceptional routes. A framing F of H is ***ample*** whenever $\mathsf{DKK}_{\mathrm{red}}(H,F)$ is complete; meaning that the union of the cones in $\mathsf{DKK}_{\mathrm{red}}(H,F)$ is the entire $\mathcal{F}(H)/E(H,F)$. In [6], the authors fully characterize the directed acyclic graphs that have an ample framing. Explicitly, they are the graphs such that every internal vertex has exactly two incoming and two outgoing edges. These graphs are called ***full***. An edge $e \in E$ between internal vertices is called ***idle*** if it is either the only edge coming out of its tail or the only edge going to its head.

Theorem 1 ([6])**.** *Let H be a directed acyclic graph with no idle edges. Ample framings of H exist if and only if H is full.*

For this reason, we only consider full graphs (acyclic or not). In this context, a framing can be described as a bi-colouring of the edges. In this case, exceptional routes are all the monochromatic ones.

Now, the key insight of [6], linking the study of flows to gentle algebras, is that, given a choice of ample framing for H, it is possible to associate to H a gentle quiver (Q_H, R_H), defining a finite-dimensional algebra. Then, the maximal cones of $\mathsf{DKK}(H)$ (or of $\mathsf{DKK}_{\mathrm{red}}(H)$) are in natural correspondence with the maximal cones of the g-vector fan of (Q_H, R_H). The g-vector fan is an invariant of a finite-dimensional algebra, introduced by Adachi, Iyama, and Reiten in [2] that has received considerable attention [13].

Making the connection between triangulations of the flow cone and the g-vector fan of a corresponding algebra is not just a curiosity: it also has implications on both sides of the connection. This has already been exploited in [6] and [8].

The main objective of this paper is to expand this theory to a bigger family of directed graphs. To do so, we relax the acyclic assumption in the study of flow cones of directed acyclic graphs and allow full graphs that have directed cycles. See the full version [1] for all the details and proofs.

We begin this abstract by defining a variant of the ample framing called ***cyclic ample framing***, requiring every minimal directed cycle to be monochromatic (therefore exceptional), that is well-suited to generalize some results of [6] on the link between framing and flows. Note that, in his study of flows for graphs with cycles, Berggren [7] has the opposite assumption; that is, every directed cycle is required to be non-monochromatic.

In Section 3, we generalize a construction of [6] that maps directed acyclic graphs to gentle algebras, to give a map that associates graphs that have cycles with complete gentle algebras. In the cyclic case, the DKK triangulation is related to a subfan of the g-vector fan that is linearly isomorphic to $\mathsf{DKK}_{\mathrm{red}}(H)$ (Theorem 11) and builds specific realizations of polytopes. For example, we have a realization of the cyclohedron.

We prove that a family of particular flows, the ***cyclic volume integer flows*** of H, are in bijection with maximal cliques of H. The cardinality of this family gives the volume of the ***flow complex*** of H; a polyhedral complex obtained by fixing, not only the flow coming out of the sources, but also the ones on the cycles. This gives a cyclic analogue of the formula mentioned above by Baldoni–Vergne and Postnikov–Stanley.

We end this extended abstract with an intriguing polytope built from the process described in Section 5, the ***mutoperhedron***, which, without being combinatorially isomorphic to the permutohedron, has the same f-vector and h-vector.

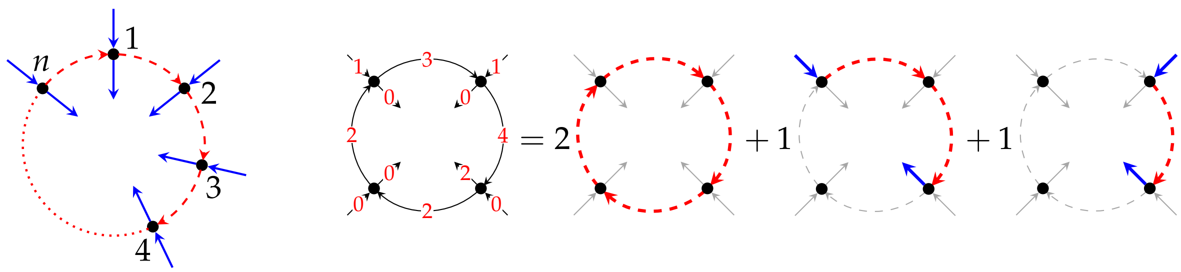
2 Flows on directed graphs

Let $H = (V, E)$ be a directed graph. The ***tail*** and ***head*** of an edge $e = (u, v) \in E$ are $\mathsf{ta}(e) := u$ and $\mathsf{he}(e) := v$; and the sets of incoming and outgoing edges at a vertex v are denoted by $\mathsf{in}(v) := \{e \in E \mid \mathsf{he}(e) = v\}$ and $\mathsf{out}(v) := \{e \in E \mid \mathsf{ta}(e) = v\}$. A vertex is a ***source*** if it has no incoming edges, a ***sink*** if it has no outgoing edges, and ***internal*** if it has both incoming and outgoing edges. We denote the set of internal edges of H by $V_0 = V_0(H)$. We only consider graphs with no idle edges and no isolated vertices.

Routes and minimal cycles A ***path*** in H is a sequence of edges $\rho = e_1, \ldots, e_k$ such that $\mathsf{he}(e_i) = \mathsf{ta}(e_{i+1})$ for all i. If $\mathsf{he}(e_k) = \mathsf{ta}(e_1)$, then ρ is called a ***cycle***. A cycle is ***minimal*** if no proper subsequence of edges forms a cycle. Unless explicitly stated otherwise, by cycle we always mean a minimal oriented cycle. We denote the set of all cycles in H as $\mathscr{C}(H)$. A ***route*** of H is either a path from a source to a sink, or a cycle $C \in \mathscr{C}(H)$.

Definition 2. Let H be a directed graph and ρ be a path from a source to a sink. The route ρ is said to be ***good*** if it does not contain any cycle $C \in \mathscr{C}(H)$; otherwise it is said to be ***bad***. We let $\mathscr{R}(H)$ denote the set of minimal cycles and good routes of H.

Example 3. The ***blossomed n-cycle*** $\circledast_n$ is the oriented n-cycle with a new sink and source attached to each vertex. It has exactly one cycle consisting of n edges (dashed in Figure 1(a)), and n^2 good routes, corresponding to an arbitrary choice of source and sink.



(a) (b)

Figure 1: (a) The blossomed n-cycle $\circledast_n$ from Example 3 with its cyclic ample framing described in Example 5. (b) A balanced flow on $\circledast_4$ and its unique decomposition as a sum of flows on compatible routes; see Theorem 7.

Flows A ***flow*** f of H is an assignment of a real number $f(e)$ to each edge $e \in E$. That is, a flow is an element of $\mathbb{R}^E$. The ***netflow*** of $f \in \mathbb{R}^E$ at a vertex $v \in V$ is

$$\mathsf{nf}_f(v) := \sum_{e \in \mathsf{out}(v)} f(e) - \sum_{e \in \mathsf{in}(v)} f(e).$$

A flow $f \in \mathbb{R}^E$ is said to be ***balanced*** if $\mathsf{nf}_f(v) = 0$ for each internal vertex $v \in V_0(H)$. The ***flow space*** of H is the subspace $\mathcal{F}(H) \subseteq \mathbb{R}^E$ consisting of balanced flows, and the ***flow cone*** is the polyhedral cone of all nonnegative balanced flows:

$$\mathcal{F}^+(H) := \{f \in \mathcal{F}(H) \mid f(e) \geq 0 \text{ for all } e \in E\} = \mathbb{R}_+ \left\{\mathbf{1}_\rho \;\middle|\; \rho \in \mathscr{R}(H)\right\},$$

where $\mathbf{1}_\rho \in \mathbb{R}^E$ denotes the indicator vector of a route ρ.

Triangulations of the flow cone When the graph H is acyclic, Danilov, Karzanov, and Koshevoy [9] construct a unimodular triangulation of $\mathcal{F}^+(H)$ for each choice of a *framing* of H. Once a framing F of H is fixed, they define what it means for routes $\rho, \rho' \in \mathscr{R}(H)$ to be *compatible*, and construct a triangulation $\mathsf{DKK}(H, F)$ whose cones are spanned by collections of pairwise compatible routes. The framing F is ***ample*** whenever the ***reduced fan*** $\mathsf{DKK}_{\mathsf{red}}(H, F)$—obtained as the quotient of $\mathsf{DKK}(H, F)$ by the span of the largest common face of its maximal cones—is complete. The authors in [6] prove that such framings exist only when the internal vertices v of H satisfy $|\mathsf{in}(v)| = |\mathsf{out}(v)| = 2$. In this case, H is said to be ***full***, and they show that framings can be understood as special bi-colourings of the edges of H. We extend these concepts to full cyclic graphs in the next section.

Definition 4. A bi-colouring $\omega : E \to \{\text{red}, \text{blue}\}$ of the edges of H is a ***cyclic ample framing*** if the following conditions hold:

(i) every internal vertex $v \in V_0(H)$ has exactly one red and one blue incoming edge, and one red and one blue outgoing edge, and
(ii) every cycle $C \in \mathscr{C}(H)$ is monochromatic (all its edges have the same colour).

Observe that if H admits a cyclic ample framing, then its (minimal oriented) cycles are necessarily edge-disjoint.

Example 5. Let H be the blossomed n-cycle in Figure 1(a). Up to exchanging colours, it admits only one cyclic ample framing: the edges on the cycle are coloured red (dashed) and the remaining edges are coloured blue (solid). We denote this framed graph by $\otimes_n$.

Definition 6. Let (H, ω) be a cyclically amply framed graph. Two routes $\rho, \rho' \in \mathscr{R}(H)$ are ***incompatible*** if both ρ and ρ' are routes from a source to a sink, and there is a common subpath (possibly a single vertex) S such that when we write $\rho = PSQ$ and $\rho' = P'SQ'$, the last edge of P and the first edge of Q' are coloured blue, and the first edge of Q and the last edge of P' are coloured red. Otherwise, ρ and ρ' are ***compatible***. A route that is compatible with any other route is called ***exceptional***. We call ***clique*** a set of compatible routes, and denote the set of (resp. maximal) cliques by $\mathcal{K}(H, \omega)$ (resp. $\mathcal{K}_{\max}(H, \omega)$). See Figure 4(c) for an example of two incompatible routes in a cyclic graph.

Observe that monochromatic paths, in particular cycles, are exceptional. When the graph H is acyclic, Definition 4 specializes to the characterization of ample framings in [6], and Definition 6 coincides with the notion of compatibility in [9]. The following two results extend the results of [9] to graphs with cyclic ample framings. The existence of cycles makes the proofs more subtle.

Theorem 7. *Let (H, ω) be a cyclically amply framed graph, and let $f \in \mathcal{F}^+(H)$ be a nonnegative flow. Then, f can be uniquely written as a sum of flows along a set of compatible routes (a clique). Moreover, if f is an integer flow, the coefficients of this sum are integers.*

Let $\mathsf{DKK} = \mathsf{DKK}(H, \omega)$ be the collection of cones $\mathbb{R}_+ \{ \mathbf{1}_\rho \mid \rho \in K \}$ for $K \in \mathcal{K}(H, \omega)$. Theorem 7 then says that DKK is a unimodular triangulation of $\mathcal{F}^+(H)$. The space $\mathcal{F}_{\mathrm{red}} = \mathcal{F}_{\mathrm{red}}(H, \omega)$ is the quotient of the flow space $\mathcal{F}$ by the subspace $\mathrm{span}_{\mathbb{R}}(\mathcal{E}) = \mathrm{span}_{\mathbb{R}}(\mathcal{E}(H, \omega))$ spanned by the indicator vectors of exceptional routes. The fan in $\mathcal{F}_{\mathrm{red}}$ obtained by projecting the cones in DKK is denoted by $\mathsf{DKK}_{\mathrm{red}} = \mathsf{DKK}_{\mathrm{red}}(H, \omega)$.

Proposition 8. *Let (H, ω) be a cyclically amply framed graph. Then, the fan $\mathsf{DKK}_{\mathrm{red}}$ is complete.*

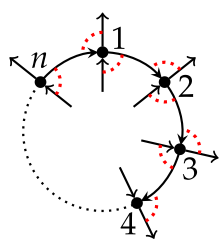
3 Relation with g-vector fans of complete gentle algebras

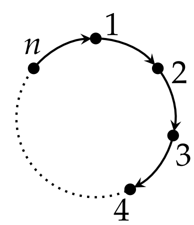
Algebra is a source of combinatorially interesting fans, going back to the Coxeter fan of a finite reflection group. Cluster fans of finite type cluster algebras are a more recent example. A wide generalization of both these classes is provided by [2], which, for any finite-dimensional algebra A, defines a fan called the g-vector fan of A, denoted $\boldsymbol{g}\mathsf{fan}(A)$. We now explain the key insight of [6], which connected the DKK-triangulation of an amply framed graph to the g-vector fan of a suitable algebra.

In the interest of efficiency, we state the key construction of [6] in an extension to our setting of cyclically amply framed graphs. When the graph H is acyclic, the definition below is precisely that of [6].

Definition 9. Given a cyclically amply framed graph $H = (H, \omega)$, let $(Q_H^{\circledast}, R_H^{\circledast})$ be the quiver obtained by reversing the arrows of H coloured blue and with relations $e_1 e_2$ whenever $\omega(e_1) \neq \omega(e_2)$. Let (Q_H, R_H) be the restriction of $(Q_H^{\circledast}, R_H^{\circledast})$ to the vertices V_0.

Example 10. When (H, ω) is the blossomed n-cycle $\otimes_n$ in Figure 1(a), the corresponding locally gentle quivers are:

$(Q_{\otimes_n}^{\circledast}, R_{\otimes_n}^{\circledast}) =$ 

and $(Q_{\otimes_n}, R_{\otimes_n}) =$ 

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If H has no cycles, then define $\Lambda_H = \mathbb{K}Q_H / \langle R_H \rangle$, i.e., the path algebra of Q_H, modulo the ideal generated by the relations from R_H. The algebra Λ_H is gentle. [6] showed that in this case, $\boldsymbol{g}\mathsf{fan}(\Lambda_H)$, which lives in $\mathbb{R}^{V_0}$, is combinatorially isomorphic to $\mathsf{DKK}_{\mathrm{red}}(H)$.

If H has oriented cycles, then the quiver Q_H is only locally gentle. If one were to consider $\Lambda_H = \mathbb{K}Q_H / \langle R_H \rangle$ as before, this would be an infinite-dimensional algebra, for which the notion of g-vector fan is not defined.

Two approaches can be taken at this point. One is to take the ***complete gentle algebra*** $\widehat{\Lambda}_H$, a certain completion of the infinite-dimensional algebra Λ_H defined above. $\widehat{\Lambda}_H$ is still infinite-dimensional, so the definition of g-vector fan from [2] does not apply, but [4] gives a suitable definition. Another, more concrete, though less obviously motivated, approach is to consider the g-vector fan of the algebra $\Lambda'_H = \mathbb{K}Q_H / \langle R_H, C_H \rangle$ where C_H consists of each path going once around any oriented cycle of Q_H. Because Λ'_H is finite-dimensional, it has a well-defined g-vector fan in the sense of [2]. It turns out that $\boldsymbol{g}\mathsf{fan}(\widehat{\Lambda}_H) = \boldsymbol{g}\mathsf{fan}(\Lambda'_H)$.

The non-kissing complex associated to (Q_H, R_H) was defined in [13] — this is a simplicial complex which extends a combinatorial model for the fan of a gentle algebra to the locally gentle setting. In [13], representation theory was an essential inspiration, but the non-kissing complex that was constructed was not given a representation-theoretic interpretation (or even a fan structure). We show that it is realized as a fan by $\boldsymbol{g}\mathsf{fan}(\widehat{\Lambda}_H)$ and (equivalently) $\boldsymbol{g}\mathsf{fan}(\Lambda'_H)$.

In the case where H contains cycles, it turns out that our DKK triangulation is related, not to the entire g-vector fan, but to the subfan contained in the linear subspace

$$W_H := \left\{ \mathbf{x} \in \mathbb{R}^{V_0} \;\middle|\; \sum_{v \in V(C)} \mathbf{x}_v = 0 \text{ for all } C \in \mathscr{C}(H) \right\}.$$

Theorem 11. *Let $H = (H, \omega)$ be a cyclically amply framed graph. The collection of cones of $\boldsymbol{g}\mathsf{fan}(Q_H, R_H)$ lying in the subspace W_H forms a complete fan inside this W_H. Moreover, this fan is linearly isomorphic to $\mathsf{DKK}_{\mathrm{red}}(H, \omega)$.*

When the graph H is acyclic, the subspace W_H coincides with the entire space $\mathbb{R}^{V_0}$. Thus, Theorem 11 implies that $\mathsf{DKK}_{\mathrm{red}}(H, \omega)$ and $\boldsymbol{g}\mathsf{fan}(Q_H, R_H)$ are linearly isomorphic. This result strengthens the combinatorial fan isomorphism from [6].

One of the reasons that it is interesting to realize a fan as a g-vector fan is that, if A is an algebra with a finite g-vector fan, then [3, 10] show that there is a polytope whose outer normal fan is $\boldsymbol{g}\mathsf{fan}(A)$ and, moreover, gives a recipe to construct a suitable polytope as a Minkowski sum of *Harder–Narasimhan polytopes* of the *brick modules* of A. See [1, §5] for details.

Applying this to Λ'_H gives an explicit realization of a polytope whose outer normal fan is $\boldsymbol{g}\mathsf{fan}(\Lambda'_H)$ as a Minkowski sum of an explicitly given and combinatorially describable collection of polytopes, generalizing the shard polytopes of [12]. The polytope in Figure 4(a) is constructed as one such Minkowski sum.

Corollary 12. *The fan $\mathsf{DKK}_{\mathrm{red}}(H, \omega)$ is polytopal.*

Proposition 13. *For the blossomed n-cycle $\otimes_n$, the fan $\mathsf{DKK}_{\mathrm{red}}(\otimes_n)$ is combinatorially isomorphic to the normal fan of the cyclohedron.*

4 Flow complex and cyclic integer flows

In this section, we define and give a correspondence between certain integer flows on H and the maximal cliques $\mathcal{K}_{\max}(H, \omega)$ using and extending the correspondence from [11] in the acyclic case. Let H be a graph with a fixed cyclic ample framing.

Definition 14. A flow $f : E(H) \to \mathbb{N}$ is a ***cyclic integer flow*** if

(i) every cycle $C \in \mathscr{C}(H)$ has exactly one edge $e_C \in C$ with $f(e_C) = 0$,

(ii) the netflow at every internal vertex is 1, except for the tails of the edges $\{e_C\}_{C \in \mathscr{C}(H)}$ where the netflow is 0.

We let $\mathcal{F}^{\circlearrowleft}(H)$ denote the collection of cyclic integer flows on H.

More generally, given $\mathbf{c} = (c_s)_s \in \mathbb{N}^{\mathsf{sources}(H)}$, we let

$$\mathcal{F}^{\circlearrowleft}_{\mathbf{c}}(H) := \left\{ f \in \mathcal{F}^{\circlearrowleft}(H) \;\middle|\; \mathsf{nf}_f(s) = c_s \text{ for all } s \in \mathsf{sources}(H) \right\}.$$

Elements of $\mathcal{F}^{\circlearrowleft}_{\mathbf{0}}(H)$ are called ***cyclic volume integer flows*** for reasons that will be clear soon (see Theorem 18).

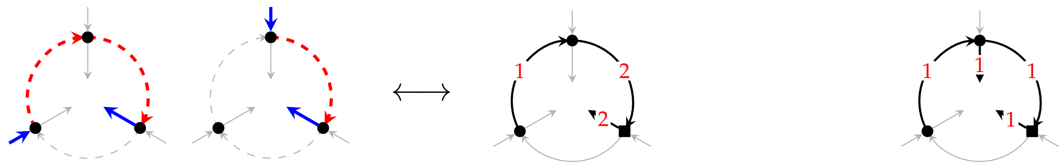

Figure 2: (Left) The non-exceptional routes of a maximal clique of $\circledast_3$ and its corresponding cyclic volume integer flow. (Right) Another cyclic volume integer flow of $\circledast_3$. We only highlight those edges with non-zero flow. The square-shaped vertices are those with netflow 0. The remaining four cyclic volume integer flows of $\circledast_3$ are obtained as rotations of the two displayed above.

Define a map $\Phi_H : \mathcal{K}_{\max}(H) \to \mathbb{N}^E$ by $\Phi_H(K)(e) = n_K(e) - 1$, where $n_K(e)$ is the number of different prefixes until the edge e for routes in K, where we set $\mathsf{prefix}(C, e) = C$ for all $e \in C$. We illustrate this map in Example 16 below.

Proposition 15. *The map Φ_H is a bijection between the maximal cliques and the cyclic volume integer flows of H.*

When H is acyclic, the definition of Φ_H and Proposition 15 first appeared in [11].

Example 16. Consider the maximal clique of the blossomed 3-cycle $\circledast_3$ whose non-exceptional routes are shown on the top left of Figure 2. The corresponding cyclic volume integer flow is shown on the top-right panel. Observe that the netflow at the internal vertices (displayed in red) is 1, except for the tail of the bottom arc, which is the edge in the cycle with flow 0. The other five cyclic volume integer flows are in the second row of the figure.

We now consider the following cyclic analogue of the *flow polytope* whose volume will be given by the number of cyclic volume integer flows.

Definition 17. For a graph H admitting a cyclic ample framing, its ***flow complex*** is

$$\widehat{\mathcal{F}}(H) := \left\{ f \in \mathcal{F}^+(H) \;\middle|\; \sum_{\substack{v \in \mathsf{sources}(H) \\ e \in \mathsf{out}(v)}} f(e) + \sum_{C \in \mathscr{C}(H)} \min\{f(e) \mid e \in C\} = 1 \right\}.$$

In general, the object $\widehat{\mathcal{F}}(H)$ is not convex (see Figure 3(b)). When H is acyclic, $\widehat{\mathcal{F}}(H)$ is convex and receives the name of *flow polytope* of H. The following result is a cyclic analogue of the formula by Baldoni–Vergne [5] and Postnikov–Stanley [14] to compute the volume of flow polytopes.

Theorem 18. *Let (H, ω) be a cyclically amply framed graph, then* $\mathrm{vol}\, \widehat{\mathcal{F}}(H) = \left| \mathcal{F}_{\mathbf{0}}^{\circlearrowleft}(H) \right|$.

Example 19. The smallest example we can consider is for the graph $\circledast_2$, in which case $\widehat{\mathcal{F}}(\circledast_2) \subseteq \mathbb{R}^6$ has dimension $6 - 2 - 1 = 3$. The graph $\circledast_2$ has two cyclic volume integer flows (see Figure 3(a)), and therefore $\operatorname{vol} \widehat{\mathcal{F}}(\circledast_2) = 2$. Concretely, $\widehat{\mathcal{F}}(\circledast_2)$ is the union of two unimodular tetrahedra sharing the common facet $\Delta_{\mathcal{E}} = \operatorname{conv}\left\{\mathbf{1}_\rho \mid \rho \in \mathcal{E}(\circledast_2)\right\}$. However, the two tetrahedra do not lie in the same 3-dimensional affine subspace of $\mathbb{R}^6$. Under a suitable projection, in which we collapse $\Delta_{\mathcal{E}}$ to a single point, this object looks like the union of two segments (shown in cyan in Figure 3(b)) sharing a common vertex.

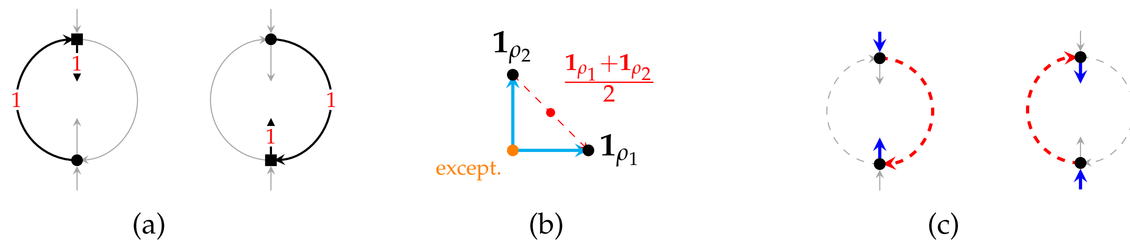

Figure 3: For $\circledast_2$: (a) its two cyclic volume integer flows, (b) its flow complex, and (c) its two good routes.

5 The mutoperhedron

Let $H_{c,p}$ denote the Cartesian product of a p-cycle and a path of length $c-1$, with sources and sinks attached appropriately so $H_{c,p}$ is full. Figure 4(b) shows this graph when $c = 3$ and $p = 2$. Note that $H_{1,n}$ is the blossomed n-cycle $\circledast_n$ from Example 3. This graph has a unique cyclic ample framing up to swapping colours. Henceforth, we fix the cyclic ample framing in which cycles are coloured red, and the remaining edges are coloured blue.

Theorem 20. *Let $H_{c,p}$ be the cyclically amply framed graph described above. Then, the reduced fan $\mathsf{DKK}_{\mathrm{red}}(H_{c,p})$ has $p^{c+1} - p$ rays and $\binom{(c+1)(p-1)}{p-1,\ldots,p-1}$ maximal cones.*

For $p = 2$, Theorem 20 says that $\mathsf{DKK}_{\mathrm{red}}(H_{c,2})$ has $2^{c+1} - 2$ rays and $(c+1)!$ maximal cones. The reader might recognize that these are also the number of rays and maximal cones, respectively, of the (essentialized) ***braid arrangement***, which is the normal fan of the c-dimensional ***permutohedron*** Π_{c+1}. In fact, since both arrangements are simplicial and polytopal, this readily implies that their f-vectors agree for $c \leq 5$.

Definition 21. The c-dimensional ***mutoperhedron*** $\mathcal{M}_c$ is the polytope whose normal fan is $\mathsf{DKK}_{\mathrm{red}}(H_{c,2})$.

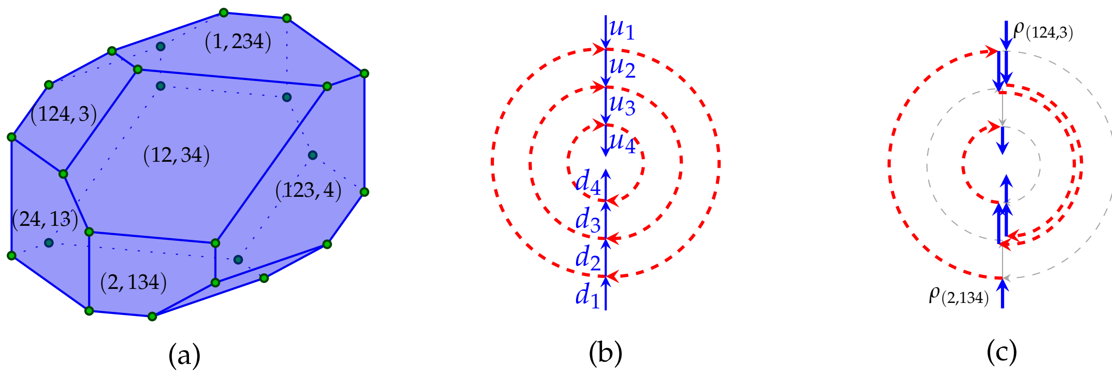

Figure 4: (a) The mutoperhedron $\mathcal{M}_3$, with f-vector $(24, 36, 14, 1)$. The facets corresponding to the route $\rho_{(12,34)} \in \mathscr{R}(H_{3,2})$ and its compatible routes are labelled. (b) The labelling of the blue edges of $H_{3,2}$ described after Corollary 23. (c) The incompatible routes $\rho_{(124,3)}, \rho_{(2,134)} \in \mathscr{R}(H_{3,2})$, witnessing that the facets of $\mathcal{M}_3$ indexed by $(124, 3)$ and $(2, 134)$ do not intersect.

One explicit realization of $\mathcal{M}_3$, constructed via Minkowski sums of projections of Harder–Narasimhan polytopes (see Section 3) is shown in Figure 4(a) above.

Theorem 22. *The mutoperhedron $\mathcal{M}_c$ and the permutohedron Π_{c+1} have the same f-vector. That is, for all $c \geq 1$ and $0 \leq i \leq c$, $f_i(\mathcal{M}_c) = f_i(\Pi_{c+1})$. However, for $c \geq 3$, these polytopes are* not *combinatorially isomorphic.*

We can equivalently rewrite the previous result in terms of h-vectors as follows.

Corollary 23. *For all $c \geq 1$ and $0 \leq i \leq c$, $h_i(\mathcal{M}_c) = A_{c+1,i}$, the Eulerian numbers.*

Proof sketch of Theorem 22 We label the blue edges of $H_{c,2}$ so that $u_1u_2 \dots u_{c+1}$ and $d_1d_2 \dots d_{c+1}$ are its two exceptional blue routes; Figure 4(b) shows this labelling when $c = 3$. In the pictures, the arrows u_i and d_i are on the top and bottom, respectively.

A good route of $H_{c,2}$ is uniquely determined by the collection of blue edges it uses. Therefore, the good non-exceptional routes of $H_{c,2}$, and consequently the facets of $\mathcal{M}_c$, are in bijection with ordered partitions of $[c + 1]$ into exactly two parts, henceforth referred to as ***bipartitions***. Explicitly, the route $\rho_{(A,B)}$ corresponding to a bipartition (A, B) of $[c + 1]$ is the unique good route using blue edges u_i for $i \in A$ and d_j for $j \in B$. See Figure 4(c) for examples. To simplify the notation, we drop the brackets and the commas from each individual part of an ordered partition.

Example 24. The 3-dimensional mutoperhedron $\mathcal{M}_3$ is shown in Figure 4(a). The route $\rho_{(12,34)} \in \mathscr{R}(H_{3,2})$, shown in Figure 5(a), is compatible with exactly five other good non-exceptional routes: $\rho_{(1,234)}$, $\rho_{(123,4)}$, $\rho_{(2,134)}$, $\rho_{(24,13)}$, and $\rho_{(124,3)}$, shown in Figure 5(b). Thus, the corresponding facet of $\mathcal{M}_3$ is a pentagon.

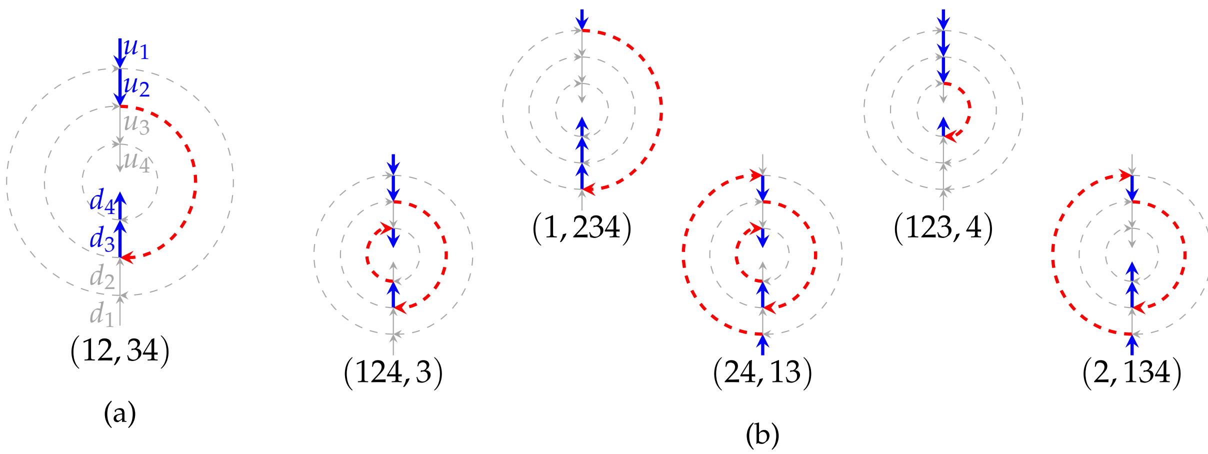

Figure 5: (a) The labels of the blue edges and the route $\rho_{(12,34)} \in \mathscr{R}(H_{3,2})$. (b) The five routes in $\mathscr{R}(H_{3,2})$ compatible with $\rho_{(12,34)}$.

There is a well-known indexing of the k-dimensional faces of the permutohedron Π_{c+1} by ordered partitions of $[c+1]$ into $c+1-k$ parts. In particular, its facets are indexed by bipartitions (A, B) of $[c+1]$, just like the mutoperhedron. Moreover, two (not necessarily distinct) facets of the permutohedron Π_{c+1} intersect if and only if the corresponding bipartitions (A_1, B_1) and (A_2, B_2) of $[c+1]$ are ***nested***: meaning that either $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. In contrast, two (not necessarily distinct) facets of the mutoperhedron $\mathcal{M}_c$ intersect if and only if the corresponding bipartitions are ***noninterfering***, which is a slightly technical condition.

Example 25. The facets of the permutohedron Π_4 indexed by $(2, 134)$ and $(124, 3)$ intersect, and their intersection is labelled by the ordered partition $(2, 14, 3)$.

In contrast, the facets of the mutoperhedron $\mathcal{M}_3$ indexed by $(2, 134)$ and $(124, 3)$ do not intersect, as Figures 4(a) and 4(c) show.

To complete the proof of Theorem 22, we show that multisets of k (not necessarily distinct) nested bipartitions are in bijection with multisets of k (not necessarily distinct) noninterfering bipartitions. Even though this bijection does not restrict to a bijection between *sets* of k nested and k noninterfering bipartitions, a simple induction argument shows that the number of such sets is equal.

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