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Skew-linked Catalan functions and Genocchi numbers

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Abstract. The discovery that k-Schur functions satisfy a striking shift invariance property and arise as a subfamily of Catalan functions resolved long-standing mysteries in the theory. These Catalan functions are symmetric functions realized as GL_ℓ-equivariant Euler characteristics on the flag variety. This suggests that the phenomena surrounding k-Schur functions might reflect a more general mechanism inside the Catalan-function framework. Indeed, we identify a broad class of *steep* Catalan functions and show that all these elements satisfy shift invariance. Moreover, the family of steep Catalan functions beautifully partitions into subclasses indexed by distinguished skew diagrams called skew-linked diagrams. We further isolate a notable subclass of steep Catalan functions that includes all k- and $(k-1)$-Schur functions and prove that they satisfy a powerful factorization property, enabling their construction from a Genocchi-number–sized subfamily. We investigate the combinatorics of Genocchi numbers and root ideals, establishing a new bijection between a natural family of root ideals and the Genocchi numbers.

Keywords: Catalan functions, k-Schur, Genocchi numbers

1 Introduction

The family of symmetric functions known as *k-Schur functions* first emerged in the study of Macdonald polynomials [11], where they appeared alongside striking combinatorial and positivity conjectures. A specialization was later shown to play a central role in quantum and affine Schubert calculus [9, 13, 14], revealing unexpected geometric depth. Yet progress on the general theory was slow, largely because the original characterization offered no mechanism for proof. A major advance came with the discovery that k-Schur functions arise naturally within a more flexible family of symmetric functions called Catalan functions, introduced by Chen and Haiman [5] and further developed by Panyushev [16]. This perspective revealed that k-Schur functions satisfy a highly structure *shift invariance* property, enabling proofs of long-standing conjectures and opening the door to new ideas.

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k-Schur functions form only a small part of a broader family of Catalan functions indexed by special skew shapes called *skew-linked diagrams*. These functions are conjectured to have a natural module-theoretic interpretation: each skew-linked Catalan function should arise as the Frobenius series of an explicit graded module over a polynomial ring, paralleling the classical Garsia-Procesi construction.

Our main contribution is to show that skew-linked Catalan functions admit a more conceptual description as equivalence classes within a broad family of steep Catalan functions. We prove that every member of this larger family satisfies shift invariance, extending a key property beyond its previously known scope. In addition, we isolate a particularly rich subclass of steep Catalan functions that contains both k- and $(k-1)$-Schur functions, and we establish stronger results for this class.

Catalan functions are defined using a Demazure operator formula and can be interpreted as GL_ℓ-equivariant Euler characteristics of vector bundles on the flag variety. They live in the ring

$$\Lambda = \mathbb{Z}[t][e_1, e_2, \dots] = \mathbb{Z}[t][h_1, h_2, \dots]$$

of symmetric functions in infinitely many variables $\mathbf{x} = (x_1, x_2, \dots)$, where e_d and h_d denote the usual elementary and complete homogeneous symmetric functions.

For our purposes, we define Catalan functions as follows. Fix a positive integer ℓ. A *root ideal* is an upper order ideal Ψ of the poset

$$\Delta_\ell^+ = \{(i,j) \mid 1 \le i < j \le \ell\}, \qquad (a,b) \le (c,d) \text{ if and only if } a \ge c \text{ and } b \le d,$$

and an *indexed root ideal* is a pair (Ψ, α) with $\alpha \in \mathbb{Z}^\ell$. For any weight α, set

$$s_\alpha(x) = \det(h_{\alpha_i + j - i}(x))_{1 \le i,j \le \ell}, \qquad h_0 = 1,\ h_d = 0 \text{ for } d < 0.$$

The *Catalan function* associated to (Ψ, α) is

$$H(\Psi;\alpha)(x;t) = \prod_{(i,j)\in\Psi} (1 - tR_{ij})^{-1} s_\alpha(x),$$

where $R_{ij} s_\alpha = s_{\alpha+\varepsilon_i-\varepsilon_j}$. In the extreme case, when Ψ consists of all positive roots and α is a partition, Catalan functions recover the Hall–Littlewood polynomials. More generally, they form a vast family of symmetric functions now known to be Schur positive for any root ideal and partition weight. Establishing this positivity spanned decades and led to many developments, e.g. [4, 8, 17, 18, 19, 20].

One direction led Chen and Haiman to a subfamily indexed by *skew-linked diagrams*. From each such diagram D, they constructed a root ideal $CH(D)$ and weight $\lambda = \mathrm{row}(D)$ [5]. They conjectured that when D is a k-skew diagram of [12], the functions $H(CH(D);\lambda)$ coincide with the k-Schur functions and arise as graded characters of simple modules. Progress came with the discovery in [3] that this subclass can be described by canonical root ideals $\Delta^k(\lambda)$ indexed by k-bounded partitions, which are simple to construct. This description revealed a *shift invariance* property, key to proving that these Catalan functions recover the k-Schur functions, and enabled proofs of conjectures such as the *k-branching conjecture*.

Motivated by these developments, we study the full collection of skew-linked Catalan functions. We show that a broad family of *steep Catalan functions* decomposes into subclasses indexed by skew-linked diagrams and recovers all skew-linked Catalan functions. By selecting a canonical root ideal in each subclass, we prove that every steep Catalan function satisfies shift invariance.

A particularly rich structure arises upon restricting to the subclass containing all k- and $(k-1)$-Schur functions. These are indexed by *k-shapes* [10]. For each k-shape γ, we construct a canonical *k-shape root ideal* $\Psi^k(\gamma)$, establish shift invariance, and prove a factorization property at $t = 1$ analogous to the k-rectangle factorization. Thus the family of k-shape Catalan functions is generated by finitely many primitive functions. We prove that the number of such functions is the Genocchi number [15] $G_{2k} = (1, 1, 3, 17, 155, \dots)$ by constructing a bijection between a canonical subfamily of root ideals and surjective pistols, building on work of Hivert and Mallet [7] and Bigeni [1].

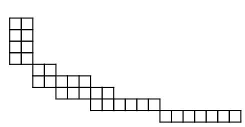
2 Skew-linked and k-Schur Catalan functions

The *row shape* of a skew diagram $D = \kappa/\eta$ is the vector of its row-lengths

$$(\kappa_1 - \eta_1, \dots, \kappa_\ell - \eta_\ell).$$

Similarly, the *column shape* of D is the vector of its column-lengths, i.e., the row-lengths of the transpose diagram $(\kappa/\eta)'$. Since D is uniquely determined by its row shape λ and column shape μ, we also write $D = (\lambda, \mu)$. A diagram $D = (\lambda, \mu)$ is said to be *skew-linked* if both λ and μ are partitions. Note that the outer-shape κ of a skew-linked diagram is uniquely determined by its row shape and column shape.

Example 2.0.1. The skew-linked diagram with row shape $\lambda = (7, 6, 5, 5, 2, 2, 2, 2, 2, 2)$ and column shape $\mu = (4, 4, 2, 2, 2, 2, 2, 2, 1^{11})$ is given by

$D =$  $\kappa = (20, 13, 9, 7, 4, 2, 2, 2, 2)$.

Chen and Haiman [5] identify a class of Catalan functions associated to a distinguished subset of skew diagrams.

Definition 2.0.2. For any skew-linked diagram $D = \kappa/\eta = (\lambda, \mu)$, define the root ideal

$$\mathrm{CH}(D) := \big\{ (i,j) \in \Delta^+_{\ell(\lambda)} \;\big|\; i \le \ell(\eta) \text{ and } j \ge \mu_{\eta_i} + i \big\}.$$

The associated *skew-linked Catalan function* is $\mathfrak{s}_D(x;t) := H\big(\mathrm{CH}(D);\ \lambda\big)(x;t)$.

In [3], k-Schur functions were identified with the following class of Catalan functions.

Definition 2.0.3. Let $\mathrm{Par}_\ell^k = \{(\mu_1, \ldots, \mu_\ell) \in \mathbb{Z}^\ell \colon k \geq \mu_1 \geq \cdots \geq \mu_\ell \geq 0\}$ denote the set of partitions contained in the $\ell \times k$-rectangle. For $\lambda \in \mathrm{Par}_\ell^k$, define the *k-shape root ideal*

$$\Delta^k(\lambda) := \{\, (i,j) \in \Delta_\ell^+ \mid k - \lambda_i + i < j\},$$

and the associated *k-Schur Catalan function*

$$s_\lambda^{(k)}(x;t) := H\big(\Delta^k(\lambda);\ \lambda\big)(x;t) = \prod_{i=1}^{\ell} \prod_{j=k+1-\lambda_i+i}^{\ell} (1 - tR_{ij})^{-1}\, s_\lambda(x).$$

A major feature of this definition is that the root ideal characterization immediately exposes the *shift invariance* property [2, 3].

For $f \in \Lambda$, let $f^\perp$ be the linear operator adjoint to multiplication by f under the Hall inner product: $\langle f^\perp(g), h\rangle = \langle g, f\,h\rangle$ for all $g, h \in \Lambda$.

Proposition 2.0.4 (Shift Invariance for k-Schur Catalan functions)**.** *For $\lambda \in \mathrm{Par}_\ell^k$, we have*

$$e_\ell^\perp\, s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}.$$

3 Steep Catalan functions

We study a broad family of Catalan functions defined by *steep root ideals*. A root ideal Ψ is steep if every row containing a root has strictly more roots than the row below it. Let $\mathfrak{R}$ denote the set of all steep root ideals. In particular, for any skew-linked shape D, the associated root ideal $\mathrm{CH}(D)$ lies in $\mathfrak{R}$. For $\Psi \in \mathfrak{R}$ and a partition λ, the Catalan function $H(\Psi;\lambda)$ is called a *steep Catalan function*. We show that the set of steep Catalan functions coincides precisely with skew-linked Catalan functions.

3.1 Steep root ideals and skew-linked diagrams

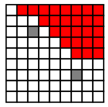
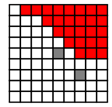
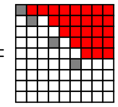
We describe a surjective map from steep root ideals to skew-linked diagrams. This construction builds on the notion of a *downpath* in a root ideal, as introduced in [2], along with a new notion of *column shape* for indexed root ideals.

Definition 3.1.1. Fix a root ideal $\Psi \in \Delta_\ell^+$ and $r \in [\ell]$. If Ψ has a non-zero number of roots in row r, define *down*$_\Psi(r)$ to be the minimum element of the set $\{j \mid (r,j) \in \Psi\}$. Otherwise, *down*$_\Psi(r)$ is undefined. For any $r \in [\ell]$, define the *path of* Ψ *starting at* r

$$\mathrm{path}_\Psi(r) = (r, \mathrm{down}_\Psi(r), \mathrm{down}_\Psi^2(r), \ldots),$$

and let $\mathrm{bot}_\Psi(r)$ (resp. $\mathrm{top}_\Psi(r)$) be the maximum (resp. minimum) element of all paths of Ψ containing r. We also set $\mathrm{downpath}_\Psi(r) = \mathrm{path}(r, \mathrm{bot}_\Psi(r))$ and $\mathrm{uppath}_\Psi(r)$ to be the reverse of $\mathrm{path}(\mathrm{top}_\Psi(r), r)$ for any $r \in [\ell]$.

Example 3.1.2. Examples of top, down, and downpath for the root ideal Ψ below:

$\Psi =$  $\Psi =$  $\Psi =$ 

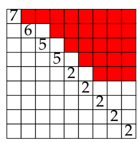
$\mathrm{top}_\Psi(7) = 3$ $\mathrm{down}_\Psi(5) = 7$ $\mathrm{downpath}_\Psi(1) = (1,2,4,6)$

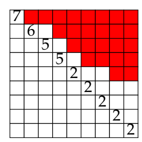
Definition 3.1.3. The *column shape* of an indexed root ideal (Ψ, λ) is defined by $\mathfrak{cs}(\Psi, \lambda) = \mu'$, where μ is determined by

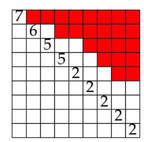
$$\mu_i = \sum_{j \in \mathrm{downpath}_\Psi(v_i)} \lambda_j\,, \tag{3.1}$$

and v is the increasing arrangement of $\{\mathrm{top}_\Psi(i) : i \in [\ell]\}$.

Example 3.1.4.







Conjugate of column shapes:

$\mu = (20, 9, 2, 2)$ $\mu = (20, 9, 2, 2)$ $\mu = (20, 7, 4, 2)$

Downpaths:

$(1,2,4,6)$, $(3,5,7)$, (8), (9) $(1,2,4,6)$, $(3,5,8)$, (7), (9) $(1,2,4,7)$, $(3,6)$, $(5,8)$, (9)

Definition 3.1.5. For partitions λ and μ, the *indexed root ideal class* of μ associated to λ is

$$[\mu]_\lambda = \{(\Psi;\lambda) \colon \Psi \in \mathfrak{R} \text{ and } \mu = \mathfrak{cs}(\Psi;\lambda)\}\,.$$

For each fixed partition λ, these sets are either empty or mutually disjoint, and form a partition of the set of indexed steep root ideals $\mathfrak{R}$ with weight λ.

Theorem 3.1.6. *Fix a partition λ. There is a one-to-one correspondence between nonempty indexed root ideal classes associated to λ and skew-linked diagrams with row shape λ.*

To prove this, we first associate each steep root ideal with a skew-linked diagram.

Lemma 3.1.7. *Given a partition λ, there is a surjective map*

$$f_\lambda \colon \{(\Psi, \lambda) : \Psi \in \mathfrak{R}\} \longrightarrow \{\textit{skew-linked shapes with row shape } \lambda\}\,, \tag{3.2}$$

defined by sending (Ψ, λ) to $D = (\lambda, \mathfrak{cs}(\Psi, \lambda))$. The outer shape of D is the partition κ where

$$\kappa_i = \sum_{j \in \mathit{downpath}_\Psi(i)} \lambda_j\,. \tag{3.3}$$

Now, fix a partition λ. All indexed root ideals in each nonempty class $[\mu]_\lambda$ have the same column shape μ; hence they have the same image under the map f_λ as in Lemma 3.1.7. In particular, for every skew-linked diagram $D = (\lambda, \mu)$, the preimage $f_\lambda^{-1}(D)$ is precisely the indexed root ideal class $[\mu]_\lambda$. Therefore, f_λ induces a bijection

$$\{\text{nonempty root ideal classes associated to } \lambda\} \leftrightarrow \{\text{skew-linked diagrams of row shape } \lambda\}.$$
$$[\mu]_\lambda \leftrightarrow (\lambda, \mu)$$

3.2 Steep Catalan functions on indexed root ideal classes

Theorem 3.2.1. *The steep Catalan functions are invariant on $[\mu]_\lambda$. That is, for a skew-linked diagram $D = (\lambda, \mu)$, we have $H(\Psi; \lambda) = H(\mathrm{CH}(D); \lambda)$ for any $\Psi \in [\mu]_\lambda$. Hence, the set of steep Catalan functions and the set of skew-linked Catalan functions are the same.*

First, we give an alternate characterization of $[\mu]_\lambda$. Recall from Definition 2.0.2 that the root ideal $\mathrm{CH}(D)$ can be interpreted as the indexed root ideal of weight λ with removable roots $(i, \mu_{\eta_i} + i)$ for $i \leq \ell(\eta)$. We then introduce the similar root ideal

$$\mathrm{CH}^+(D) := \{(i, j) \in \Delta^+_{\ell(\kappa)} \mid i \leq \ell(\eta) \text{ and } j \geq \mu_{\eta_i+1} + i\},$$

with removable roots $(i, \mu_{\eta_i+1} + i)$ for all $i \leq \ell(\eta)$. Note that $\mathrm{CH}(D) \subset \mathrm{CH}^+(D)$.

Lemma 3.2.2. *For any skew-linked shape $D = (\lambda, \mu)$,*

$$[\mu]_\lambda = \{(\Psi; \lambda) \colon \Psi \in \mathfrak{R},\ CH(D) \subset \Psi \subset CH^+(D)\}.$$

With Lemma 3.2.2 in hand, we verify our claim by building a chain of intermediate root ideals $\Psi_0 := \mathrm{CH}(D) \subset \Psi_1 \subset \Psi_2 \subset \cdots \subset \Psi$, and show that at each step adding or removing a root in a specific row does not change the associated Catalan function. The argument uses properties of steep root ideals together with repeated applications of the key equality criteria developed in [2, 3].

3.3 Shift invariance for skew-linked Catalan functions

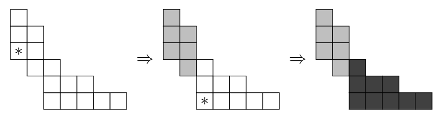
We establish an analogous shift invariance property for skew-linked Catalan functions.

Definition 3.3.1. Let D be a skew-linked shape with ℓ rows. Define a map π that removes the first column of each part D^i in the decomposition of D as follows.

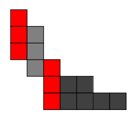
> Let $*$ denote the bottom cell of the leftmost column of D, and let m be the number of cells in its row. Set D^1 to be the first m columns of D, then iteratively apply the same process to $D \setminus (D^1 \sqcup \cdots \sqcup D^i)$ to define D^{i+1} until the remaining diagram is empty.

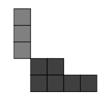
Remark 3.3.2. If D is a skew-linked diagram of ℓ rows, then $|\pi(D)| = |D| - \ell$.

Example 3.3.3. Consider the skew-linked diagram $D = (7,5,3,2,2,1)/(2,2,1)$.



D decomposes into $D^1 \sqcup D^2$.





The first column of each part D^i shown in red   The diagram $\pi(D)$

Theorem 3.3.4. *For any skew-linked shape D with ℓ rows, we have $e_\ell^\perp \mathfrak{s}_D(x;t) = \mathfrak{s}_{\pi(D)}(x;t)$.*

Idea of the Proof. Fix $D = (\lambda, \mu)$ and let $\pi(D) = (\tilde{\lambda}, \tilde{\mu})$. By Lemma 4.11 of [2], we have

$$e_\ell^\perp\, \mathfrak{s}_D = e_\ell^\perp\, H\big(\mathrm{CH}(D); \lambda\big) = H\big(\mathrm{CH}(D);\ \lambda - 1^\ell\big) \quad \text{and} \quad \mathfrak{s}_{\pi(D)} = H\big(\mathrm{CH}\big(\pi(D)\big); \tilde{\lambda}\big).$$

Since $\tilde{\lambda} = \lambda - 1^\ell$, it suffices to show that $H\big(\mathrm{CH}(D); \tilde{\lambda}\big) = H\big(\mathrm{CH}(\pi(D)); \tilde{\lambda}\big)$. We obtain $\mathrm{CH}(\pi(D)) \subset \mathrm{CH}(D) \subset \mathrm{CH}^+(\pi(D))$ by tracing how the deletion map π affects the non-roots in each row. Then the result follows from Theorem 3.2.1 and Lemma 3.2.2. □

4 k-shape Catalan functions

Next, we study a family of skew-linked Catalan functions called *k-shape Catalan functions*. This family of symmetric functions include all k-Schur and $(k-1)$-Schur functions.

4.1 k-shapes

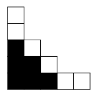
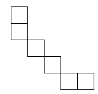
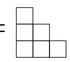
Let $\mathrm{Par}_\ell^k = \{(\mu_1, \ldots, \mu_\ell) \in \mathbb{Z}^\ell : k \geq \mu_1 \geq \cdots \geq \mu_\ell \geq 0\}$ denote the set of partitions contained in the $\ell \times k$-rectangle, and let Par^k denote the set of partitions μ with $\mu_1 \leq k$. The trailing zeros are a useful bookkeeping device for elements of Par_ℓ^k, whereas for Par^k we adopt the more common convention: each $\mu \in \mathrm{Par}^k$ is identified with $(\mu_1, \ldots, \mu_{\ell(\mu)}, 0^i)$ for any $i \geq 0$, where the *length* $\ell(\mu)$ is the number of nonzero parts of μ.

Definition 4.1.1 (k-shape partitions)**.** The *k-interior* of a partition γ, denoted by $\mathrm{Int}^k(\gamma)$, is the subpartition of cells with hook length greater than k. The *k-boundary* of γ, denoted by $\partial_k(\gamma)$, is the skew shape of cells with hook length at least k.

We define the *k-row shape* $\mathrm{rs}^k(\gamma) \in \mathbb{Z}_{\geq 0}^\infty$ (resp. *k-column shape* $\mathrm{cs}^k(\gamma) \in \mathbb{Z}_{\geq 0}^\infty$) of γ to be the sequence giving the numbers of cells in the rows (resp. columns) of $\partial_k(\gamma)$.

A partition γ is a *k-shape* if $\mathrm{rs}^k(\gamma)$ and $\mathrm{cs}^k(\gamma)$ are partitions. Thus for every k-shape γ, $\partial_k(\gamma)$ is a skew-linked diagram. As every k-shape is uniquely determined by its k-row and k-column shapes, we will often denote γ by its boundary $\partial_k(\gamma) = (\mathrm{rs}^k(\gamma), \mathrm{cs}^k(\gamma))$.

Example 4.1.2. Here is an example of a 2-shape γ:

$\gamma =$  $\partial_2(\gamma) =$  $\mathrm{Int}^2(\gamma) =$ 

4.2 k-shape root ideals and Catalan functions

In §3, we proved that the set of steep root ideals $\mathfrak{R}$ is partitioned by the indexed root ideal classes $[\mu]_\lambda$ over skew-linked shapes (λ, μ). We now refine this partition in the context of k-shapes. For an indexed root ideal $(\Psi; \lambda)$ of length ℓ, the *bandwidth* of row $i \in [\ell]$ is

$$\mathrm{band}(\Psi;\lambda)_i := \left|\{ (i,j) \in \Delta_\ell^+ \setminus \Psi \colon j \in [\ell]\}\right| + \lambda_i.$$

Fix a k-bounded partition λ, and define the set of *indexed k-root ideals*

$$\mathfrak{R}^k := \{(\Psi;\lambda) \colon \Psi \in \mathfrak{R},\ \mathrm{band}(\Psi;\lambda)_i \in \{k, k-1\} \text{ for all } i\},$$

and for a k-bounded partition μ, define the *k-root ideal class*

$$[\mu]_\lambda^k = [\mu]_\lambda \cap \mathfrak{R}^k = \{(\Psi;\lambda) \in [\mu]_\lambda \colon \mathrm{band}(\Psi;\lambda)_i \in \{k, k-1\} \text{ for all } i\}.$$

We claim that the k-root ideal classes are in bijection with k-shapes.

Theorem 4.2.1. *Fix $\lambda \in Par^k$. There is a one-to-one correspondence between nonempty k-root ideal classes associated to λ and k-shapes with row shape λ.*

$$\begin{aligned} \{\textit{nonempty k-root ideal classes associated to } \lambda\} &\leftrightarrow \{\textit{k-shapes with row shape } \lambda\} \\ [\mu]_\lambda^k &\leftrightarrow (\lambda, \mu) \end{aligned}$$

Idea of the Proof. A key ingredient in proving this theorem is the surjective map f_λ introduced in Lemma 3.1.7. Restricting f_λ to the subset of steep root ideals whose bandwidth of each row is either k or $k-1$, every image under f_λ is a k-shape. Hence f_λ induces a surjection $\widetilde{f_\lambda} : \mathfrak{R}^k \twoheadrightarrow \{k\text{-shapes with row shape } \lambda\}$, and the preimage of $\gamma = (\lambda, \mu)$ is $\widetilde{f_\lambda}^{-1}(\gamma) = f_\lambda^{-1}(\gamma) \cap \mathfrak{R}^k = [\mu]_\lambda \cap \mathfrak{R}^k = [\mu]_\lambda^k$. □

Next, we can identify the associated k-root ideal class with a k-shape $\gamma = (\lambda, \mu)$ as:

$$\begin{aligned} [\mu]_\lambda^k &= [\mu]_\lambda \cap \mathfrak{R}^k \\ &= \{(\Psi;\lambda) \colon \Psi \in \mathfrak{R},\ \mathrm{CH}(\gamma) \subset \Psi \subset \mathrm{CH}^+(\gamma),\ \Delta^k(\lambda) \subset \Psi \subset \Delta^{k-1}(\lambda)\} \\ &= \{(\Psi;\lambda) \colon \Psi \in \mathfrak{R},\ \Psi^k_{\min}(\gamma) \subset \Psi \subset \Psi^k_{\max}(\gamma)\}, \end{aligned}$$

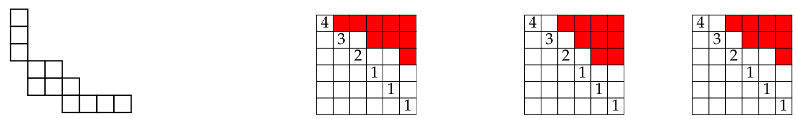
where $\Psi^k_{\min}(\gamma) := \Delta^k(\lambda) \cup \mathrm{CH}(\gamma)$ and $\Psi^k_{\max}(\gamma) := \mathrm{CH}^+(\gamma) \cap \Delta^{k-1}(\lambda)$.

Definition 4.2.2. For a k-shape γ, define the (canonical) *k-shape root ideal*

$$\Psi^k(\gamma) := \Psi^k_{\max}(\gamma) = \mathrm{CH}^+(\gamma) \cap \Delta^{k-1}(\lambda),$$

and the *associated k-shape Catalan function* $\mathfrak{s}^{(k)}_\gamma(x;t) := H(\Psi^k(\gamma); \mathrm{rs}^k(\gamma))(x;t)$.

Example 4.2.3. Consider $k = 5$ and $\gamma = 743111$.



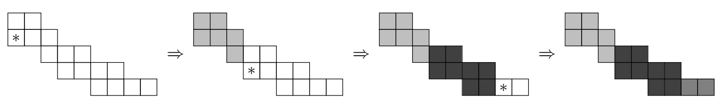
k-skew diagram of γ $\quad$ $\Delta^{k-1}(\mathrm{rs}^k(\gamma))$ $\quad$ $\mathrm{CH}^+(\gamma)$ $\quad$ $\Psi^k(\gamma)$

4.3 Shift invariance property for k-shape Catalan functions

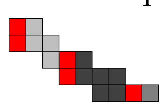
Miraculously, the deletion map from §3.3 induces a shift invariance property k-shape Catalan functions, extending the result known for k-Schur functions.

Theorem 4.3.1. *Let γ be a k-shape and $\lambda = rs^k(\gamma)$ be its k-row shape with $\ell = \ell(\lambda)$. The diagram $\pi(\gamma)$ is a $(k-1)$-shape and $e^\perp_\ell \mathfrak{s}^k_\gamma = \mathfrak{s}^{k-1}_{\pi(\gamma)}$.*

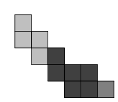
Example 4.3.2. This example illustrates the action of the deletion map on a 5-shape.



5-shape γ decomposes into $\gamma^1 \sqcup \gamma^2 \sqcup \gamma^3$



The first column of each part γ^i shown in red



4-shape $\pi(\gamma)$

5 Irreducible k-Shape Catalan functions

A powerful property of k-Schur functions is that the infinite family can be constructed from a subset of size $k!$. This property follows from a simple factorization property.

Proposition 5.0.1. *[12] Let λ be a k-bounded partition. For $d \in [k]$, we have $s_{R_d}\mathfrak{s}^{(k)}_\lambda = \mathfrak{s}^{(k)}_{\lambda \sqcup R_d}$, where R_d is a partition of the form $(k-d+1)^d$ for $d \in [k]$ and $\lambda \sqcup R_d$ denotes the partition rearrangement of the parts of λ and R_d.*

All k-Schur functions can be constructed from the *irreducible* ones, those indexed by partitions from which no R_d can be removed, and there are $k!$ irreducibles. Since the set of k-shape Catalan functions contains all k-Schur functions and $(k-1)$-Schur functions, it was conjectured in [10] that they satisfy a similar factorization property. Hivert and Mallet [7] conjectured that the number of "irreducibles" involves the Genocchi numbers.

5.1 The rectangle property

Using the composition law for Catalan operators [3, Proposition 3.2] and applying the equality criteria of [2, 3], we analyze the structure of our newly introduced root ideals and establish a factorization property for k-shape Catalan functions at $t = 1$.

Theorem 5.1.1. *Let γ be a k-shape, and let $R = (u^v)$ denote a k- or $(k-1)$-rectangle, i.e., $u + v \in \{k+1, k\}$. Then*

$$s_R \, \mathfrak{s}_\gamma^{(k)}(x;1) = \mathfrak{s}_{\gamma+R}^{(k)}(x;1),$$

where $\gamma + R$ denotes the k-shape obtained by inserting a rectangle into γ as described in [7].

Remark 5.1.2. The general case involving t requires the use of the more general Catalan operators, namely the Garsia–Jing operators B_R [6]. In analogy with the k-Schur case, we conjecture that $B_R \, \mathfrak{s}_\gamma^{(k)}(x;t) = t^n \, \mathfrak{s}_{\gamma+R}^{(k)}(x;t)$ for some positive integer n.

This motivates the study of the minimal set of k-shapes needed to construct all k-shape Catalan functions. Hivert and Mallet [7] defined an intricate operation that inserts a k- or $(k-1)$-rectangle into a k-shape, producing another k-shape. We say that a k-shape is *irreducible* if it cannot be obtained from another k-shape via such a rectangle insertion. They conjectured that the number of irreducible k-shapes is equal to the Genocchi number G_{2k}. This was proven by Bigeni [1] via a bijection between irreducible k-shapes and surjective pistols of height $k-1$, where the latter is known to give a set of cardinality G_{2k} as is defined as follows. For each positive integer n, a *surjective pistol of height n* is a surjective map $f\colon \{1, 2, \ldots, 2n\} \to \{2, 4, \ldots, 2n\}$ satisfying $f(j) \geq j$ for all j.

5.2 Irreducible k-root ideals and Genocchi numbers

We conclude with an enumerative study from the point of view of root ideals, avoiding the irreducible k-shape construction; we identify a distinguished family of *irreducible k-root ideals* and construct a bijection between this family and surjective pistols.

Definition 5.2.1. A k-root ideal Ψ is *irreducible* if, for each $i = 1, 2, \ldots, k-1$, the number of rows of length i in Ψ with bandwidth $k-1$ (resp. k) is at most $k-i-1$ (resp. $k-i$).

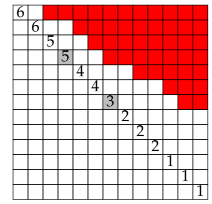
Definition 5.2.2. Fix an irreducible k-root ideal Ψ. The sequences $(a_i)_{i\in[k-1]}$ and $(b_i)_{i\in[k-1]}$ of non-negative integers are defined as follows:

Fix $i \in [k-1]$. Let z be the bottom row in Ψ of length i and bandwidth $k-1$. Let $y = \mathrm{top}_\Psi(z)$. If $y = 1$, or $\mathrm{up}_\Psi(y+1)$ is undefined, or the row $\mathrm{up}_\Psi(y+1)$ has bandwidth $k-1$, then set $a_i = 0$. Otherwise, set $a_i = \lambda_{\,\mathrm{up}_\Psi(y+1)}$.

Similarly, let z' be the bottom row in Ψ of length i and bandwidth k. Let $y' = \mathrm{top}_\Psi(z')$. If $y' = 1$, or $\mathrm{up}_\Psi(y'+1)$ is undefined, or the row $\mathrm{up}_\Psi(y'+1)$ has bandwidth $k-1$, then set $b_i = 0$. Otherwise, set $b_i = \lambda_{\,\mathrm{up}_\Psi(y'+1)}$.

Example 5.2.3.

Consider the following irreducible 8-root ideal Ψ. To find a_3, we have $z = 7, y = \mathrm{top}_\Psi(7) = 4, \mathrm{up}_\Psi(y+1) = \mathrm{up}_\Psi(5) = 2$. The second row of Ψ has bandwidth equal to 8. Therefore, we get $a_3 = \lambda_2 = 6$.



From irreducible root ideals to surjective pistols

Fix an irreducible k-root ideal Ψ and consider the corresponding sequences $(a_i)_{i\in[k-1]}$ and $(b_i)_{i\in[k-1]}$. We construct a map $f\colon \{1,2,\ldots,2k-2\} \to \mathbb{N}$ as follows:

$$f(2i-1) = \begin{cases} 2(a_i - 1) & \text{if } a_i > 0 \text{ and } a_i \notin \{f(1), f(2), \ldots, f(2i-2)\}, \\ 2i + 2x_i & \text{otherwise;} \end{cases}$$

$$f(2i-2) = \begin{cases} 2(b_i - 1) & \text{if } b_i > 0 \text{ and } b_i \notin \{f(1), f(2), \ldots, f(2i-3)\}, \\ 2i + 2y_i - 2 & \text{otherwise,} \end{cases}$$

where x_i (resp. y_i) is the number of rows of length i in Ψ with bandwidth $k-1$ (resp. k).

From surjective pistols to irreducible root ideals

Fix a surjective pistol f of height $k-1$. It suffices to construct sequences of nonnegative integers $(x_i)_{i\in[k-1]}$ and $(y_i)_{i\in[k-1]}$ satisfying $x_i \le k-i-1$ and $y_i \le k-i$.

Set $x_{k-1} = 0$ and $y_{k-1} = \frac{f(2k-2)-(2k-2)}{2}$. Next, suppose x_i and y_i have already been determined for all $i > n$. Consider an indexed root ideal $\widetilde{\Psi}$ where x_i (resp. y_i) is the number of rows of length i with bandwidth $k-1$ (resp. k). We then define x_n by

$$x_n = \begin{cases} \frac{f(2n-1)-2n}{2}, & \text{if } f(2n-1) \in \{f(1), f(2), \ldots, f(2n-2)\}, \\ T - \sum_{i>n}(x_i + y_i), & \text{otherwise,} \end{cases}$$

where j is the index of the top row of length $\frac{f(2n-1)}{2} + 1$ of $\widetilde{\Psi}$ with bandwidth k, $z = \mathrm{bot}_{\widetilde{\Psi}}(\mathrm{down}_{\widetilde{\Psi}}(j) - 1)$, and (z, T) is the removable root of $\widetilde{\Psi}$. Define y_n analogously.

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