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Invariant theory for left regular bands

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Abstract. Left regular bands are a special family of semigroups which arise in algebraic combinatorics. In this abstract, we explore the representation theory of the *invariant subalgebras* of left regular band semigroup algebras carrying symmetries. A key role is played by the topology of partially ordered sets and regular CW complexes.

Keywords: Poset topology, monoids, semigroups, finite-dimensional algebras, group actions, regular CW complexes, CAT(0)-cube complexes, derangements

1 Introduction

A **left regular band**—or an **LRB**—is a finite semigroup B which satisfies (i) $x^2 = x$ and (ii) $xyx = xy$ for all $x, y \in B$. There are numerous combinatorial objects which carry an LRB structure, including matroids, real and complex hyperplane arrangements, CAT(0)-cube complexes, COMs, and more (see [6, 13]). Many of these objects have natural symmetry groups which act on the associated LRB by semigroup automorphisms.

For a field $\mathbf{k}$ and an LRB B, the **semigroup algebra** $\mathbf{k}B$ is the algebra with $\mathbf{k}$-basis B, and with multiplication extended $\mathbf{k}$-linearly. The representation theory of LRB semigroup algebras is well-studied. It was initially examined for its connections with the spectra of Markov chains [3, 4, 7, 6], but it has since become a topic of independent interest [1, 12, 13, 14, 16, 17]. One tool which has proved especially fruitful in this area is *combinatorial topology*, namely the topology of posets and of regular cell complexes, as developed Margolis–Saliola–Steinberg [12, 13].

If a finite group G acts on B by semigroup automorphisms, one can examine the **invariant subalgebra** $(\mathbf{k}B)^G$ of $\mathbf{k}B$, consisting of the elements fixed by the action of all $g \in G$. The invariant subalgebras of LRB algebras can have interesting combinatorics; most famously, Bidigare [4] proved that the invariant subalgebra of the LRB associated to a reflection arrangement is anti-isomorphic to the widely studied *Solomon's descent algebra.* Inspired by Margolis–Saliola–Steinberg, we explore the representation theory of the invariant subalgebras $(\mathbf{k}B)^G$ through the lens of group-equivariant combinatorial topology.

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Our main contributions include a combinatorial characterization of when $(\mathbf{k}B)^G$ is semisimple and commutative (Theorem 1) and topological formulas for the G-representation structure of certain subspaces of $\mathbf{k}B$ called *invariant Peirce components* (Theorem 3 and Theorem 4). The invariant Peirce components provide information about the *Cartan invariants* of $(\mathbf{k}B)^G$ as well as the structure of $\mathbf{k}B$ as a module over both G and $(\mathbf{k}B)^G$ (Proposition 1). We discuss applications of our results to CAT(0)-cube LRBs (Corollary 1) and to the *derangement symmetric function* of Désarménien–Wachs [10] (Corollary 2).

This is an extended abstract of an upcoming paper by the authors. Meanwhile, the details for most of these results can be found in the first author's dissertation [9].

2 Background

2.1 Two posets associated to LRBs

Each LRB B has two associated posets. The **semigroup poset** $(B, \leq)$ consists of the elements of B ordered by the relation $x \leq y$ if and only if $yx = x$. The **support semilattice** $(\Lambda(B), \leq)$ is the poset of **principal left ideals** $Bb := \{xb : x \in B\}$ as b varies in B. The intersection of two principal left ideals is again a principal left ideal, making the poset a meet-semilattice. The map $\sigma\colon B \to \Lambda(B)$ sending $b \mapsto Bb$ is called the **support map**.

For LRBs arising from combinatorial structures, the posets tend to reflect the combinatorics of the structures, as we will see in the upcoming examples. We note that if a finite group acts on an LRB by semigroup automorphisms, then it also induces an action on both the semigroup poset and the support semilattice by poset automorphisms.

2.2 Examples

2.2.1 Real, central hyperplane arrangement face monoids

Each hyperplane in a real, central arrangement partitions the ambient vector space into three subsets: two open half spaces and the hyperplane itself. The **faces** of the arrangement are the nonempty sets formed by intersecting a choice of one of these subsets over *all* hyperplanes. For example, the arrangement in Figure 1 (left) has thirteen faces.

Tits [20] considered a multiplication operation on the faces which satisfies the LRB axioms. Geometrically, the product $f \cdot g$ of faces f and g is the face entered upon taking an "epsilon" step from any point on f to any point on g (see left picture in Figure 1). The central face is forms an identity element, so the faces actually form a *monoid*.

The semigroup poset of the face monoid $\mathcal{F}(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$ is isomorphic to the face poset of its **zonotope**—a particular polytope "dual" to $\mathcal{A}$. The support semilattice $\Lambda(\mathcal{F}(\mathcal{A}))$ is isomorphic to the **lattice of intersections** of $\mathcal{A}$, consisting of the intersections of hyperplanes, ordered under reverse inclusion (see Figure 1).

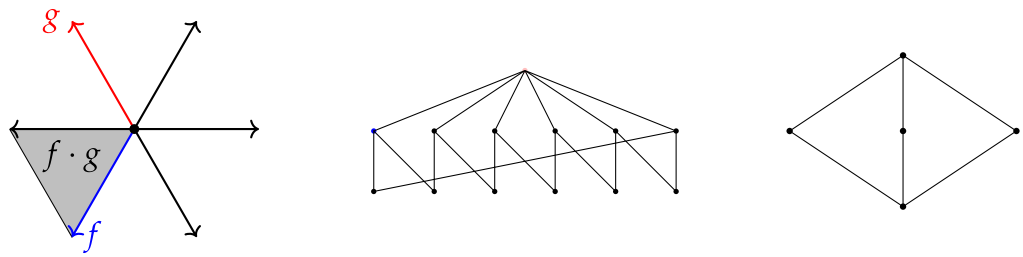

Figure 1: A hyperplane face monoid (left), its semigroup poset (middle) which is the face poset of its zonotope (a filled in hexagon), and its support semilattice (right).

Finite groups of linear transformations preserving the arrangement—i.e. reflection groups acting on their arrangements—act on the associated LRB by automorphisms.

2.2.2 CAT(0)-cube complex LRBs

A simplicial complex is a **flag complex** if whenever the one-skeleton of a simplex belongs to the complex, so does the whole simplex. The **link** of a vertex in a cube complex is the simplicial complex on the edges incident to the vertex, where a set of edges form a simplex if they belong to a common cube. A CAT(0)**-cube complex** is a simply-connected, regular cube complex for which the link of each of its vertices is a flag complex.

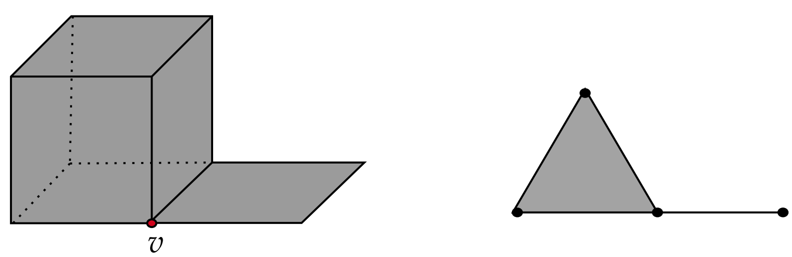

Figure 2: A CAT(0)-cube complex and the link of the vertex labeled v.

Independently, Margolis–Saliola–Steinberg [13] and Bandelt–Chepoi–Knauer [2] proved there is an LRB structure on the faces $\mathcal{F}(\mathcal{C})$ of any finite CAT(0)-cube complex $\mathcal{C}$. Geometrically, the product $f \cdot g$ for faces f and g is the subface of f induced by the vertices of f closest to g in the path metric of the 1-skeleton of $\mathcal{C}$ (see Figure 3).

The semigroup poset $\mathcal{F}(\mathcal{C})$ is isomorphic to the face poset of $\mathcal{C}$ where we do *not* include an empty face. CAT(0)-cube complexes have a notion of hyperplanes (see Figure 4 for the idea, and [13, §3.5] for details). The support semilattice $\Lambda(\mathcal{F}(\mathcal{C}))$ is isomorphic to the semilattice of intersections of these hyperplanes ordered under reverse inclusion.

Let $\mathcal{C}$ be a finite CAT(0)-cube complex. We say that a finite group G acts on $\mathcal{C}$ by an **LRB action** if it acts by isometries, permutes the cubes of $\mathcal{C}$, and if *all* of G fixes a face c

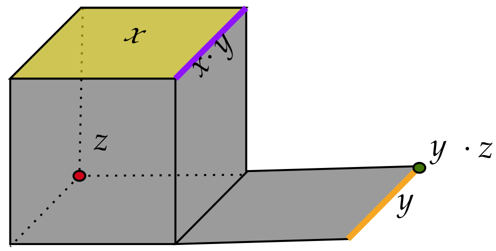

Figure 3: Example of multiplication in a CAT(0)-cube complex

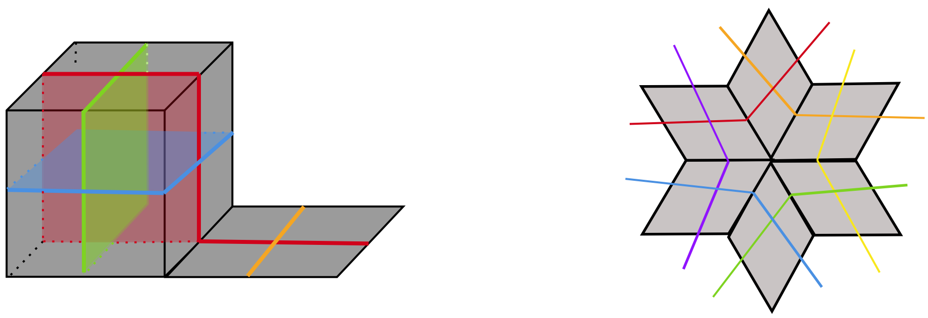

Figure 4: The hyperplanes of two CAT(0)-cube complexes

of $\mathcal{C}$ setwise, then all of G must fix c pointwise.[1] We show that if G acts by an LRB action on $\mathcal{C}$, it acts on $\mathcal{F}(\mathcal{C})$ by semigroup automorphisms.

Example 1 (Cubulated n-gon)**.** Let $n \geq 4$. Consider the CAT(0)-cube complex formed from a regular n-gon by adding a vertex to the center and adding edges from the central vertex to the midpoints of each of the edges of the n-gon, which we now also consider vertices. The resulting cube complex $\mathfrak{C}_n$ has $2n+1$ vertices, $3n$ edges, n squares, and n hyperplanes. See Figure 4 (right) for $\mathfrak{C}_6$. The natural action of the dihedral group D_{2n} of order $2n$ is an LRB action.

One very interesting and well-studied CAT(0)-cube complex is the *space of phylogenetic n-trees*, which carries a natural LRB action by the symmetric group. We do not have the space to explain this LRB here, but we refer the curious reader to [9] for the details.

2.2.3 Geometric lattice LRBs

Let $\mathcal{L}$ be a finite geometric lattice. In [6, §6.2], Brown defined an associated LRB[2] $S(\mathcal{L})$, which is a special case of the Karnofsky–Rhodes expansion of a lattice (see [12, §3.4]).

[1] This is weaker than the standard condition that if $g \in G$ fixes c setwise, then g fixes c pointwise.

[2] Brown defined two LRBs associated to *matroids*. We use the one he calls $\overline{S}$, which only depends on the lattice of flats.

The elements of $S(\mathcal{L})$ are prefixes of maximal chains of $\mathcal{L}$. Let $\lessdot$ denote a covering relation in $\mathcal{L}$ and write $\vee$ to indicate join. The multiplication in $S(\mathcal{L})$ is given by

$$(\hat{0} \lessdot x_1 \lessdot x_2 \lessdot \cdots \lessdot x_k)\cdot(\hat{0} \lessdot y_1 \lessdot y_2 \lessdot \cdots \lessdot y_\ell) \\ = (\hat{0} \lessdot x_1 \lessdot x_2 \lessdot \cdots \lessdot x_k \leq y_1 \vee x_k \leq y_2 \vee x_k \leq \cdots \leq y_\ell \vee x_k)^\wedge,$$

where $\wedge$ indicates removing repeated elements, as one reads from left to right. The element $(\hat{0})$ is an identity. If $\mathcal{L}$ is the Boolean lattice, $S(\mathcal{L})$ is the well-studied **free LRB**.

Within the semigroup poset $S(\mathcal{L})$, two chains x, y satisfy $x \leq y$ precisely if y is a *prefix* of x; this poset is always a tree. The support semilattice $\Lambda(S(\mathcal{L}))$ is isomorphic to $\mathcal{L}^{\mathrm{opp}}$.

Groups acting on $\mathcal{L}$ by poset automorphisms act on $S(\mathcal{L})$ by monoid automorphisms.

2.3 Classes of LRBs

For an LRB B and a support $X \in \Lambda(B)$, the **contraction** of the semigroup poset to X is $B_{\geq X} := \{b \in B : \sigma(b) \geq X\}$. An LRB is called **connected** if the Hasse diagrams of all its contractions are connected as graphs. For any field $\mathbf{k}$, Margolis–Saliola–Steinberg [13, Theorem 4.15] showed the semigroup algebra $\mathbf{k}B$ has a multiplicative identity if and only if B is connected. All of our examples so far are connected LRBs, even though CAT(0)-cube complex LRBs rarely have an identity!

A **CW LRB** is an LRB for which each of its contractions is the face poset of a *regular CW complex*. Face monoids of real hyperplane arrangements and the face semigroups of CAT(0)-cube complexes are both CW LRBs. Other examples include LRBs arising from oriented matroids, affine and complex hyperplane arrangements, and COMs.

We call an LRB **k-hereditary** if $\mathbf{k}B$ is a hereditary algebra, meaning each of its left ideals are projective. Margolis–Saliola–Steinberg proved [13, Corollary 5.24] that if the semigroup poset of an LRB is a tree, then for any $\mathbf{k}$, B is $\mathbf{k}$-hereditary, generalizing a result of Brown for the free LRB [14, Theorem 8.1]. The LRBs associated to geometric lattices are thus $\mathbf{k}$-hereditary, as are the Karnofsky-Rhodes expansions of lattices.

3 Conventions

Henceforth, let B be a connected LRB carrying an action by a finite group G by semigroup automorphisms, and let $\mathbf{k}$ be a field with char $\mathbf{k} \nmid |G|$.

4 When is $(\mathbf{k}B)^G$ semisimple or commutative?

We combinatorially characterize when $(\mathbf{k}B)^G$ is semisimple or commutative.

Theorem 1. *The following are equivalent:*

- *The invariant subalgebra $(\mathbf{k}B)^G$ is semisimple.*
- *The number of G-orbits of the two posets are equal: $|\Lambda(B)/G| = |B/G|$.*
- *The invariant subalgebra $(\mathbf{k}B)^G$ is commutative.*

The proof of this theorem is quite straightforward, so we sketch the idea here. Using the well-known fact that $\sigma\colon \mathbf{k}B \to \mathbf{k}\Lambda(B)$ is a G-equivariant surjective map with $\ker\sigma$ the radical $\mathrm{rad}(\mathbf{k}B)$ of $\mathbf{k}B$, we show $(\mathbf{k}B)^G/\mathrm{rad}((\mathbf{k}B)^G) \cong (\mathbf{k}\Lambda(B))^G \cong \mathbf{k}^{\Lambda(B)/G}$, which gives the equivalence of the first two statements and their implication of the third. We then show the third implies the first using the Peirce decomposition of the algebra.

Example 2. *The reader should verify that $|\Lambda(\mathfrak{C}_n)/D_{2n}| = 3 \neq 6 = |\mathfrak{C}_n/D_{2n}|$, and hence the invariant subalgebra $(\mathbf{k}\mathfrak{C}_n)^{D_{2n}}$ is nonsemisimple and noncommutative.*

Theorem 1 also provides a simple explanation for why Solomon's descent algebra is noncommutative and non-semisimple in the majority of Coxeter types. Additionally, it gives a straightforward reason for why the S_n-invariant subalgebra of the free LRB algebra *is* commutative and semisimple, as was shown in [5, Theorem 1.3].

5 Representation theory of $(\mathbf{k}B)^G$ and Peirce components

By imitating the idempotent constructions in [13, 15], we explain the structure of the *complete families of primitive, orthogonal idempotents* of $(\mathbf{k}B)^G$. The members of any such family are indexed by the G-orbits of supports $[X] \in \Lambda(B)/G$, and so we denote such a family by $\{E_{[X]} : [X] \in \Lambda(B)/G\}$. The idempotents determine the structure of the simple modules $M_{[X]}$ and projective indecomposable modules $P_{[X]}$ of $(\mathbf{k}B)^G$, which are also indexed by the G-orbits $\Lambda(B)/G$. In particular,

$$P_{[X]} := (\mathbf{k}B)^G \cdot E_{[X]} \text{ and } M_{[X]} := \mathrm{top}(P_{[X]}) = \Big((\mathbf{k}B)^G \cdot E_{[X]}\Big) \,/\, \Big(\mathrm{rad}((\mathbf{k}B)^G) \cdot E_{[X]}\Big).$$

The **Cartan invariants** of $(\mathbf{k}B)^G$ are the Jordan-Hölder **composition multiplicities** $[P_{[X]} : M_{[Y]}]$ of the simples $M_{[Y]}$ within the projective indecomposables $P_{[X]}$, as $[X], [Y]$ vary in $\Lambda(B)/G$; these are an important statistic for finite dimensional algebras. In order to understand the Cartan invariants of $(\mathbf{k}B)^G$, we consider a particular G-representation decomposition of $\mathbf{k}B$ which we call its **invariant Peirce decomposition**:

$$\mathbf{k}B = \bigoplus_{[X],[Y] \in \Lambda(B)/G} E_{[Y]} \cdot \mathbf{k}B \cdot E_{[X]}. \tag{5.1}$$

We call each summand in Equation (5.1) an **invariant Peirce component** of $\mathbf{k}B$. One easy consequence of our idempotent study is that the component $E_{[Y]} \cdot \mathbf{k}B \cdot E_{[X]}$ vanishes unless there exists $Y' \in [Y]$ and $X' \in [X]$ for which $X' \leq Y'$ in $\Lambda(B)$.

We are particularly interested in understanding the *G-representation structure* of each invariant Peirce component. Our motivation comes from wanting to understand the Cartan invariants of $(\mathbf{k}B)^G$, as well as the structure of $\mathbf{k}B$ as a module over G and $(\mathbf{k}B)^G$. This relationship is summarized in the following proposition.

Proposition 1. *Assume further that* $\mathbf{k}$ *is algebraically closed. Then,*

- *The Cartan invariants of* $(\mathbf{k}B)^G$ *are* $[P_{[X]} : M_{[Y]}] = \left\langle \mathbb{1}, E_{[Y]} \cdot \mathbf{k}B \cdot E_{[X]} \right\rangle$.
- *More generally, let* χ *be an irreducible character of* G. *As a* $(\mathbf{k}B)^G$*-module, the* χ*-isotypic component* $(\mathbf{k}B)^\chi$ *decomposes as* $(\mathbf{k}B)^\chi = \bigoplus (\mathbf{k}B)^\chi \cdot E_{[X]}$ *for* $[X] \in \Lambda(B)/G$ *and the multiplicity of the composition factor of* $M_{[Y]}$ *in each* $(\mathbf{k}B)^G$*-submodule* $(\mathbf{k}B)^\chi \cdot E_{[X]}$ *is*
$$[(\mathbf{k}B)^\chi \cdot E_{[X]} : M_{[Y]}] = (\dim \chi) \left\langle \chi,\ E_{[Y]} \cdot \mathbf{k}B \cdot E_{[X]} \right\rangle .$$

Our formulas for the G-representation structure of invariant Peirce components will be in terms of **poset cohomology**. For a poset P, the reduced cohomology groups $\widetilde{H}^*(P; \mathbf{k})$ are the cohomology groups of its **order complex** $\Delta(P)$ (see [21]). When a group G acts on a poset P by poset automorphisms, each $\widetilde{H}^*(P; \mathbf{k})$ is a G-representation.

6 Invariant Peirce components for CW LRBs

6.1 Motivation: real, central hyperplane arrangement face monoids

In [8], the first author studied the invariant Peirce components for the LRB associated to the braid arrangement. A crucial tool in her work was the following translation into poset cohomology, following from work of Saliola [15] and Aguiar–Mahajan [1, Proposition 14.44].

Let $\mathcal{F}$ be the face monoid of a real, central arrangement and let G be a finite subgroup of the general linear group preserving the arrangement. Write G_X for the setwise stabilizer of $X \in \Lambda(B)$. Recall that the supports in $\Lambda(\mathcal{F})$ can be identified with the intersections of hyperplanes. We write $\det(X)$ for the one-dimensional G_X-character mapping g to the determinant of its action on the intersection corresponding to X.

Theorem 2 (Saliola, Aguiar–Mahajan)**.** *Let* $X \leq Y$ *in* $\Lambda(\mathcal{F})$*, and let* (X, Y) *be the subposet given by its open interval. Set* $k = \mathrm{rk}(Y) - \mathrm{rk}(X)$*. As* G*-representations,*
$$E_{[Y]} \cdot \mathbf{k}\mathcal{F} \cdot E_{[X]} \cong \bigoplus_{[X' \leq Y']} \det(X') \otimes \widetilde{H}^{k-2}((X', Y'); \mathbf{k}) \otimes \det(Y') \Big\uparrow_{G_{X'} \cap G_{Y'}}^{G},$$
where the direct sum is over G*-orbits of pairs* $[X' \leq Y']$ *satisfying* $X' \in [X]$ *and* $Y' \in [Y]$.

Theorem 2 was especially powerful as the support semilattice for the braid arrangement LRB is the *set partition lattice*, whose topology is well-studied, i.e. [18, 19, 21].

6.2 Generalization to CW LRBs

Assume B is a CW LRB. Our aim is to generalize Theorem 2 to understand the invariant Peirce components of $\mathbf{k}B$. One step in doing so involves coming up with the correct analogue of the twisting $\det(X)$ character.

Let $y \in B$ with $\sigma(y) = Y$, and define the subposet $B^{<y} := \{x \in B : x < y\}$. There is an unusual, but well-defined, action of G_Y on $B^{<y}$, which we call the $*$-action, given by

$$g * x := y \cdot g(x),$$

with $g(x)$ the ordinary action of G. We write $\widetilde{B^{<y}}$ to indicate we are viewing $B^{<y}$ under this special $*$-action of G_Y. Since B is the face poset of a regular CW complex, the order complex $\Delta(B^{<y})$ is a sphere. The twisting character we need turns out to be the degree character of the $*$-action of G_Y on the sphere $\Delta(\widetilde{B^{<y}})$:

Lemma 1 (Definition of $\deg(Y)$)**.** *Let $Y \in \Lambda(B)$ and fix $y \in B$ with $\sigma(y) = Y$. There is a well-defined G_Y-representation (meaning it is independent of our choice of y) given by*

$$\deg(Y) := \widetilde{H}^{\mathrm{rk}(y)-2}(\widetilde{B^{<y}}).$$

Our generalization of Theorem 2 is as follows:

Theorem 3. *Let B be a connected CW LRB. Consider G-orbits $[X], [Y]$ in $\Lambda(B)/G$ and set $k := \mathrm{rk}(Y) - \mathrm{rk}(X)$. As G-representations,*

$$E_{[Y]} \cdot \mathbf{k}B \cdot E_{[X]} \cong \bigoplus_{[X' \leq Y']} \left[\deg(X') \otimes \widetilde{H}^{k-2}\left((X', Y'); \mathbf{k}\right) \otimes \deg(Y')\right] \Big\uparrow_{G_{X'} \cap G_{Y'}}^{G},$$

where the direct sum is over G-orbits of pairs $[X' \leq Y']$ satisfying $X' \in [X]$ and $Y' \in [Y]$.

The proof of Theorem 3 is a bit too involved to sketch here, but two key tools are cellular homology and the theory of CW LRB semigroup algebras developed in [13].

6.3 Application of Theorem 3: $\mathrm{CAT}(0)$-cube complexes

By applying Theorem 3 to $\mathrm{CAT}(0)$-cube LRBs carrying LRB actions, we obtain the interesting corollary that their invariant Peirce components are *permutation representations*. This supplies an especially explicit method for computing their Cartan invariants:

Corollary 1. *Let G act on a finite $\mathrm{CAT}(0)$-cube complex $\mathcal{C}$ by an LRB action. Then, for G-orbits $[X]$, $[Y]$ in $\Lambda(\mathcal{F}(\mathcal{C}))/G$, as G-representations*

$$E_{[Y]} \cdot \mathbf{k}\mathcal{F}(\mathcal{C}) \cdot E_{[X]} \cong \mathrm{span}_{\mathbf{k}}\{X' \leq Y' : X' \in [X], Y' \in [Y]\}.$$

Hence, the Cartan invariants are

$$\left[P_{[X]} : M_{[Y]}\right] = \#\{G\text{-orbits } [X' \leq Y'] \ : X' \in [X], Y' \in [Y] \ \}.$$

For CAT(0)-cube LRBs, the order complexes of the open intervals $\Lambda(\mathcal{F}(\mathcal{C}))$ turn out to be spheres of the appropriate dimension, so Theorem 3 reduces Corollary 1 to showing that $\deg(X') \otimes \widetilde{H}^{k-2}((X',Y');\mathbf{k}) \otimes \deg(Y')$ is the trivial representation.

Example 3 (Cartan invariants of $(\mathbf{k}\mathfrak{C}_n)^{D_{2n}}$ and $(\mathbf{k}\mathfrak{C}_n)^{C_n}$)**.** Let C_n be the cyclic subgroup of D_{2n} consisting of the rotations. For both $G = C_n, D_{2n}$, there are three orbits in $\Lambda(\mathfrak{C}_n)/G$: the set $[\varnothing]$ containing the single empty intersection, the set $[h]$ containing each individual hyperplane, and the set $[i]$ containing all the nonempty intersections between two hyperplanes. Using Corollary 1, the Cartan matrices for both algebras are in Figure 5.

	$P_{[\varnothing]}$	$P_{[h]}$	$P_{[i]}$
$M_{[\varnothing]}$	1	0	0
$M_{[h]}$	1	1	0
$M_{[i]}$	1	1	1

	$P_{[\varnothing]}$	$P_{[h]}$	$P_{[i]}$
$M_{[\varnothing]}$	1	0	0
$M_{[h]}$	1	1	0
$M_{[i]}$	1	2	1

Figure 5: Cartan matrix for $(\mathbf{k}\mathfrak{C}_n)^{D_{2n}}$ (left) and $(\mathbf{k}\mathfrak{C}_n)^{C_n}$ (right)

7 Hereditary LRBs and derangements

7.1 Our general result: LRBs with hereditary semigroup algebras

Unfortunately, dimension-counting reveals that similar analogues of Theorem 3 do not hold for general LRBs. However, for $\mathbf{k}$-hereditary LRBs, we still manage to obtain a formula for the invariant Peirce components in terms of poset topology.

Theorem 4. *Let B be a $\mathbf{k}$-hereditary LRB and let $[X],[Y] \in \Lambda(B)/G$ with $[X] < [Y]$. For each support Z with $[X] < [Z] \leq [Y]$, fix an element b_Z with $\sigma(b_Z) = Z$. As G-representations,*

$$E_{[Y]} \cdot \mathbf{k}B \cdot E_{[X]} \cong \bigoplus_{\substack{m \geq 1, \\ [X_0 < X_1 < \cdots < X_m]: \\ X_0 \in [X], X_m \in [Y]}} \left(\bigotimes_{i=1}^{m} \widetilde{H}^0 \left(\widetilde{B^{<b_{X_i}}_{\geq X_{i-1}}} \right) \right) \Bigg\uparrow_{\bigcap_i G_{X_i}}^{G}.$$

Here, $\widetilde{B^{<b_{X_i}}_{\geq X_{i-1}}}$ refers to the $*$-action from Section 6.2.

We prove Theorem 4 with a homological argument, making use of the structure of the Ext-spaces between simple $\mathbf{k}B$-modules discovered by Margolis–Saliola–Steinberg [13].

7.2 Application of Theorem 4: Derangement representations

7.2.1 Ordinary derangement numbers

A permutation in the symmetric group S_n is a **derangement** if it has no fixed points. The number of derangements of S_n, written d_n, is usually counted by Möbius inversion:

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k} \xrightarrow[\text{Möbius inversion}]{} d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!. \tag{7.1}$$

One drawback of this formula is the negative signs. A seemingly lesser-known, but manifestly positive formula for derangements can be deduced from work of Gessel [11, Eqn (6)] (who gives a bijective proof) or Brown [6, inductively apply Prop. 10] (for an inductive proof). Here, $\alpha \vDash n$ denotes an integer **composition** $(\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$ summing to n with each $\alpha_i > 0$ and $\binom{n}{\alpha}$ is the **multinomial coefficient** $\frac{n!}{\prod_i \alpha_i!}$.

$$d_n = \sum_{\alpha \vDash n} \binom{n}{\alpha} \prod_{i=1}^{\ell(\alpha)} (\alpha_i - 1). \tag{7.2}$$

7.2.2 The derangement symmetric function

The **derangement symmetric function** is a well-studied symmetric function introduced by Desarménien and Wachs in [10]. Using h for the complete homogeneous symmetric functions and e for the elementary symmetric functions, the derangement symmetric function is the following categorification of Equation (7.1):

$$\mathfrak{d}_n := \sum_{k=0}^{n} (-1)^k e_k \cdot h_{1^{n-k}}. \tag{7.3}$$

The symmetric function $\mathfrak{d}_n$ and its corresponding S_n-representation (the **derangement representation**) have numerous interesting reformulations and connections (see [5, §3C] and the references therein). Relevantly, [5, Proposition 4.2] can be used to show that for the free LRB $S(B_n)$, where B_n the Boolean lattice on n letters, the "maximal" invariant Peirce component $E_{\{1,2,\ldots,n\}} \cdot \mathbf{k}S(B_n) \cdot E_{\varnothing}$ carries the derangement representation.

7.2.3 Generalized derangement representations and categorification of Equation (7.2)

For any poset P, Brown [6, Appendix C] introduced and studied a **generalized derangement number** d_P, which satisfies enumerative formulas related to Equation (7.1) and Equation (7.2) in [6, Equation (42) and Proposition 10]. In the case that P is a geometric lattice $\mathcal{L}$, Brown's definition of $d_{\mathcal{L}}$ can be translated in terms of the "maximal" invariant Peirce decompositions, namely

$$d_{\mathcal{L}} := \dim_{\mathbf{k}} E_{\hat{0}_{\mathcal{L}}} \cdot \mathbf{k}S(\mathcal{L}) \cdot E_{\hat{1}_{\mathcal{L}}}.$$

This inspires a natural generalization of the Desarménien–Wachs derangement representation for any geometric lattice $\mathcal{L}$ with a group G acting by poset automorphisms. Specifically, we define the generalized derangement representation $\mathrm{Der}(\mathcal{L})$ to be

$$\mathrm{Der}(\mathcal{L}) := E_{\hat{0}_{\mathcal{L}}} \cdot \mathbf{k}S(\mathcal{L}) \cdot E_{\hat{1}_{\mathcal{L}}}.$$

By applying our Theorem 4 to geometric lattices, we obtain the following:

Theorem 5. *Let $\mathcal{L}$ be a geometric lattice. As G-representations,*

$$\mathrm{Der}(\mathcal{L}) \cong \bigoplus_{\substack{m \geq 1, \\ [\hat{1}_{\mathcal{L}} = X_0 > \cdots > X_m = \hat{0}_{\mathcal{L}}]}} \left(\bigotimes_{i=0}^{m-1} V_{X_i, X_{i+1}} \right) \Bigg\uparrow_{\bigcap_i G_{X_i}}^{G}, \tag{7.4}$$

where $V_{X,Y}$ is the true $G_X \cap G_Y$-representation

$$V_{X,Y} := \mathrm{span}_{\mathbf{k}}\{Z \in \mathcal{L} : Y \lessdot Z \leq X\} - \mathbb{1}.$$

We point out that Theorem 5 categorifies and generalizes Equation (7.2). Namely, applying it to $\mathcal{L} = B_n$, we obtain the following formula for $\mathfrak{d}_n$, which, when taking the dimensions of the corresponding S_n-representations, recovers Equation (7.4).

Corollary 2. *Using s for the Schur symmetric function, and using the convention that $s_{(0,1)} = 0$,*

$$\mathfrak{d}_n = \sum_{\alpha \vDash n} \prod_{i=1}^{\ell(\alpha)} s_{(\alpha_i - 1, 1)}.$$

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