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Which Polytopes are Positively Curved?

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Abstract. We study Forman–Ricci and effective resistance curvatures on the skeleta of convex polytopes. Our guiding questions are: how frequently do polytopal graphs exhibit everywhere positive curvature, and what structural constraints does positivity impose? For Forman–Ricci curvature we derive an exact identity for the average edge curvature in terms of flag f-numbers and establish the existence of infinite families of Forman–Ricci-positive polytopes in every fixed dimension $d \geq 6$. We prove finiteness results in low dimension: there are only finitely many Forman–Ricci-positive 3- and 4-polytopes; for $d = 5$ we show finiteness in the simplicial case, and conjecture its extension to 5-polytopes more generally. For the resistance curvature $\kappa(v)$ we establish the existence of infinite families for all $d \geq 3$, and we provide a quantitative lower bound for $\kappa(v)$ in a simple 3-polytope in terms of the lengths of the three 2-faces incident to v. This bound leads to constructions of non-vertex-transitive, resistance-positive 3-polytopes via Δ-operations, and a degree-based obstruction showing that if each neighbor of v has degree at most $d_v - 2$, then $\kappa(v) \leq 0$. Our results suggest that positive curvature on polytopal skeletons is rare and constrained.

Keywords: discrete curvature, convex polytopes, diameter, Forman–Ricci curvature, effective resistance, graph Laplacians.

1 Introduction

Curvature of graphs and other discrete structures is a rapidly developing field rooted in deep questions concerning how well discrete models capture the geometric structure of continuous spaces. Although the notion of *discrete curvature* may seem counterintuitive

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at first, there exist many notions of curvature on graphs and complexes which have been shown to satisfy discrete analogues of well-known results from differential geometry. Examples include Bonnet-Myers-type theorems relating curvature to diameter bounds (see, e.g., [7, Thm. 1], [10, Thm. 6.3]) and Lichnerowicz-type bounds relating curvature to Laplacian eigenvalues (see, e.g., [15, Thm. 4.2], [24, Thm. 3]). Additionally, discrete curvatures have been used in applied data science and the analysis of networks, demonstrating their versatility and importance. For example, Weber and others [28, 9] connected Forman–Ricci curvature to the analysis of complex networks, while Ollivier and others related the curvature of Markov chains to their mixing rates and spectral gaps, showing that positive curvature ensures fast convergence (see e.g., [18, 17] and references therein).

Meanwhile, in polyhedral geometry, many longstanding open questions remain which concern the very quantities investigated in the theory of discrete curvatures. The *Hirsch conjecture* (see [29]), for example, predicted that the largest diameter $f(d,n)$ of a polytope of dimension $d \geq 1$ defined by no more than $n \geq 1$ linear inequalities satisfies $f(n,d) \leq n-d$. The Hirsch conjecture was disproved in general by Santos [21], but variations of the conjecture remain open and of great interest to the community. Another example is *Barnette's conjecture* (see [1]), which hypothesizes that every cubic bipartite 3-dimensional polyhedral graph is Hamiltonian. It was shown recently by Devriendt [5] that Hamiltonian graphs have positive curvature with respect to a weighted variant of resistance curvature.

It is therefore natural to consider the properties of discrete curvatures within the category of convex polytopes and, in particular, their graphs (i.e., their 1-skeleta). Little effort has been made in this research direction and we are not aware of any prior work. In this article, we investigate the discrete curvature of polytopes with emphasis on *Forman–Ricci curvature* (see Definition 2) and *effective resistance curvature* (see Definition 3). In both cases, we are interested in the basic question of *how many positively curved combinatorial types of d-polytopes exist for various values of the dimension d*. Somewhat surprisingly, in both cases, we show that positive curvature polyhedra appear to be rare.

1.1 Related work

Forman–Ricci curvature, introduced for general cell complexes [10], specializes to an edge-based invariant on graphs that is local and computationally cheap. Subsequent work has seen the theory develop further in various directions: Watanabe [26] established a Gauss-Bonnet-type theorem for graphs and 2-complexes; Bloch [2] analyzed structural limitations of the edge-only definition in dimension 2 and proposed a poset-theoretic extension that restores a Gauss-Bonnet analogue and clarifies the failure of ubiquitous negativity on surfaces; Jost and Münch [12] characterized lower bounds of Forman–Ricci curvature via the contractivity of the Hodge-Laplacian semigroup and related (optimized) Forman–Ricci and Ollivier curvatures, yielding refined diameter bounds and a bridge to heat semigroup techniques. Extensions of this notion to weighted graphs [23], directed graphs [23], and hypergraphs [14] have also been considered.

Resistance curvature, on the other hand, is comparatively newer and continues to be an active topic of research. Effective resistance (see [13]), more generally, is a metric on the vertices of a graph and is related to the simple random walk on the graph [25, 8], spanning trees, and graph sparsification [22]. Using resistance as the basis for a notion of curvature was originally proposed by Devriendt and Lambiotte [6], and was followed shortly thereafter by a closely related notion by Devriendt, Ottolini, and Steinerberger [7]. Subsequent work by Devriendt considered a relaxed notion of positive resistance curvature for graphs [5] and its connections to combinatorial properties of graphs satisfying this condition.

1.2 Notation and mathematical background

We follow the notation and conventions of the classical books [11, 29]. Let P be a d-dimensional polytope. For each $0 \leq k \leq d-1$, we denote by $\mathcal{F}_k = \mathcal{F}_k(P)$ the collection of k-dimensional faces of P. We write $f_k = f_k(P)$ to refer to the cardinality of $\mathcal{F}_k$. If $0 \leq i < j \leq d-1$ we denote by f_{ij} the number of pairs (F, G) with $F \in \mathcal{F}_i$, $G \in \mathcal{F}_j$ and $F \subseteq G$. The *k-skeleton* of P consists of the collection of all faces of dimension at most k. The *graph* of P is the combinatorial graph $G = G(P) = (\mathcal{F}_0(P) =: V(P), \mathcal{F}_1(P) =: E(P))$, i.e., the 1-skeleton of P. Note that in general polytopes are not characterized by their graphs: polytopes whose graph is isomorphic to the complete graph are known as *neighborly* and are abundant (see, e.g., [29, Ch. 8]). In general, the graph of a d-dimensional polytope is known to be d-vertex-connected (Balinski's theorem), and in dimension 3, the graphs of 3-polytopes are characterized combinatorially as exactly those graphs which are planar and 3-vertex-connected (Steinitz's theorem).

If $e \in E(P)$ is any edge, we denote by $\mathcal{F}\uparrow(e) \subseteq \mathcal{F}_2(P)$ the collection of 2-faces of P that contain e and by $\mathcal{F}\downarrow(e) \subseteq \mathcal{F}_0(P)$ the set of vertices of P contained in e.

Definition 1. *Let P be a polytope and $e, e' \in E(P)$ fixed edges. We say that e and e' are* parallel neighbors *if one of the following statements holds:*

i) $\mathcal{F}\downarrow(e) \cap \mathcal{F}\downarrow(e') \neq \emptyset$, but $\mathcal{F}\uparrow(e) \cap \mathcal{F}\uparrow(e') = \emptyset$, i.e., if e and e' share a vertex, but are not contained in a common two face.

ii) $\mathcal{F}\uparrow(e) \cap \mathcal{F}\uparrow(e') \neq \emptyset$, but $\mathcal{F}\downarrow(e) \cap \mathcal{F}\downarrow(e') = \emptyset$, i.e., e and e' are vertex disjoint edges that are contained in a two dimensional face.

The collection of parallel edges of e is denoted by $\mathcal{E}(e)$.

(a) (b)

Figure 1: *(a)* Parallel neighbors (blue) of an edge e (red) in the case where e is an edge of a heptagon (left), and in the case where e is adjacent to a vertex of degree 7 (right). *(b)* A 3-dimensional square cupola polytope with edges labeled according to their Forman–Ricci curvature.

Definition 2. *Let P be a polytope and let $e \in E(P)$ be any fixed edge. The* Forman–Ricci curvature *of e, denoted $\kappa_F(e)$, is given by*

$$\kappa_F(e) := |\mathcal{F} \uparrow (e)| + 2 - |\mathcal{E}(e)|.$$

We illustrate Definition 1 and Definition 2 in Figure 1a and Figure 1b, respectively. Definition 2 originally appeared (for cell complexes) in 2003 in a work of Forman (see [10]) and is known in the literature by this name. The same paper contains a Bonnet-Myers-type diameter bound, which is set up as follows. The distance between two vertices $v, v' \in \mathcal{F}_0$, denoted $d(v, v')$, is the length of any shortest path from v to v' in $G(P)$. The diameter of G is the largest distance between a pair of vertices and is denoted by $\operatorname{diam}(G)$. The degree of a vertex v, denoted d_v, is the number edges incident to v. The following theorem is a Bonnet-Meyers type result which motivates our study of positively curved polytopes.

Theorem 1 (Bonnet-Myers Theorem for Forman–Ricci curvature (see [10]))**.** *Let P be a polytope. Suppose there exists $c > 0$ such that for $\kappa_F(e) \geq c$ for each edge $e \in E(P)$. Then the following hold:*

(i) If $v_1, v_2 \in V(P)$ and $e_1, e_2 \in E(P)$ occur on any shortest v_1-v_2 path, then the distance $d(v_1, v_2)$ satisfies

$$d(v_1, v_2) \leq \frac{1}{c}\left(2 + |\mathcal{F} \uparrow (e_1)| + |\mathcal{F} \uparrow (e_2)|\right).$$

(ii) Consequently,

$$\operatorname{diam}(P) \leq \frac{2}{c}\left(1 + \max_{e \in E(P)} |\mathcal{F} \uparrow (e)|\right).$$

Some of our results call only for a combinatorial graph which is not necessarily derived as the 1-skeleton of a polytope; in such cases we consider graphs of the form

$G = (V, E)$ where V is any finite set of vertices and $E \subseteq \binom{V}{2}$. If $\{i, j\} \in E$ we write $i \sim j$. If G is any graph, we denote by $n \geq 1$ the number of vertices in G and $m \geq 1$ the number of edges in G. We denote by $\mathbf{A} = \mathbf{A}(G)$ (resp. $\mathbf{D} = \mathbf{D}(G)$) the adjacency matrix (resp. diagonal vertex degree matrix) of G. The matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is known as the *combinatorial Laplacian matrix* of G. We denote by $E' \subseteq V \times V$ any fixed but otherwise arbitrary orientation of the edges E, i.e., any set containing exactly one ordered representative for each $e = \{v_1, v_2\} \in E$. The *vertex-edge oriented incidence matrix* $\mathbf{B} \in \mathbb{R}^{n \times m}$ is defined entrywise by the values

$$\mathbf{B}_{v_i, e_j} = \begin{cases} 1 & \text{if } e_j = (v_i, \cdot) \\ -1 & \text{if } e_j = (\cdot, v_i) \\ 0 & \text{otherwise} \end{cases}, \quad v_i \in V,\ e_j \in E'. \tag{1.1}$$

Note that regardless of choice of orientation on the edges, $\mathbf{L} = \mathbf{B}\mathbf{B}^\top$ (see, e.g., [3]). We choose to orient the edges with respect to the indexing on the nodes only for concreteness. We recall the well known facts that L is symmetric and positive semidefinite, and as long as G is connected, $\mathbf{L}$ has rank $n - 1$. We denote the Moore-Penrose inverse of $\mathbf{L}$ by $\mathbf{L}^\dagger$ (see [16] for an historic reference). *Effective resistance* is a metric on V which is defined by the formula

$$r_{v_1 v_2} = (\mathbf{1}_{v_1} - \mathbf{1}_{v_2})^\top \mathbf{L}^\dagger (\mathbf{1}_{v_1} - \mathbf{1}_{v_2}), \quad v_1, v_2 \in V. \tag{1.2}$$

Here, $\mathbf{1}_v$ is the indicator vector of $v \in V$. We note that by writing $\widetilde{\mathbf{L}} = \mathbf{L} + \frac{1}{n}\mathbf{J}_n$, where $\mathbf{J}_n \in \mathbb{R}^{n \times n}$ is the all ones matrix), which is nonsingular, one may also write

$$r_{v_1 v_2} = (\mathbf{1}_{v_1} - \mathbf{1}_{v_2})^\top \widetilde{\mathbf{L}}^{-1} (\mathbf{1}_{v_1} - \mathbf{1}_{v_2}), \quad v_1, v_2 \in V. \tag{1.3}$$

The following variational characterization of effective resistance is useful in practice.

Lemma 1. *For each $u, v \in V$, the effective resistance r_{uv} is given by*

$$r_{uv} = \inf \left\{ \|\mathbf{J}\|_2^2 \ : \ \mathbf{J} \in \mathbb{R}^{E'},\ \mathbf{B}\mathbf{J} = \mathbf{1}_u - \mathbf{1}_v \right\}.$$

Definition 3. *Let $G = (V, E)$ be any fixed graph and let $v \in V$ be fixed. Then the* effective resistance curvature *at v, denoted $\kappa_R(v)$, is given by*

$$\kappa_R(v) = 1 - \frac{1}{2} \sum_{\substack{u \in V \\ u \sim v}} r_{uv}.$$

This notion of curvature originally appeared in a 2022 paper of Devriendt and Lambiotte (see [6]). A subsequent notion, also known as effective resistance curvature, was introduced in a 2024 paper of Devriendt, Ottolini, and Steinerberger (see [7]). The latter notion can be considered a modification of the former, as although it in principle is motivated by an equilibrium measure of the effective resistance matrix, the two are the same up to a global scaling factor. The latter paper obtained a Bonnet-Myers-type result, which we state below, having been adjusted to be consistent with our chosen convention Definition 3.

Theorem 2 (Bonnet-Myers Theorem for Resistance Curvature (see [7])). *Let $G = (V, E)$ be a connected graph with maximum degree Δ and effective resistance matrix $\mathbf{R} = (r_{uv})_{u,v \in V}$. Assume the node resistance curvature $\boldsymbol{\kappa} = (\kappa_R(v))_{v \in V}$ satisfies $\kappa_R(v) \geq K > 0$ for each $v \in V$. Then*

$$\operatorname{diam}(G) \leq \left\lceil \sqrt{\frac{\Delta\, \boldsymbol{\kappa}^\top \mathbf{R} \boldsymbol{\kappa}}{K}}\, \log |V| \right\rceil.$$

1.3 Our Contributions

We study various aspects of the Forman–Ricci and effective resistance curvatures for skeleta of polytopes. We start by analyzing the average curvature and derive an equation to compute average curvature in terms of face numbers of the polytope. By analyzing polytopes whose 2-skeletons admit an edge-transitive group action we obtain the following result. We say a polytope P is *Forman–Ricci-positive* provided $\kappa_F(e) > 0$ for each $e \in E(P)$. We remind the reader that we consider polytopes up to combinatorial equivalence.

Theorem 3. *For each $d \geq 6$, there are infinitely many Forman–Ricci-positive polytopes with dimension d.*

Further analysis of the average curvature yields the following result which is useful for studying positive polytopes in smaller dimensions.

Theorem 4. *Let $d \geq 3$ be fixed and let $\Delta \geq 3$ be a real number. The set of Forman–Ricci-positive d-polytopes with the property that the average degree of a vertex is at most Δ is finite.*

This theorem has several consequences and essentially says that the edge density of Forman–Ricci-positive graphs has to be rather large. As a consequence, the number of Forman–Ricci-positive *simple* d-dimensional polytopes is finite for all d.

Next we turn to the situation in low dimensions.

Theorem 5. *The set of Forman–Ricci positive 3-polytopes is finite. Polytopes in this collection have no more than 15 vertices.*

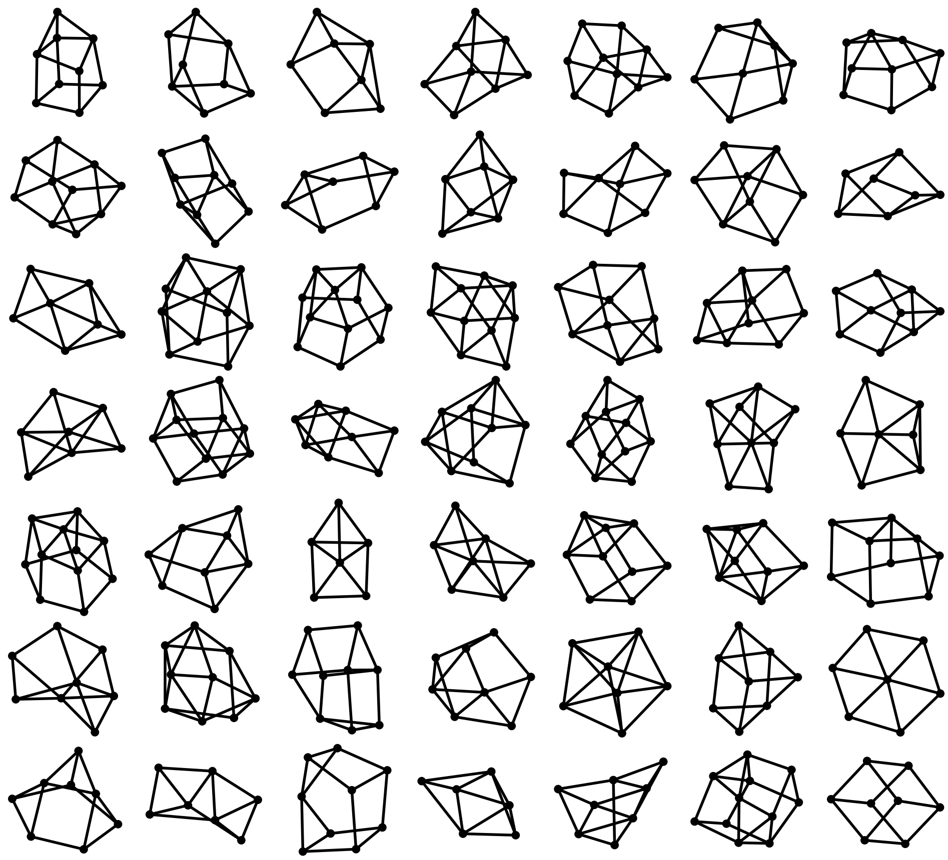

Figure 2: An illustration of the graphs of 49 Forman–Ricci-positive 3-polytopes, obtained as a random sample of the Forman–Ricci-positive 3-polytopes on at most twelve vertices.

It is interesting to compare Theorem 5 with a similar result on a different combinatorial curvature in [4].

Next, in dimension 4, using the proof of Theorem 4 and known structural results about the graphs of 3-polytopes, we obtain the following result.

Theorem 6. *The set of Forman–Ricci positive* 4*-polytopes is finite.*

The result says little about how to classify such 4-polytopes, but the proof shows that, in particular, Forman–Ricci positive 4-polytopes have no vertex of degree greater than 12. The case of $d = 5$ is less well understood, and we conjecture that Theorem 5 and Theorem 6 extend to this setting.

Conjecture 1. *The set of Forman–Ricci positive* 5*-polytopes is finite.*

In dimension 5, we are able make progress in the special case of simplicial polytopes.

Theorem 7. *The set of Forman–Ricci positive simplicial* 5*-polytopes is finite.*

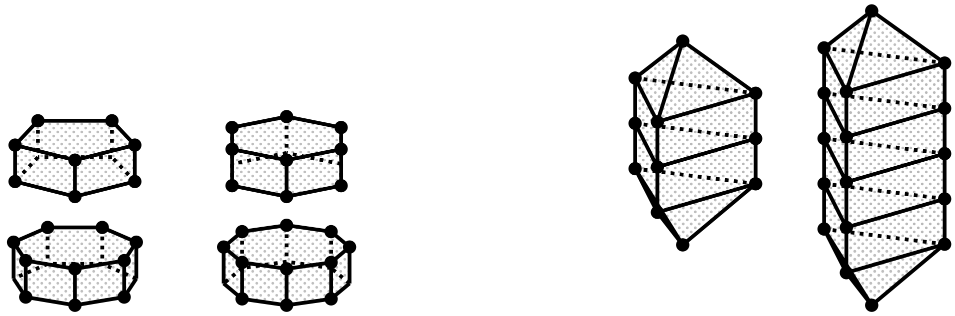

Figure 3: *(a)* The prism 3-polytopes and their graphs with faces consisting of (in clockwise order) five, six, seven, and eight vertices. *(b)* The 3-polytope constructed via its graph G_k in **??**, here shown for $k = 3, 5$.

This appears to be strong evidence in favor of Conjecture 1, since 2-faces that are not triangles contribute at most 0 to the curvature computation.

Next, we describe our results on resistance curvature. We follow a similar program with a more quantitative angle and obtain several results which, although spiritually analogous, have different conclusions and implications. We call a polytope P *resistance positive* if $\kappa_R(v) > 0$ for each $v \in V(P)$.

Theorem 8. *For each $d \geq 2$, there are infinitely many resistance positive polytopes with dimension d.*

Theorem 8 follows from the existence of an infinite family of d-polytopes with vertex transitive graphs; namely, in the case of $d = 2, 3$, polygons and polygonal prisms (see Figure 3a); and in the case of $d \geq 4$, the existence of d-polytopes whose graphs are isomorphic to the complete graph K_n. We note, however, that a complete characterization of resistance positive 3-polytopes seems out of reach at present; in particular, we identify a family of resistance positive 3-polytopes which have graphs that are not vertex transitive (see Figure 3b).

On the quantitative side we obtain the following curvature bound for a vertex in a simple polytope in terms of the lengths of the polygonal cycles incident to a given vertex.

Theorem 9. *Let $G = (V, E)$ be the 1-skeleton of a simple 3-polytope. Fix $v \in V$, and let $\mathcal{C}(v)$ denote the set of 2-faces incident to v. For each $C \in \mathcal{C}(v)$, let $\ell_C := |E(C)|$ be the length (edge count) of C. Then the resistance curvature of G at v satisfies*

$$\kappa_R(v) \geq 1 - \frac{1}{2} \sum_{C \in \mathcal{C}(v)} \frac{\ell_C - 1}{(\ell_C - 1)\left(\sum_{C' \in \mathcal{C}(v)} \frac{1}{\ell_{C'} - 1} + 1\right) - 1}.$$

This bound is used to identify families of non-vertex-transitive 3-polytopes obtained as Δ-expansions of known simple 3-polytopes. We also investigate quantitative lower bounds for the resistance curvature in generic graphs, and obtain the following degree-based criterion for the existence of a vertex with negative resistance curvature.

Theorem 10. *Let $G = (V, E)$ be a graph and fix adjacent vertices $u, v \in V$. Assume for simplicity that either $d_u \geq 2$ or $d_v \geq 2$ (i.e., that $\{u, v\}$ is not a connected component of G). Then it holds*

$$r_{uv} \geq \max\left\{\frac{d_v + d_u - 2}{d_u d_v - 1}, \frac{4}{d_u + d_v + 2}\right\}.$$

Corollary 1. *Let $G = (V, E)$ be any graph, and suppose $v \in V$ satisfies the following two conditions:*

(i) *$d_v \geq 2$, and*

(ii) *For each $u \sim v$, $d_u \leq d_v - 2$.*

Then the resistance curvature $\kappa_R(v)$ satisfies $\kappa_R(v) \leq 0$.

Corollary 1 can be used to rule out resistance positivity for many polytopes; pyramids are natural examples in the case of $d = 3$ since their apexes generally meet the hypotheses of Corollary 1. Moreover, Corollary 1 establishes that resistance positive graphs must, in a weak sense, be "close" to degree regular, and in doing so lends credence to the overall picture that resistance positive polytopes are often rare.

We finish with a conjecture about resistance curvature and simple 3-dimensional polytopes that would imply the scarcity of the most relevant resistance positive polytopes in dimension 3. If the vast majority of faces have at least six sides, their incidences may guarantee induced subgraphs that may be utilized to guarantee the negativity. Eberhard's theorem implies that several large incidences make it plausible that the following is true:

Conjecture 2. *There are finitely many simple 3-dimensional polytopes that are resistance positive and are not isomorphic to a prism over a polygon.*

We remark that there are many other notions of curvature we did not discuss here, including those of Ollivier [18], Lin-Lu-Yau [15], or Steinerberger [24], which have been studied at length in the literature. Similarly, we are aware of only a few papers trying to compare curvatures [20, 27, 19]. We believe that exploring the variants of our results for other curvatures could be of interest. Our experiments suggest that the same kind of results will hold.

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