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Tree metrics and log-concavity for matroids

Federico Ardila-Mantilla*[1], Sergio Cristancho †[2], Graham Denham‡[3],
Christopher Eur§[4], June Huh¶[2], and Botong Wang||[5]

[1]*San Francisco State University and Universidad de los Andes*
[2]*Princeton University*
[3]*University of Western Ontario*
[4]*Carnegie Mellon University*
[5]*University of Wisconsin - Madison*

Abstract. We characterize $\mathrm{M}^\natural$-concave functions on sets in terms of Lorentzian polynomials. We achieve this by showing that a certain rank 1 modification of the distance matrix of a ultrametric tree is positive semi-definite, generalizing classical results of Schoenberg and of Graham and Pollak on tree metrics. Using the characterization, we resolve two open questions that extend Mason's log-concavity conjectures for the number of independent sets of a matroid: one posed by Giansiracusa, Rincón, Schleis, and Ulirsch for valuated matroids, and another by Dowling and Zhao for ordinary matroids.

Keywords: matroid, valuated matroid, Lorentzian polynomial, log-concave, tree metric, ultrametric

1 Introduction

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. A *matroid* M on $[n]$ consists of a nonempty set $\mathrm{IN}(\mathsf{M})$ of subsets of $[n]$ satisfying the following axioms:

- if $I \subseteq J \subseteq [n]$ and $J \in \mathrm{IN}(\mathsf{M})$ then $I \in \mathrm{IN}(\mathsf{M})$ also, and
- if $I, J \in \mathrm{IN}(\mathsf{M})$ such that $|I| > |J|$ then there exist $i \in I \setminus J$ such that $J \cup i \in \mathrm{IN}(\mathsf{M})$.

Here we write i for $\{i\}$ as is customary in matroid theory. The elements of $\mathrm{IN}(\mathsf{M})$ are called the *independent sets* of M. The maximal independent sets of M are called the *bases* of M, denoted by $\mathrm{B}(\mathsf{M})$, and the *rank* of M is the common cardinality of its bases.

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*federico@sfsu.edu. Federico Ardila-Mantilla was supported by NSF-DMS2154279.
†sergio.cris@princeton.edu. Sergio Cristancho was supported by the Simons Investigator Grant.
‡gdenham@uwo.ca. Graham Denham was supported by NSERC of Canada.
§ceur@cmu.edu. Chris Eur was supported by NSF-DMS2246518.
¶huh@princeton.edu. June Huh was supported by the Simons Investigator Grant.
||wang@math.wisc.edu. Botong Wang was supported by NSF-DMS1926686.

Let $I_k = I_k(\mathsf{M}) = |\mathrm{IN}(\mathsf{M})_k|$ be the number of independent sets of size k, for $0 \leq k \leq \operatorname{rank}(\mathsf{M})$. Mason [12] conjectured the following three sets of inequalities for all $0 < k < \operatorname{rank}(\mathsf{M})$:

$$I_k^2 \geq I_{k-1}I_{k+1} \tag{M1}$$

$$I_k^2 \geq \Big(1+\frac{1}{k}\Big)I_{k-1}I_{k+1} \tag{M2}$$

$$I_k^2 \geq \Big(1+\frac{1}{k}\Big)\Big(1+\frac{1}{n-k}\Big)I_{k-1}I_{k+1}. \tag{M3}$$

They are equivalent to the statements that the sequences $(I_k)_k$, $(k!I_k)_k$ and $(I_k/\binom{n}{k})_k$ are *log-concave*. The last property is referred to as the *ultra log-concavity* of the sequence $(I_k)_k$.

Inequality (M1) was proved for realizable matroids in [10] by building on [8], and for arbitrary matroids in [1]. Inequality (M2) was established in [9], while (M3) was proved independently in [2] and [3]. These proofs were achieved by the development of combinatorial Hodge theory [1] and Lorentzian polynomials [3].

We extend (M3) to the more general realm of $\mathrm{M}^\natural$-concave functions, set functions $\nu : 2^{[n]} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$ satisfying certain discrete convexity axioms. For a parameter $0 < q \leq 1$, consider the weighted sum

$$I_{q;k} = I_{q;k}(\nu) := \sum_{I \in \binom{[n]}{k}} q^{-\nu(I)} \tag{1.1}$$

for $0 \leq k \leq n$, where by convention $q^\infty = 0$. In Section 3, we will prove that

$$I_{q;k}^2 \geq \Big(1+\frac{1}{k}\Big)\Big(1+\frac{1}{n-k}\Big)I_{q;k-1}I_{q;k+1} \tag{VM3}$$

holds for any $\mathrm{M}^\natural$-concave function μ, all $0 < k < n$ and any $0 < q \leq 1$. This affirmatively answers a conjecture of J. Giansiracusa, Rincón, Schleis and Ulirsch [6] as a special case. Along the way we prove a characterization of $\mathrm{M}^\natural$-concave functions in terms of Lorentzian polynomials (Theorem 4):

> A set function ν on E satisfies the gross substitutes property if and only if its homogeneous generating polynomial $Z_{q,\nu}$ is a Lorentzian polynomial for all positive $q \leq 1$.

This answers a question of Eur and Huh [5] (see Remark 3).

In Section 4, we prove multivariate polynomial inequalities that significantly refine (M2). Concretely, we prove that for any matroid M of rank r the following polynomial inequality holds for any $0 < k < r$:

$$I_k(w)^2 \succeq \left(1+\frac{1}{k}\right) I_{k-1}(w)I_{k+1}(w). \tag{PM2}$$

where $I_k(w)$ is the generating polynomial of the independent sets of M of size k. This proves a conjecture of both Dowling [4] and Zhao [22], recently popularized by Igor Pak [17].

The fundamental ingredient is Section 5, where we show that a natural matrix associated with any ultrametric tree is positive semi-definite, generalizing classical results on tree metrics:

> For any ultrametric tree T with n leaves whose common distance from the root is 1, the matrix $2\left(1-\frac{1}{n}\right)\mathbf{1}_{n\times n} - D$ is positive definite, where D is the distance matrix of T and $\mathbf{1}_{n\times n}$ is the all ones $n \times n$ matrix.

We will see that this serves as the base case for our characterization of $\mathrm{M}^\natural$-concave functions.

2 Lorentzian polynomials and log-concavity

Let us begin with a brief introduction to the theory of *Lorentzian polynomials* and their relation to matroid theory and log-concave sequences.

Let $\mathrm{H}_n^d \subset \mathbb{R}[w_1, \dots, w_n]$ be the space of degree d homogeneous polynomials in n variables with real coefficients, endowed with its usual metric topology. Denote $\partial_i = \frac{\partial}{\partial w_i}$ for any $i \in [n]$, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for any $\alpha \in \mathbb{Z}^n_{\geq 0}$.

In degree $d = 2$, the set of *strictly Lorentzian polynomials* $\mathring{\mathrm{L}}_n^2 \subseteq \mathrm{H}_n^2$ consists of all quadratic forms f with positive coefficients whose Hessian matrix $\mathcal{H}_f = (\partial_i \partial_j f)_{ij}$ is nonsingular and has the Lorentzian signature $(+, -, \cdots, -)$. For $d > 2$, we define $\mathring{\mathrm{L}}_n^d \subseteq \mathrm{H}_n^d$ recursively by letting

$$\mathring{\mathrm{L}}_n^d := \left\{ f \in \mathrm{H}_n^d \colon \partial_i f \in \mathring{\mathrm{L}}_n^{d-1} \text{ for all } 1 \leq i \leq n. \right\}.$$

The *Lorentzian polynomials* $\mathrm{L}_n^d \subseteq \mathrm{H}_n^d$ are defined to be the closure of $\mathring{\mathrm{L}}_n^d$.

Some properties of Lorentzian polynomials that will be relevant are the following:

1. If $f(x) \in \mathrm{L}_n^d$ and A is an $n \times m$ nonnegative matrix, then $f(Ay) \in \mathrm{L}_m^d$ [3, Theorem 2.10].
2. If $f \in \mathrm{L}_n^d$, then $\sum_i a_i \partial_i f \in \mathrm{L}_n^{d-1}$ for any $a_i \geq 0$ [3, Corollary 2.11].

We define the *rank d simplex* on $[n]$ to be $\Delta_n^d := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{[n]} \colon \alpha_1 + \cdots + \alpha_n = d \right\}$, and denote by e_i the i-th standard basis vector of $\mathbb{Z}^{[n]}$.

An *M-convex set* of rank d on $[n]$ is a collection $\mathrm{J} \subseteq \Delta_n^d$ satisfying the *symmetric exchange property*: For any $\alpha, \beta \in \mathrm{J}$ and $i \in [n]$ such that $\alpha_i > \beta_i$, there exists a $j \in [n]$ such that

$\beta_j > \alpha_j$ for which both $\alpha + e_j - e_i$ and $\beta + e_i - e_j$ belong to J. Identifying the power set $2^{[n]}$ with the cube $\{0,1\}^{[n]}$ in $\mathbb{Z}^{[n]}$, one finds that a subset $\mathrm{J} \subset \binom{[n]}{d}$ defines the bases of a matroid if and only if J is M-convex. In this sense, M-convex sets are 'multiset' generalizations of matroids, also known as *polymatroids* in the literature.

Given a degree d homogeneous polynomial $f = \sum_{\alpha \in \Delta_n^d} c_\alpha w^\alpha \in \mathbb{R}[w_1, \dots, w_n]$, its *support* is defined as

$$\operatorname{supp}(f) = \left\{ \alpha \in \Delta_n^d \colon c_\alpha \neq 0 \right\}.$$

The following characterization of Lorentzian polynomials outlines their relationship to matroids and M-convexity:

Theorem 1. *[3, §2] Let $f \in \mathrm{H}_n^d$ be a homogeneous polynomial with non-negative coefficients. The following are equivalent:*

1. *The polynomial $f \in \mathrm{L}_n^d$ is Lorentzian.*
2. *Its support $\operatorname{supp} f \subseteq \Delta_n^d$ is M-convex and for all $\alpha \in \Delta_n^{d-2}$, the Hessian of $\partial^\alpha f$ has at most one positive eigenvalue.*

This characterization yields a very concrete relation between log-concave sequences and Lorentzian binary forms.

Corollary 2. *[3, Ex. 2.3] If $f \in \mathrm{H}_2^d$ is a bivariate polynomial of the form*

$$f(x,y) = \sum_{k=0}^{d} c_k x^k y^{d-k},$$

with all $c_k \geq 0$, then f is Lorentzian if and only if the sequence $(c_k / \binom{d}{k})_k$ is log-concave and has no internal zeros.

3 Inequalities for $\mathrm{M}^\natural$-concave set functions

A set function $\nu : 2^{[n]} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$ on the ground set $[n]$ is said to be *$\mathrm{M}^\natural$-concave* if it satisfies the following:

1. ν is not everywhere $-\infty$.
2. *The exchange axiom*: For any $I, J \in 2^{[n]}$ and $i \in I \setminus J$, we have either
   (a) $\nu(I) + \nu(J) \leq \nu(I \setminus i) + \nu(J \cup i)$, or
   (b) there exists some $j \in J \setminus I$ such that $\nu(I) + \nu(J) \leq \nu(I \setminus i \cup j) + \nu(J \setminus j \cup i)$.

The *domain* of such a function is the set $\mathrm{dom}(\nu) := \{S \subseteq [n] : \nu(S) \neq -\infty\}$.

Introduced by Murota and Shioura [15], $\mathrm{M}^\natural$-concave functions are central objects in discrete convex analysis and discrete optimization. For instance, $\mathrm{M}^\natural$-concavity is equivalent to the *gross substitutes property* in economics [18]. For a comprehensive introduction to $\mathrm{M}^\natural$-concavity and the related notion of M-convexity, we refer the reader to [14, Ch.6].

For our purposes, $\mathrm{M}^\natural$-concave functions play the role of the simultaneous generalization of several notions in matroid theory including: matroid independent sets (Example 1), matroid rank functions (Example 2), and valuated matroids (Example 3).

Example 1 (Matroid independent sets). Let M be a matroid of rank r on $[n]$. The trivial valuation on the collection of independent sets of M, denoted $\mathrm{IN}(\mathsf{M})$, determines an $\mathrm{M}^\natural$-concave function

$$\nu_{\mathsf{M}}(S) = \begin{cases} 0 & \text{if } S \in \mathrm{IN}(\mathsf{M}) \\ -\infty & \text{otherwise,} \end{cases}$$

defined precisely so that $\mathrm{dom}(\nu_{\mathsf{M}}) = \mathrm{IN}(\mathsf{M})$.

Example 2 (Matroid rank functions). Let M be a matroid on $[n]$, then its corresponding *rank function* $\mathrm{rk}_{\mathsf{M}} \colon 2^{[n]} \to \mathbb{Z}_{\geq 0}$ is an $\mathrm{M}^\natural$-concave function satisfying $\mathrm{dom}(\mathrm{rk}_{\mathsf{M}}) = 2^{[n]}$. Further, if $\boldsymbol{M} = (\mathsf{M}_1, \dots, \mathsf{M}_\ell)$ is a *flag matroid* on $[n]$, for any $c_0 \in \mathbb{R}$ and $c_1, \dots, c_\ell \in \mathbb{R}_{\geq 0}$, the function

$$r = c_0 + c_1 \,\mathrm{rk}_{\mathsf{M}_1} + \cdots + c_\ell \,\mathrm{rk}_{\mathsf{M}_\ell},$$

is an $\mathrm{M}^\natural$-concave function by a theorem of Shioura [20, Thm. 3].

Example 3 (Valuated matroid independent sets). A *valuated matroid* of rank r on $[n]$ is a function $\nu \colon \binom{[n]}{r} \to \overline{\mathbb{R}}$ defined on the collection $\binom{[n]}{r}$ of subsets of $[n]$ of size r that is not everywhere $-\infty$ and that satisfies the *symmetric exchange property*: for any $B, B' \in \binom{[n]}{r}$ and $b \in B \setminus B'$, there exists a $b' \in B' \setminus B$ for which

$$\nu(B) + \nu(B') \leq \nu(B \setminus b \cup b') + \nu(B' \setminus b' \cup b).$$

Any valuated matroid ν has an *underlying matroid* M whose set of bases is

$$\mathrm{B}(\mathsf{M}) = \left\{ B \in \binom{[n]}{r} : \nu(B) \neq -\infty \right\}.$$

We denote a valuated matroid by the pair (M, ν) of a matroid and corresponding valuation.

Murota [13] showed that any valuated matroid (M, ν) admits a cryptomorphic characterization by an $\mathrm{M}^\natural$-concave function defined on independent sets, also denoted ν by abuse of notation, by setting

$$\nu(S) := \max \{\nu(B) \colon S \subseteq B \text{ and } |B| = r\},$$

for all $S \subseteq [n]$; here we set the maximum of the empty set to be $-\infty$, so that $\nu(S) = -\infty$ if $|S| > r$. In particular, the collection of independent sets of the underlying matroid M is given by

$$\mathrm{IN}(\mathsf{M}) = \mathrm{dom}(\nu) = \{S \subseteq [n] \colon \nu(S) \neq -\infty\}\,.$$

It is worth noting, that if $\nu\colon 2^{[n]} \to \overline{\mathbb{R}}$ is $\mathrm{M}^{\natural}$-concave and $\varnothing \in \mathrm{dom}(\nu)$, then $\mathrm{dom}(\nu)$ is necessarily the collection of independent sets of a matroid M_{ν} [16, Cor. 1.4], although ν might not define a valuated matroid on M_{ν} in the sense we have established here.

Recall that for a set function $\nu : 2^{[n]} \to \overline{\mathbb{R}}$ and a parameter $0 < q \leq 1$, we defined the weighted sum

$$I_{q;k} = I_{q;k}(\nu) := \sum_{I \in \binom{[n]}{k}} q^{-\nu(I)} \tag{3.1}$$

for $0 \leq k \leq n$, where by convention $q^{\infty} = 0$. This sequence was first considered by Giansiracusa, Rincón, Schleis and Urlisch [6] in the context of valuated matroids. Giansiracusa et al. [6, Thm. A] proved that a valuated matroid (M, ν) satisfies Mason's mid-strength conjecture, generalizing (M2):

$$I_{q;k}^2 \geq \left(1 + \frac{1}{k}\right) I_{q;k-1} I_{q;k+1} \qquad \text{for all } k \text{ and } 0 < q \leq 1, \tag{VM2}$$

and conjectured that the ultra-log concave inequalities (generalizing (M3)) also hold. We prove their conjecture in the more general setting of $\mathrm{M}^{\natural}$-concave functions:

Theorem 3 (Ultra log-concavity for $\mathrm{M}^{\natural}$-concave functions)**.** *An $\mathrm{M}^{\natural}$-concave function $\nu : 2^{[n]} \to \overline{\mathbb{R}}$ satisfies*

$$I_{q;k}^2 \geq \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) I_{q;k-1} I_{q;k+1} \tag{VM3}$$

for all $0 < k < n$ and $0 < q \leq 1$.

We deduce the theorem by showing that $\mathrm{M}^{\natural}$-concave functions characterize the set functions that give rise to two special Lorentzian polynomials:

Given a set function $\nu : 2^{[n]} \to \overline{\mathbb{R}}$, we define the *homogeneous generating polynomial* $Z_{q,\nu}(w)$ of ν with parameter $0 < q \leq 1$ and its respective *normalization* to be

$$\begin{aligned} Z_{q,\nu}(w) &:= \sum_{S \subseteq [n]} q^{-\nu(S)} w^S w_0^{n-|S|} \in \mathbb{R}[w_0, w_1, \ldots, w_n], \\ N(Z_{q,\nu})(w) &:= \sum_{S \subseteq [n]} \frac{q^{-\nu(S)}}{(n-|S|)!} w^S w_0^{n-|S|} \in \mathbb{R}[w_0, w_1, \ldots, w_n], \end{aligned}$$

where $w^S := \prod_{i \in S} w_i$ for $S \subseteq [n]$, and N is the linear map defined by $N(w^{\alpha}) = \frac{w^{\alpha}}{\alpha!}$ and $\alpha! = \prod_{i=0}^{n} \alpha_i!$.

The homogeneous generating polynomial specializes to well-known polynomials of interest in matroid theory. For example:

Example 4 (Independent set generating polynomials). When $\nu = \nu_{\mathsf{M}}$ is the trivial valuation on the independent sets of a matroid M (Example 1), we recover the *homogeneous independent set generating polynomial*:

$$I_{\mathsf{M}}(w) = \sum_{S \in \mathrm{IN}(\mathsf{M})} w^S w_0^{n-|S|},$$

which in turn specializes to the usual independence polynomial of the matroid M.

Example 5 (Multivariate Tutte polynomials). When $\nu = \mathrm{rk}_{\mathsf{M}}$ is the rank function of a matroid M (Example 2), we obtain the *homogeneous multivariate Tutte polynomial* of the matroid:

$$T_{q,\mathsf{M}}(w) = \sum_{S \subseteq [n]} q^{-\mathrm{rk}_{\mathsf{M}}(S)} w^S w_0^{n-|S|},$$

as defined in [5, §3] and [21].

We prove the following characterization of $\mathrm{M}^\natural$-concave functions in terms of Lorentzian polynomials:

Theorem 4. *Let $\nu : 2^{[n]} \to \overline{\mathbb{R}}$ be an arbitrary set function on $[n]$. The following are equivalent:*

1. *ν is $M^\natural$-concave.*
2. *For any parameter $0 < q \leq 1$, the generating polynomial $Z_{q,\nu}$ is Lorentzian.*
3. *For any parameter $0 < q \leq 1$, the normalized generating polynomial $N(Z_{q,\nu})$ is Lorentzian.*

As a corollary we obtain theorem 3 as follows: the Lorentzian property is preserved by non-negative changes of variable, so the binary form

$$N(Z_{q,\nu})(x, y, \ldots, y) = \sum_{S \subseteq [n]} \frac{q^{-\nu(S)}}{(n-|S|)!} y^{|S|} x^{n-|S|} = \sum_{k=0}^{n} I_{q;k}(\nu) y^k x^{n-k}$$

is Lorentzian, and by corollary 2 the sequence $(I_{q;k}(\nu))_k$ is ultra log-concave.

There are further noteworthy remarks:

Remark 1. By Example 5, Theorem 4 not only proves Mason's strongest conjecture (M3) for matroids by means of the independence polynomial, but also recovers the Brändén and Huh's proof [3, Thm. 4.14] by means of the homogeneous multivariate Tutte polynomial.

Remark 2. The equivalence of (1) and (2) in Theorem 4 is a special case of [3, Thm. 3.14] that holds for the larger class of *M-concave functions*. The equivalence of (1) and (3) is new and special to $\mathrm{M}^\natural$-concave functions, for it fails for M-concave functions in general.

Remark 3. In [5, §5], Eur and Huh ask which functions ν belong to the class $\mathcal{L}_n$ of functions on $2^{[n]}$ whose homogeneous generating polynomial $Z_{q,\nu}$ is Lorentzian for any $0 < q \leq 1$. Theorem 4 shows that $\mathcal{L}_n$ is precisely the class of $\mathrm{M}^\natural$-concave functions on n elements.

4 Polynomial inequalities

For a matroid M on $[n]$, let

$$I_k(w) := \sum_{I \in \mathrm{IN}(\mathsf{M})_k} w^I \in \mathbb{Z}[w_1, \ldots, w_n] \qquad \text{for } 0 \le k \le \operatorname{rank}(\mathsf{M}). \tag{4.1}$$

For multivariate polynomials $f, g \in \mathbb{Z}[w_1, \ldots, w_n]$, we write $f \preceq g$ and $g \succeq f$ if all coefficients of their difference $g - f$ are nonnegative.

Igor Pak [17] asked whether polynomial version of Mason's conjecture (M1) holds. That is, for all $k \ge 0$, does the following polynomial inequality hold?

$$I_k(w)^2 \succeq I_{k-1}(w) I_{k+1}(w). \tag{PM1}$$

We affirmatively answer the question by deducing from Theorem 4 a stronger polynomial inequality for $\mathrm{M}^\natural$-concave functions. For an $\mathrm{M}^\natural$-concave function ν on $[n]$, define

$$I_{q;k}(w) = \sum_{S \in \binom{[n]}{k}} q^{-\nu(S)} w^S \in \mathbb{R}[w_1, \ldots, w_n].$$

Theorem 5. *Given an $\mathrm{M}^\natural$-concave function $\nu : 2^{[n]} \to \overline{\mathbb{R}}$, for all $0 \le i < j \le k < l \le n$ with $i + l = j + k$, we have that*

$$j! I_{q;j}(w) k! I_{q;k}(w) \succeq i! I_{q;i}(w) l! I_{q;l}(w) \quad \textit{for all } 0 < q \le 1.$$

for all $0 < q \le 1$. In particular, for all $0 < k < n$,

$$I_{q;k}(w)^2 \succeq \Big(1 + \frac{1}{k}\Big) I_{q;k-1}(w) I_{q;k+1}(w) \quad \textit{for all } 0 < q \le 1.$$

When ν is the $\mathrm{M}^\natural$-concave function ν_{M} of a matroid M described in Example 1, Theorem 5 states that the polynomial version (PM2) of Mason's stronger inequality (M2) holds, and in particular answers Dowling and Zhao's conjecture affirmatively. Moreover, in this case the first part of Theorem 5 admits a combinatorial interpretation described in Corollary 6 below.

Corollary 6. *Consider a subset $S \subseteq [n]$ of the ground set of a matroid. If $a, b \ge 0$ are integers with $a + b = |S|$, we let $N_S(a, b)$ be the number of partitions $S = A \sqcup B$ of S into ordered independent sets A and B of sizes a and b, respectively. For any integers $0 \le i < j \le k < l \le n$ with $|S| = i + l = j + k$, we have the inequality*

$$N_S(i, l) \le N_S(j, k).$$

Remark 4. Having proved the polynomial version of Mason's middle inequality (PM1), we note that the polynomial version of Mason's strongest inequality (M3) is false. For the uniform matroid of rank 2 on 2 elements $U_{2,2}$, we compute

$$\left(I_1(w)/\binom{2}{1}\right)^2 - I_0(w)/\binom{2}{0}\cdot I_2(w)/\binom{2}{2} = \frac{1}{4}(w_1^2+w_2^2-2w_1w_2),$$

so the polynomials $(I_k)_k$ are in general not coefficient-wise ultra log-concave.

Remark 5. Let M be a matroid on $[n]$. For any $\lambda \in \mathbb{R}^n_{\geq 0}$, since $I_{\mathsf{M}}(\lambda_1 x, \lambda_2 x, \cdots, \lambda_n x, y)$ is Lorentzian (see Example 4) and

$$I_{\mathsf{M}}(w) = \sum_{k=0}^{\mathrm{rank}(\mathsf{M})} I_k(w) w_0^{n-k},$$

we see that the sequences $(I_k(\lambda))_k$, $(k!\cdot I_k(\lambda))_k$ and $(I_k(\lambda)/\binom{n}{k})_k$ are log-concave for all λ.

This means that for $0<k<\mathrm{rank}(\mathsf{M})$ the polynomial

$$I_k(w)^2 - \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right) I_{k-1}(w) I_{k+1}(w) \tag{4.2}$$

is nonnegative everywhere on $\mathbb{R}^n_{\geq 0}$, although in general it contains monomials with negative coefficients (Remark 4 above). By Theorem 5, on the other hand,

$$I_k(w)^2 - \left(1+\frac{1}{k}\right) I_{k-1}(w) I_{k+1}(w)$$

has nonnegative coefficients.

5 Ultrametric trees

A central ingredient in the proof of both Theorem 4 is a result on tree distance matrices that we now describe. In their design of efficient address systems for communication networks, Graham and Pollak [7] introduced the *distance matrix* of a graph, whose *ij* entry is the distance between vertices *i* and *j*. They proved that the signature of this matrix gives a lower bound for the addresses in their design. In particular, they proved that the distance matrix of *any tree* has the Lorentzian signature $(+,-,\cdots,-)$. Our proofs of the matroid inequalities above rely on a refinement of their result in the case of ultrametric trees.

An *ultrametric tree* is a rooted tree whose n leaves are at the same distance from the root. Let D be the $n\times n$ *leaf distance matrix*, whose *ij* entry is the distance between leaves i and j. On one hand, Graham and Pollak's result implies that $-D$ has signature

$(+, \cdots, +, -)$. On the other hand, a theorem of Schoenberg [19, Theorem 1] implies that for any ultrametric tree T, the matrix $-D$ is *conditionally positive semi-definite*, meaning that for any vector $v = (v_1, \ldots, v_n)$ with $v_1 + \cdots + v_n = 0$, we have $-v^T Dv \geq 0$. In particular this implies that D has signature $(+, \cdots, +, -)$ because it has non-positive entries.

The previous classical results imply that one can make these matrices positive semi-definite after adding a multiple of the all-ones $n \times n$ matrix $\mathbf{1}_{n\times n}$. We prove that only a "small" perturbation is necessary. Concretely, we prove:

Theorem 7. *The following hold:*

1. *The matrix* $2(1 - \frac{1}{n})\mathbf{1}_{n\times n} - D$ *is positive semi-definite for any ultrametric tree with* n *leaves whose common distance from the root is* 1*.*
2. *The scalar* $2(1 - \frac{1}{n})$ *is the smallest number that makes the above statement true.*

Theorem 7 serves as the base case in our inductive argument for the proof of Theorem 4. This connection stems from the fact that every $\mathrm{M}^\natural$-concave function whose domain contains $\binom{[n]}{\leq 2}$ naturally gives rise to an ultrametric tree. This is a minor variation of the fact that the space of uniform valuated matroids of rank 2 is equal to the space of phylogenetic trees [11, Thm. 4.3.5].

We also characterize the trees for which the scalar $2(1 - \frac{1}{n})$ is optimal in the following sense. For real numbers $c \geq 0$, we consider the $n \times n$ matrix $A_c = c\mathbf{1}_{n\times n} - D$. There exists a smallest positive constant $c = \hat{c}_T \leq 2(1 - 1/n)$ for which the matrix A_c is positive semi-definite.

Example 6. Suppose S is the star tree with n leaves of radius 1. Then

$$(A_c)_{ij} = \begin{cases} c - 2 & i \neq j \\ c & i = j. \end{cases}$$

Note that, if $\mathbf{1}_n$ is the all-ones n-vector, then

$$\mathbf{1}_n^t A_c \mathbf{1}_n = cn^2 - 2n(n-1),$$

which is non-negative only if $c \geq 2(1 - \frac{1}{n})$. Therefore $\hat{c}_S = 2(1 - \frac{1}{n})$.

In fact, this turns out to be the only possibility.

Theorem 8 (Equality case of Theorem 7)**.** *Let* T *be an ultrametric tree. Then* $\hat{c}_T = 2(1 - \frac{1}{n})$ *if and only if* T *is a star tree.*

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