Séminaire Lotharingien de Combinatoire **XX** (2026)
Article #35, 12 pp.

Proceedings of the 38th Conference on Formal Power Series and Algebraic Combinatorics (Seattle)

Rowmotion and Echelonmotion

Colin Defant*[1], Yuhan Jiang†[1], Rene Marczinzik‡[2], Adrien Segovia§[3], David E Speyer¶[4], Hugh Thomas||[3], and Nathan Williams**[5]

[1]*Department of Mathematics, Harvard University, Cambridge, MA 02138, USA*
[2]*Mathematical Institute of the University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany*
[3]*Lacim, UQAM, Montréal, QC, H3C 3P8 Canada*
[4]*Department of Mathematics, University of Michigan, Ann Arbor, MI, 48109, USA*
[5]*Department of Mathematical Sciences, University of Texas at Dallas, Richardson, TX, 75080, USA*

Abstract. Given a linear extension σ of a finite poset R, we consider the permutation matrix indexing the Bruhat cell containing the Cartan matrix of R with respect to σ. This yields a bijection $\mathrm{Ech}_\sigma \colon R \to R$ that we call *echelonmotion*; it is the inverse of the Coxeter permutation studied by Klász, Marczinzik, and Thomas. Those authors proved that echelonmotion agrees with rowmotion when R is a distributive lattice. We generalize this result to semidistributive lattices. In addition, we prove that every trim lattice has a linear extension with respect to which echelonmotion agrees with rowmotion. We also show that echelonmotion on an Eulerian poset (with respect to any linear extension) is an involution. Finally, we initiate the study of *echelon-independent posets*, which are posets for which echelonmotion is independent of the chosen linear extension. We prove that a lattice is echelon-independent if and only if it is semidistributive. Moreover, we show that echelon-independent connected posets are bounded and have semidistributive MacNeille completions.

Keywords: rowmotion, Bruhat decompotion, linear extension, Eulerian poset, semidistributive lattice, trim lattice

*colindefant@gmail.com. Supported by the NSF under Award No. 2201907.

†yhj1761@gmail.com.

‡marczire@math.uni-bonn.de.

§adrien.segovia@gmail.com. Supported by NSERC Discovery Grants RGPIN-2022-03960 and RGPIN-2024-04465.

¶speyer@umich.edu. Partially supported by the NSF under Award No. 2246570.

|| thomas.hugh_r@uqam.ca. Partially supported by NSERC Discovery Grant RGPIN-2022-03960 and the Canada Research Chairs program, grant number CRC-2021-00120.

**nathan.f.williams@gmail.com. Partially supported by the NSF under Award No. 2246877

1 Introduction

Rowmotion. Let Q be a finite poset, and consider the distributive lattice $J(Q)$ of lower order ideals of Q, partially ordered by containment. For $X \subseteq Q$, let

$$\nabla_Q(X) = \{q \in Q : q \geq x \text{ for some } x \in X\}$$

and

$$\Delta_Q(X) = \{q \in Q : q \leq x \text{ for some } x \in X\}$$

be the upper and lower order ideals generated by X. *Rowmotion* is the bijective map $\mathrm{Row}_{J(Q)} \colon J(Q) \to J(Q)$ defined by

$$\mathrm{Row}_{J(Q)}(I) = Q \setminus \nabla_Q(\max(I)), \tag{1.1}$$

where $\max(I)$ is the set of maximal elements of I. Birkhoff's representation theorem states that every finite distributive lattice is isomorphic to the lattice of lower order ideals of a finite poset, so one can define a bijective rowmotion operator on any distributive lattice. The left-hand side of Figure 1 shows the action of rowmotion on the lower order ideals of a 3-element poset.

Rowmotion has been discovered and rediscovered in several contexts, including matroid theory [12] and quiver representation theory [16, 20]. It has been studied extensively in dynamical algebraic combinatorics [1, 3, 14, 23, 28, 29, 32]. Rowmotion has been extended to larger families of posets besides distributive lattices [3, 9, 31, 32].

We consider a broad generalization of rowmotion to any finite poset.

Echelonmotion. The new incarnation of rowmotion we consider was recently discovered by Klász, Marczinzik, and Thomas while studying the grade bijection for Auslander–Gorenstein algebras [18], and is defined using the Bruhat decomposition of the general linear group $\mathrm{GL}_n(\mathbb{C})$. Let B be the set of upper-triangular invertible complex matrices. Recall that

$$\mathrm{GL}_n(\mathbb{C}) = \bigsqcup_{P \in \mathfrak{S}_n} BPB, \tag{1.2}$$

where $\mathfrak{S}_n$ denotes the symmetric group on n letters, which we identify with the set of $n \times n$ permutation matrices. (See [22] for more details.)

Let R be an n-element poset, and let $[n] = \{1, \ldots, n\}$. A *linear extension* of R is a bijection $\sigma \colon R \to [n]$ such that $\sigma(x) \leq \sigma(y)$ for all $x, y \in R$ satisfying $x \leq y$. Let $\mathcal{L}(R)$ denote the set of linear extensions of R and fix $\sigma \in \mathcal{L}(R)$. The *Cartan matrix* of R with respect to σ is the $n \times n$ matrix $W^{R,\sigma}$ whose entry in row i and column j is

$$W^{R,\sigma}_{i,j} = \begin{cases} 1 & \text{if } \sigma^{-1}(i) \geq \sigma^{-1}(j) \\ 0 & \text{if } \sigma^{-1}(i) \not\geq \sigma^{-1}(j). \end{cases}$$

Because $W^{R,\sigma}$ is lower-triangular with ones on its diagonal, it is invertible—so by (1.2), there is a unique permutation matrix $P^{R,\sigma} \in \mathfrak{S}_n$ such that $W^{R,\sigma} \in BP^{R,\sigma}B$. Using σ, we can view $P^{R,\sigma}$ as a bijection from R to itself.

Definition 1.1. Given a linear extension σ of a finite poset R, define *echelonmotion* with respect to σ to be the bijection $\mathrm{Ech}_\sigma \colon R \to R$ such that $\mathrm{Ech}_\sigma(x) = y$ if and only if $P^{R,\sigma}$ has a 1 in row $\sigma(y)$ and column $\sigma(x)$. We say a poset R is *echelon-independent* if $\mathrm{Ech}_\sigma = \mathrm{Ech}_{\sigma'}$ for all linear extensions σ, σ' of R.

Klász, Marczinzik, and Thomas [18] proved that if $R = J(Q)$ is a distributive lattice, then $\mathrm{Ech}_\sigma = \mathrm{Row}_R$ for any linear extension σ of R.[1] In particular, this shows that distributive lattices are echelon-independent.

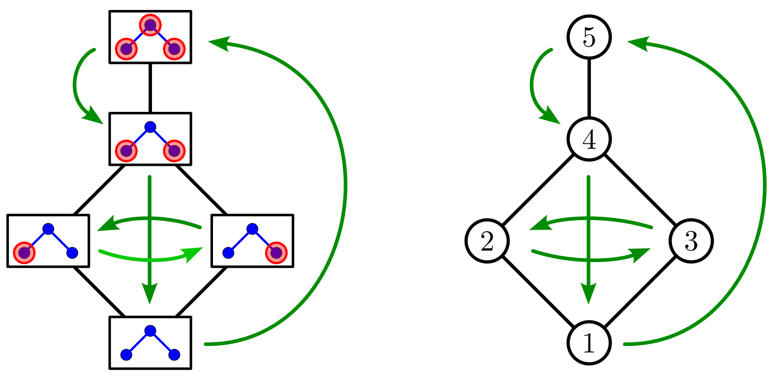
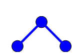


Figure 1: On the left is the distributive lattice of lower order ideals of a 3-element poset, where each lower order ideal is represented as a collection of elements shaded in red. On the right is the same lattice labeled according to a linear extension. In each depiction, a green arrow is drawn from an element to its image under rowmotion.

Example 1.2. Let Q be the 3-element poset . The left side of Figure 1 shows the lattice $R = J(Q)$, with the action of rowmotion represented by green arrows. The right side of the figure shows the same lattice with elements labeled by a linear extension σ. The Cartan matrix and the permutation matrix in its Bruhat decomposition are

$$W^{R,\sigma} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad P^{R,\sigma} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

[1] They actually worked with the bijection Ech_σ^{-1}, which they called the *Coxeter permutation*. This name comes from the fact that it corresponds to the permutation matrix appearing in the Bruhat decomposition of the *Coxeter matrix* $-(W^{R,\sigma})^{-1}(W^{R,\sigma})^\top$. The definition of rowmotion used in [18] is the inverse of the one we use here.

Indeed, this follows from the factorization

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

The permutation matrix $P^{R,\sigma}$, viewed as a permutation of R, agrees with rowmotion.

Our aim is to study echelonmotion for several classes of posets, as detailed in Figure 2. This extended abstract is a shortened form of [8].

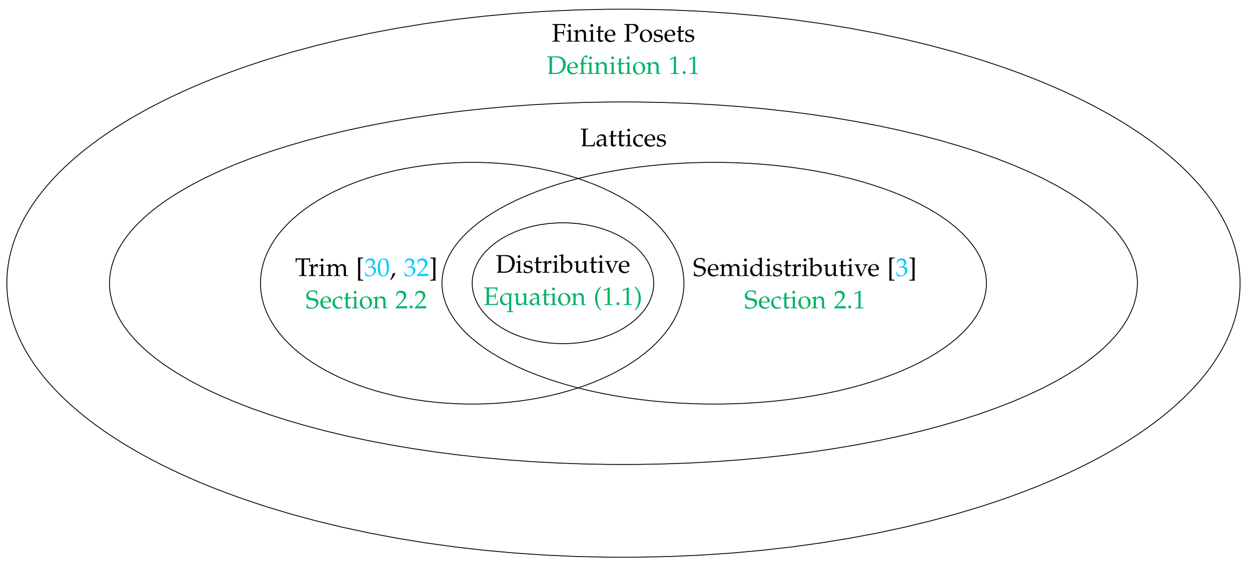


Figure 2: Some of the classes of posets considered in this abstract (figure not to scale).

2 Main Results

All posets in this article are assumed to be finite. We write $\mathcal{J}_L$ and $\mathcal{M}_L$ for the set of join-irreducible elements and meet-irreducible elements, respectively, of a lattice L.

2.1 Semidistributive Lattices

A lattice L is called *meet-semidistributive* if for all $x, y \in L$ such that $x \leq y$, the set $\{z \in L : z \wedge y = x\}$ has a maximum element. A lattice is *join-semidistributive* if its dual is meet-semidistributive. A lattice is *semidistributive* if it is both meet-semidistributive and join-semidistributive.

Semidistributive lattices generalize distributive lattices. They have a representation theorem that mimics Birkhoff's representation theorem for distributive lattices [27]. Notable examples of semidistributive lattices include intervals in weak order on Coxeter groups [6, 26], the facial weak orders of simplicial hyperplane arrangements [11], Cambrian lattices [25], ν-Tamari lattices [5, 24], framing lattices [4], and lattices of torsion classes of finite-dimensional algebras [10].

Suppose L is semidistributive. We can label each cover relation $x \lessdot y$ in L with the minimum element $j_{x,y}$ of the set $\{z \in L : z \vee x = y\}$; this element is necessarily join-irreducible. For $x \in L$, define $\mathcal{U}_L(x) = \{j_{x,w} : x \lessdot w\}$ and $\mathcal{D}_L(x) = \{j_{w,x} : w \lessdot x\}$. Barnard [3] defined the *rowmotion* operator $\mathrm{Row}_L \colon L \to L$ by the equation

$$\mathcal{U}_L(\mathrm{Row}_L(x)) = \mathcal{D}_L(x). \tag{2.1}$$

Note that (2.1) agrees with (1.1) when L is distributive.

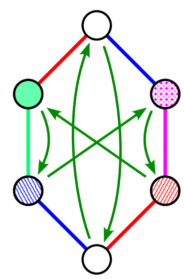


Figure 3: A semidistributive lattice. Each join-irreducible element has its own color, which is also used to color the edges that it labels. A green arrow is drawn from each element to its image under rowmotion.

Our first theorem generalizes the main result of [18] to semidistributive lattices.

Theorem 2.1. *Let L be a semidistributive lattice. We have $\mathrm{Ech}_\sigma = \mathrm{Row}_L$ for every linear extension σ of L. In particular, L is echelon-independent. Furthermore, every echelon-independent lattice is semidistributive.*

2.2 Trim Lattices

We next turn our attention to trim lattices, which were introduced by Thomas [30]. Following [21], we say a lattice L is *extremal* if it has a maximum-length chain of cardinality $k+1$, where $k = |\mathcal{J}_L| = |\mathcal{M}_L|$. An element $x \in L$ is called *left modular* if for all $y, z \in L$ satisfying $y \leq z$, we have the equality $(y \vee x) \wedge z = y \vee (x \wedge z)$. A lattice is *left modular* if it has a maximal chain of left modular elements. A lattice is *trim* if it is both extremal and left modular [30, 32]. Notable examples of trim lattices include distributive lattices, Cambrian lattices [25], ν-Tamari lattices [5, 24], and certain lattices of torsion classes of finite-dimensional algebras [32, 2].

Thomas and Williams [32] defined a rowmotion operator $\mathrm{Row}_L\colon L \to L$ whenever L is trim. They define a labeling of the edges of L by join-irreducible elements of L, and they define rowmotion so that the labels of the edges extending up from $\mathrm{Row}_L(x)$ are the same as the labels of the edges extending down from x. When L is both trim and semidistributive, their definition agrees with Barnard's definition and, therefore, coincides with echelonmotion on L (with respect to any linear extension) by Theorem 2.1. When L is trim but not semidistributive, Theorem 2.1 tells us that we cannot have $\mathrm{Ech}_\sigma = \mathrm{Row}_L$ for *every* linear extension σ of L. Nonetheless, we have the following theorem.

Theorem 2.2. *Let L be a trim lattice. There exists $\sigma \in \mathcal{L}(L)$ such that $\mathrm{Ech}_\sigma = \mathrm{Row}_L$.*

Remark 2.3. In [8], we provide an explicit recursive recipe for constructing certain linear extensions of a trim lattice L that we call *vertebral* linear extensions [8, Definition 6.7]. We then prove that $\mathrm{Ech}_\sigma = \mathrm{Row}_L$ whenever σ is vertebral.

2.3 Eulerian Posets

A *rank function* on a poset R is a map $\mathrm{rk}\colon R \to \mathbb{Z}$ such that $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$ for every cover relation $x \lessdot y$; if such a map exists, we say R is *graded*. We say R is *Eulerian* if it is graded and its Möbius function satisfies $\mu_R(x, y) = (-1)^{\mathrm{rk}(y)-\mathrm{rk}(x)}$ for all $x, y \in R$ with $x \leq y$. Notable examples of Eulerian posets include face lattices of polytopes and intervals of Bruhat order on Coxeter groups. We have the following theorem regarding echelonmotion on Eulerian posets.

Theorem 2.4. *Let R be an Eulerian poset. For every $\sigma \in \mathcal{L}(R)$, the map Ech_σ is an involution.*

2.4 Echelon-Independence

Echelon-independent posets are especially nice because they come equipped with an intrinsic echelonmotion bijection (without the additional specification of a linear extension, see Definition 1.1). While we do not have a characterization of such posets, we do have some necessary conditions and some sufficient conditions for echelon-independence.

Recall that a poset is *bounded* if it has a minimum element and a maximum element. A poset is called *connected* if its Hasse diagram is a connected graph.

Theorem 2.5. *Every echelon-independent connected poset is bounded.*

Theorem 2.6. *Let R be an echelon-independent connected poset of cardinality at least 2. Then echelonmotion on R has no fixed points.*

Theorem 2.7. *The MacNeille completion of an echelon-independent connected poset is a semidistributive lattice.*

The converse of Theorem 2.7 is false. Figure 4 shows a poset labeled by two different linear extensions σ and σ' such that $\mathrm{Ech}_\sigma \neq \mathrm{Ech}_{\sigma'}$. However, the MacNeille completion of this poset is a Boolean lattice of cardinality 16, which is semidistributive.

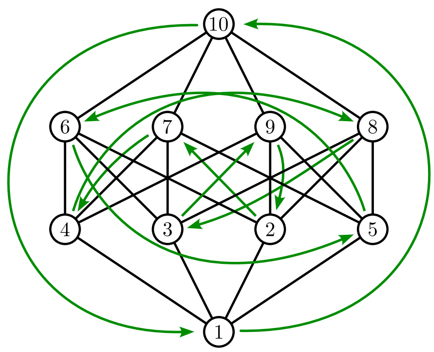


Figure 4: Two linear extensions of a poset whose MacNeille completion is distributive. Echelonmotion with respect to each linear extension is represented by green arrows; note that the two echelonmotion maps are different.

We now provide an algorithm to test whether a poset is echelon-independent. This algorithm is useful in practice; for example, it allowed us to verify that Bruhat order on the symmetric group S_n is echelon-independent when $n \leq 5$ and is not echelon-independent when $n = 6$. Let σ be a linear extension of an n-element poset R. For $x \in R$, let

$$\mathrm{Pre}_\sigma(x) = \sigma^{-1}([1, \sigma(x)]) \quad \text{and} \quad \mathrm{Suc}_\sigma(x) = \sigma^{-1}([\sigma(x), n]).$$

If we view σ as a total order on R, then $\mathrm{Pre}_\sigma(x)$ is the set of elements that weakly precede x in σ, while $\mathrm{Suc}_\sigma(x)$ is the set of elements that weakly succeed x in σ.

Suppose $x, y \in R$. If x and y are comparable, define

$$\Lambda_1(x,y) = \{\lambda \in \mathcal{L}(R) : \mathrm{Pre}_\lambda(x) = \Delta_R(x) \text{ and } \mathrm{Pre}_\lambda(y) = \Delta_R(y)\},$$

$$\Lambda_2(x,y) = \{\lambda \in \mathcal{L}(R) : \mathrm{Suc}_\lambda(x) = \nabla_R(x) \text{ and } \mathrm{Suc}_\lambda(y) = \nabla_R(y)\}.$$

If x and y are incomparable, let

$$\Xi_1(x,y) = \{\xi \in \mathcal{L}(R) : \mathrm{Pre}_\xi(x) = \Delta_R(x) \text{ and } \mathrm{Pre}_\xi(y) = \Delta_R(x) \cup \Delta_R(y)\},$$
$$\Xi_2(x,y) = \{\xi \in \mathcal{L}(R) : \mathrm{Pre}_\xi(x) = \Delta_R(x) \cup \Delta_R(y) \text{ and } \mathrm{Pre}_\xi(y) = \Delta_R(y)\},$$
$$\Xi_3(x,y) = \{\xi \in \mathcal{L}(R) : \mathrm{Suc}_\xi(x) = \nabla_R(x) \text{ and } \mathrm{Suc}_\xi(y) = \nabla_R(x) \cup \nabla_R(y)\},$$
$$\Xi_4(x,y) = \{\xi \in \mathcal{L}(R) : \mathrm{Suc}_\xi(x) = \nabla_R(x) \cup \nabla_R(y) \text{ and } \mathrm{Suc}_\xi(y) = \nabla_R(y)\}.$$

Proposition 2.8. *Let $\sigma^\#$ be a linear extension of a poset R. Fix $x \in R$, and let $y = \mathrm{Ech}_{\sigma^\#}(x)$. Suppose x and y are comparable. Fix $\lambda_1 \in \Lambda_1(x,y)$ and $\lambda_2 \in \Lambda_2(x,y)$. If $\mathrm{Ech}_{\lambda_1}(x) = \mathrm{Ech}_{\lambda_2}(x) = y$, then $\mathrm{Ech}_\sigma(x) = y$ for every linear extension σ of R.*

Proposition 2.9. *Let $\sigma^\#$ be a linear extension of a poset R. Fix $x \in R$, and let $y = \mathrm{Ech}_{\sigma^\#}(x)$. Suppose x and y are incomparable. For each $k \in \{1,2,3,4\}$, fix a linear extension $\xi_k \in \Xi_k(x,y)$. If $\mathrm{Ech}_{\xi_1}(x) = \mathrm{Ech}_{\xi_2}(x) = \mathrm{Ech}_{\xi_3}(x) = \mathrm{Ech}_{\xi_4}(x) = y$, then $\mathrm{Ech}_\sigma(x) = y$ for every linear extension σ of R.*

To use Propositions 2.8 and 2.9 in practice to test whether or not an n-element poset R is echelon-independent, one first chooses an arbitrary linear extension $\sigma^\#$ and computes $\mathrm{Ech}_{\sigma^\#}$. For each $x \in R$ such that x and $\mathrm{Ech}_{\sigma^\#}(x)$ are comparable, one can use Proposition 2.8 to test whether or not $\mathrm{Ech}_\sigma(x) = \mathrm{Ech}_{\sigma^\#}(x)$ for every linear extension σ of R; this requires one to test whether $\mathrm{Ech}_{\lambda_1}(x) = \mathrm{Ech}_{\lambda_2}(x) = y$ for just two specific linear extensions λ_1 and λ_2. For each $x \in R$ such that x and $\mathrm{Ech}_{\sigma^\#}(x)$ are not comparable, one can use Proposition 2.9 to test whether or not $\mathrm{Ech}_\sigma(x) = \mathrm{Ech}_{\sigma^\#}(x)$ for every linear extension σ of R; this requires testing whether $\mathrm{Ech}_{\xi_1}(x) = \mathrm{Ech}_{\xi_2}(x) = \mathrm{Ech}_{\xi_3}(x) = \mathrm{Ech}_{\xi_4}(x) = y$ for just four specific linear extensions $\xi_1, \xi_2, \xi_3, \xi_4$.

3 Further directions

3.1 Echelon-Independence and Auslander-Regular Posets

Recall that Theorems 2.1, 2.5 and 2.7 make progress toward the following problem.

Problem 3.1. *Classify echelon-independent posets.*

More generally, one may wonder how many different echelonmotions a given poset induces, and when two permutations σ result in the same echelonmotion Ech_σ.

Let us briefly discuss an interesting connection between the rowmotion operator and homological algebra of finite-dimensional algebras. Let K be a field, and let A be a finite-dimensional K-algebra of finite global dimension. Then A is called *Auslander-regular* if the minimal injective coresolution

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots \to I^n \to 0$$

has the property that the projective dimension of I^i is at most i for all $i \geq 0$. This is the non-commutative generalization of the classical commutative regular rings due to Auslander; see the surveys [7, 17]. Iyama and Marczinzik [16] proved that a finite lattice L is distributive if and only if the incidence algebra of L is Auslander-regular. Every Auslander-regular algebra A comes with a natural permutation π, called the *Auslander–Reiten permutation*, which is defined by the condition that the indecomposable projective module $P_{\pi(i)}$ is the last term in the minimal projective resolution of the indecomposable injective module I_i. For other equivalent descriptions of the Auslander–Reiten permutation, see [18]. It was shown in [16] that the Auslander–Reiten permutation on the incidence algebra of a distributive lattice coincides with the inverse of the rowmotion operator on the elements of this distributive lattice. We call a finite poset R *Auslander-regular* if the incidence algebra KR is Auslander-regular when K has characteristic 0. The classification of Auslander-regular posets is an open problem with some partial progress in the forthcoming article [15]. Examples of Auslander-regular posets that are not lattices include the strong Bruhat order on the symmetric groups S_3 and S_4. Many more examples related to number theory will be given in [15]. We have the following conjecture:

Conjecture 3.2. *Let R be a connected Auslander-regular poset. Then R is echelon-independent, and echelonmotion on R is the inverse of the Auslander–Reiten permutation of R.*

We have verified this conjecture for posets with at most 10 elements. Note that a connected Auslander-regular poset is bounded and has a distributive MacNeille completion; see [15]. As a final remark, we note that we can associate to every finite-dimensional algebra of finite global dimension an echelonmotion using the fact that the Cartan matrix is invertible in that case. Echelonmotion on a poset with respect to a linear extension σ is then obtained as a special case by considering the Cartan matrix of the incidence algebra of the poset with respect to σ. It turns out that this definition is also interesting for other finite-dimensional algebras associated to combinatorial objects such as Dyck paths, which correspond bijectively to linear Nakayama algebras; see [19].

3.2 Canonical Bruhat Factorizations

Let σ be a linear extension of a lattice L, and let $W = W^{L,\sigma}$ be the corresponding Cartan matrix. Recall that the *Coxeter matrix* of L with respect to σ is defined to be $C = -W^{-1}W^{\top}$. Klász, Marczinzik, and Thomas [18] proved that L is distributive if and only if there exists an upper-triangular matrix U and a permutation matrix P such that $C = PU$. In other words, L is distributive if and only if C has a Bruhat decomposition in which the first upper-triangular matrix is the identity matrix. Furthermore, the matrix U in this case is an involution [20]. One can view this factorization as a "canonical Bruhat factorization" of the Coxeter matrix of a distributive lattice. This leads us to the following (somewhat vague) question.

Question 3.3. *Let σ be a linear extension of a semidistributive lattice L. Is there a canonical Bruhat factorization of the Coxeter matrix of L with respect to σ?*

3.3 Independence Posets

Thomas and Williams [31] introduced *independence posets* as a generalization of trim lattices, and they explained how to define a bijective rowmotion operator on an independence poset. This leads us naturally to the following question.

Question 3.4. *Let R be an independence poset. Does there exist a linear extension σ of R such that Ech_σ is the same as rowmotion on R?*

3.4 Modular Lattices

A lattice L is *modular* if for all $a, b, x \in L$ such that $a \leq b$, we have $a \vee (x \wedge b) = (a \vee x) \wedge b$. Every distributive lattice is modular. There are numerous other notable examples of modular lattices, including the lattice of normal subgroups of a finite group and the lattice of submodules of a module over a ring.

In a seminal article, Dilworth [13] proved that for every modular lattice L and every nonnegative integer k, the number of elements of L that are covered by k elements equals the number of elements of L that cover k elements. That is,

$$\left|\{x \in L : |\mathrm{Cov}_L^{\uparrow}(x)| = k\}\right| = \left|\{x \in L : |\mathrm{Cov}_L^{\downarrow}(x)| = k\}\right|.$$

The following conjecture states that for any linear extension σ of L, echelonmotion with respect to σ explicitly realizes Dilworth's theorem.

Conjecture 3.5. *Let σ be a linear extension of a modular lattice L. For every $x \in L$, we have*

$$\left|\mathrm{Cov}_L^{\uparrow}(\mathrm{Ech}_\sigma(x))\right| = \left|\mathrm{Cov}_L^{\downarrow}(x)\right|.$$

We have tested Conjecture 3.5 for all modular lattices with at most 9 elements. For each prime power $q \leq 4$ and each positive integer $d \leq 3$, we have tested the conjecture for 200 random linear extensions of the lattice of subspaces of $\mathbb{F}_q^d$.

3.5 Beyond Linear Extensions

Let R be an n-element poset. The article [16] considers bijections $\sigma \colon R \to [n]$ that need not be linear extensions. Given any such bijection, one can define the Cartan matrix $W^{R,\sigma}$ and the echelonmotion map $\mathrm{Ech}_\sigma \colon R \to R$ just as in Definition 1.1. It could be fruitful to explore this more general setting.

Acknowledgements

This work began at the 2025 workshop on Lattice Theory at the Banff International Research Station. We thank BIRS for the excellent working conditions that they provided. We thank Tianyi Yu for detailed comments on an earlier draft.

References

[1] D. Armstrong, C. Stump, and H. Thomas. “A uniform bijection between nonnesting and noncrossing partitions”. *Trans. Amer. Math. Soc.* **365**.8 (2013), pp. 4121–4151. DOI.

[2] S. Asai, O. Iyama, K. Mousavand, and C. Paquette. “Brick-splitting Torsion Pairs and Trim Lattices”. *arXiv preprint arXiv:2506.13602* (2025).

[3] E. Barnard. “The canonical join complex”. *Electron. J. Combin.* **26**.1 (2019), Paper No. 1.24, 25 pp. DOI.

[4] M. von Bell and C. Ceballos. “Framing lattices and flow polytopes”. *Sém. Lothar. Combin.* **91B** (2024), Art. 98, 12.

[5] M. von Bell, R. S. González D’León, F. A. Mayorga Cetina, and M. Yip. “A unifying framework for the ν-Tamari lattice and principal order ideals in Young’s lattice”. *Combinatorica* **43**.3 (2023), pp. 479–504. DOI.

[6] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*. Vol. 231. Graduate Texts in Mathematics. Springer, New York, 2005, pp. xiv+363.

[7] J. Clark. “Auslander-Gorenstein rings for beginners”. *International symposium on ring theory. Proceedings of the 3rd Korea-China-Japan international symposium, held jointly with the 2nd Korea-Japan joint ring theory seminar, Kyongju, Korea, June 28–July 3, 1999*. Boston, MA: Birkhäuser, 2001, pp. 95–115.

[8] C. Defant, Y. Jiang, R. Marczinzik, A. Segovia, D. E. Speyer, H. Thomas, and N. Williams. “Rowmotion and Echelonmotion”. Preprint, arXiv:2507.18230. 2025. Link.

[9] C. Defant and N. Williams. “Semidistrim lattices”. *Forum Math. Sigma* **11** (2023), Paper No. e50, 35 pp. DOI.

[10] L. Demonet, O. Iyama, N. Reading, I. Reiten, and H. Thomas. “Lattice theory of torsion classes: beyond τ-tilting theory”. *Trans. Amer. Math. Soc. Ser. B* **10** (2023), pp. 542–612. DOI.

[11] A. Dermenjian, C. Hohlweg, T. McConville, and V. Pilaud. “The facial weak order on hyperplane arrangements”. *Discrete Comput. Geom.* **67**.1 (2022), pp. 166–202. DOI.

[12] M. M. Deza and K. Fukuda. “Loops of clutters”. *Coding Theory and Design Theory: Part I Coding Theory*. Springer, 1990, pp. 72–92.

[13] R. P. Dilworth. “Proof of a conjecture on finite modular lattices”. *Ann. of Math. (2)* **60** (1954), pp. 359–364. DOI.

[14] S. Hopkins. “Order polynomial product formulas and poset dynamics”. *Open problems in algebraic combinatorics*. Vol. 110. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2024, pp. 135–157.

[15] O. Iyama, V. Klász, M. Kleinau, and R. Marczinzik. “2-Gorenstein algebras and dissective posets, work in progress”.

[16] O. Iyama and R. Marczinzik. “Distributive lattices and Auslander regular algebras”. *Adv. Math.* **398** (2022). Id/No 108233, 27 pp. DOI.

[17] V. Klász and R. Marczinzik. “A survey on Auslander-Gorenstein algebras”. Preprint, arXiv:2508.19079 [math.RT] (2025). 2025. Link.

[18] V. Klász, R. Marczinzik, and H. Thomas. “Auslander regular algebras and Coxeter matrices”. Preprint, arXiv:2501.09447. 2025. Link.

[19] V. Klász, M. Kleinau, and R. Marczinzik. “Classification of Auslander-Gorenstein monomial algebras: The acyclic case, work in progress”.

[20] R. Marczinzik, H. Thomas, and E. Yıldırım. “On the interaction of the Coxeter transformation and the rowmotion bijection”. *J. Comb. Algebra* **8**.3-4 (2024), pp. 359–374. DOI.

[21] G. Markowsky. “Primes, irreducibles and extremal lattices”. *Order* **9**.3 (1992), pp. 265–290. DOI.

[22] O. H. Odeh, D. D. Olesky, and P. van den Driessche. “Bruhat decomposition and numerical stability”. *SIAM J. Matrix Anal. Appl.* **19**.1 (1998), pp. 89–98. DOI.

[23] D. I. Panyushev. “On orbits of antichains of positive roots”. *European J. Combin.* **30**.2 (2009), pp. 586–594. DOI.

[24] L.-F. Préville-Ratelle and X. Viennot. “An extension of Tamari lattices”. *Trans. Amer. Math. Soc.* **369**.7 (2017), pp. 5219–5239. DOI.

[25] N. Reading. “Cambrian lattices”. *Adv. Math.* **205**.2 (2006), pp. 313–353. DOI.

[26] N. Reading. “Finite Coxeter groups and the weak order”. *Lattice theory: special topics and applications. Vol. 2*. Birkhäuser/Springer, Cham, 2016, pp. 489–561.

[27] N. Reading, D. E. Speyer, and H. Thomas. “The fundamental theorem of finite semidistributive lattices”. *Selecta Math. (N.S.)* **27**.4 (2021), Paper No. 59, 53. DOI.

[28] T. Roby. “Dynamical algebraic combinatorics and the homomesy phenomenon”. *Recent trends in combinatorics*. Vol. 159. IMA Vol. Math. Appl. Springer, 2016, pp. 619–652. DOI.

[29] J. Striker and N. Williams. “Promotion and rowmotion”. *European J. Combin.* **33**.8 (2012), pp. 1919–1942. DOI.

[30] H. Thomas. “An analogue of distributivity for ungraded lattices”. *Order* **23**.2-3 (2006), pp. 249–269. DOI.

[31] H. Thomas and N. Williams. “Independence posets”. *J. Comb.* **10**.3 (2019), pp. 545–578. DOI.

[32] H. Thomas and N. Williams. “Rowmotion in slow motion”. *Proc. Lond. Math. Soc. (3)* **119**.5 (2019), pp. 1149–1178. DOI.