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Grothendieck polynomials of fireworks and inverse fireworks permutations

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Abstract. Grothendieck polynomials were introduced by Lascoux and Schützenberger to represent *K*-classes of structure sheaves of Schubert varieties. Their supports and top-degree components have recently become active areas of research. Using pipe dreams, combinatorial objects that compute Grothendieck polynomials, we prove several results for the Grothendieck polynomials of two families of permutations. For fireworks permutations, we show that the support of the Grothendieck polynomial is "as large as possible" and is M-convex after homogenization. For inverse fireworks permutations, we show that every monomial of non-maximal degree appearing in the Grothendieck polynomial divides another monomial of degree exactly one higher. Our work also yields the first direct combinatorial formula for the top-degree components of Grothendieck polynomials.

Keywords: Pipe dreams, Grothendieck polynomial, Castelnuovo–Mumford polynomial, Newton polytope

1 Introduction

Lascoux and Schützenberger [14, 15] introduced explicit representatives of the cohomology classes and *K*-classes of structure sheaves of Schubert varieties in flag varieties, called the *Schubert polynomials* $\mathfrak{S}_w(\mathbf{x})$ and *Grothendieck polynomials* $\mathfrak{G}_w(\mathbf{x})$, respectively. In general, $\mathfrak{G}_w(\mathbf{x})$ is not homogeneous, and its lowest-degree component equals $\mathfrak{S}_w(\mathbf{x})$.

Recently, the top-degree homogeneous component $\mathfrak{G}_w^{\mathrm{top}}$ of the Grothendieck polynomial $\mathfrak{G}_w$ has attracted increasing attention [2, 9, 10, 21, 22]. We emphasize two reasons for its importance. First, the degree of $\mathfrak{G}_w^{\mathrm{top}}$ determines the *Castelnuovo–Mumford regularity* of matrix Schubert varieties [24]. Second, the support of $\mathfrak{G}_w$, which is the set of monomials appearing in $\mathfrak{G}_w$, is conjectured to be governed by the support of $\mathfrak{G}_w^{\mathrm{top}}$. The main goal of this extended abstract is to make partial progress toward this conjecture.

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Conjecture 1.1 ([18]). *For a permutation w,*

$$\operatorname{supp}(\mathfrak{G}_w) = \bigcup_{\substack{\alpha \in \operatorname{supp}(\mathfrak{G}_w) \\ \beta \in \operatorname{supp}(\mathfrak{G}_w^{\mathrm{top}})}} [\alpha, \beta], \tag{1.1}$$

where $[\alpha, \beta]$ denotes the component-wise interval $\{\gamma \in \mathbb{Z}^n : \alpha_i \leq \gamma_i \leq \beta_i \text{ for all } i\}$.

Conjecture 1.1 was previously known only for vexillary permutations [9]. Motivated by this conjecture, we study the polynomial $\mathfrak{G}_w^{\mathrm{top}}$ and its support. We focus on two families of permutations:

Definition 1.2. A permutation w is called *fireworks* if the initial terms of the decreasing runs of w are increasing. A permutation w is *inverse fireworks* if w^{-1} is fireworks.

Example 1.3. The permutation $w = \underline{3}1\underline{6}\underline{7}542$ is fireworks and the permutation $u = \underline{3}1\underline{7}5\underline{6}42$ is not fireworks. The initial terms of each decreasing run are underlined.

When w lies in one of these two families, $\mathfrak{G}_w^{\mathrm{top}}$ exhibits remarkable structure via the following two theorems.

Theorem 1.4 ([18]). *When w is fireworks, $\mathfrak{G}_w^{\mathrm{top}}$ consists of a single monomial. Specifically, it is a positive integer multiple of $x^{\mathrm{wt}(\overline{D(w)})}$, where $\overline{D(w)}$ denotes the up-closure of the Rothe diagram of w (see Definition 2.1).*

Theorem 1.5 ([22]). *For all $w \in S_n$, there is a unique inverse fireworks permutation $\pi(w)$ with $\mathfrak{G}_w^{\mathrm{top}} = c_w \cdot \mathfrak{G}_{\pi(w)}^{\mathrm{top}}$ for some $c_w \in \mathbb{Z}$. In fact, $c_w = |\mathfrak{G}_{\pi(w^{-1})}^{\mathrm{top}}(1, \ldots, 1)|$.*

The map $\pi(\cdot)$ can be computed explicitly (see Section 4.2 of [22]). Consequently, to understand all $\mathfrak{G}_w^{\mathrm{top}}$, it suffices to study the case of inverse fireworks permutations.

Based on these two theorems, we prove several results regarding $\mathfrak{G}_w^{\mathrm{top}}$ and $\operatorname{supp}(\mathfrak{G}_w)$ when w is fireworks or inverse fireworks. The following summarized results are based on joint work of the first and second authors on fireworks [4], and of the first and third authors on inverse fireworks [3].

Theorem 1.6 ([4]). *Conjecture 1.1 holds when w is fireworks:*

$$\operatorname{supp}(\mathfrak{G}_w) = \bigcup_{\alpha \in \operatorname{supp}(\mathfrak{G}_w)} [\alpha, \mathrm{wt}(\overline{D(w)})].$$

A surprising corollary of Theorem 1.6 is that *homogenized Grothendieck polynomials* $\widetilde{\mathfrak{G}}_w$ of fireworks permutations have M-convex support. In general, M-convexity of $\operatorname{supp}(\widetilde{\mathfrak{G}}_w)$ is far from a formal consequence of Conjecture 1.1; it is even an open problem to show that $\operatorname{supp}(\mathfrak{G}_w^{\mathrm{top}})$ is M-convex. Conversely, in the case that $\mathfrak{S}_w$ has zero-one coefficients, $\operatorname{supp}(\widetilde{\mathfrak{G}}_w)$ is known to be M-convex ([1]) while Conjecture 1.1 remains open.

Theorem 1.7 ([4])**.** *For fireworks $w \in S_n$, the homogenized Grothendieck polynomial $\widetilde{\mathfrak{G}}_w$ has M-convex support. In particular, $\mathfrak{G}_w$ has SNP, each degree component $\mathfrak{G}_w^{(c)}$ has M-convex support, and the sizes $|\mathrm{supp}(\mathfrak{G}_w^{(c)})|$ of their supports form a log-concave sequence.*

Because taking convex hull commutes with homogenization, Theorem 1.7 implies that ordinary Grothendieck polynomials $\mathfrak{G}_w$ of fireworks permutations have SNP.

As far as the authors are aware, M-convexity of $\mathrm{supp}(\widetilde{\mathfrak{G}}_w)$ was previously known for only exponentially many permutations in n (cf. [6]). Theorem 1.7 supplies superexponentially many permutations for which $\mathrm{supp}(\widetilde{\mathfrak{G}}_w)$ is M-convex.

A special subclass of fireworks permutations called *layered* permutations has previously appeared in various maximization problems in Schubert calculus [16, 20, 19, 8, 25]. The support formula in Theorem 1.6 implies that layered permutations maximize Newton polytopes of fireworks Grothendieck polynomials with respect to inclusion.

Corollary 1.8 ([4])**.** *Let w be a fireworks permutation. Then $\pi(w)$ is the layered permutation whose block sizes are the lengths of descending runs of w, and $\mathrm{supp}(\mathfrak{G}_{\pi(w)}) \supseteq \mathrm{supp}(\mathfrak{G}_w)$.*

When w is inverse fireworks, we can prove a weaker version of Conjecture 1.1, which is [18, Conj. 1.2]:

Theorem 1.9 ([3])**.** *Let w be an inverse fireworks permutation. If $\mathbf{x}^\alpha$ appears in $\mathfrak{G}_w$ with $\deg(\mathbf{x}^\alpha) < \deg(\mathfrak{G}_w)$, then there exists i such that $x_i \cdot \mathbf{x}^\alpha$ also appears in $\mathfrak{G}_w$.*

Finally, we define a novel combinatorial object which we call *bumpless vertical-less pipe dreams* (BVPDs), labeled by inverse fireworks permutations. By Theorem 1.5, BVPDs yield a combinatorial formula for all $\mathfrak{G}_w^{\mathrm{top}}$.

Theorem 1.10 ([3])**.** *For any $w \in S_n$,*

$$\mathfrak{G}_w^{\mathrm{top}} = |\mathrm{BVPD}(\pi(w^{-1}))| \cdot \sum_{P \in \mathrm{BVPD}(\pi(w))} \mathbf{x}^{\mathrm{wt}(P)}.$$

As far as the authors are aware, no direct combinatorial formula for $\mathfrak{G}_w^{\mathrm{top}}$ was previously known. Theorem 1.10 provides the first such formula.

2 Background

Our main combinatorial tool is pipe dreams. A *pipe dream* is a filling of a staircase grid $\{(i,j) \in [n] \times [n] \colon i + j \leq n\}$ using *cross tiles* $\boxplus$ and *bump tiles* . By placing half-bump tiles on the antidiagonal $\{(i,j) \in [n] \times [n] \colon i + j = n + 1\}$, a pipe dream can be viewed as a network of n pipes with an associated permutation (see Figure 2). Given a pipe dream P, there are two different ways to trace pipes in P. The *top-to-left trace* is obtained by tracing the pipes from top border to the left, ignoring any repeat crossings between pairs of pipes that has already crossed. The pipe dream's associated permutation is then obtained by reading the pipe indices on the left from top to bottom. The *left-to-top trace* is obtained by tracing the pipes from the left border to the top, again ignoring any repeat crossings between pairs of pipes that have already crossed. The pipe dream's associated permutation is obtained by taking the inverse of the pipe indices on the top read from left to right. It is a well-known fact that, although the paths traced by the two traces may differ, they yield the same permutation.

Let $\mathrm{PD}(w)$ be the set of pipe dreams associated with w and let $\mathrm{wty}(P)$ be the set of its cross tiles $\boxplus$. Following [7, 11], we define the *Grothendieck polynomial* $\mathfrak{G}_w(\mathbf{x})$ to be

$$\mathfrak{G}_w(\mathbf{x}) := \sum_{P \in \mathrm{PD}(w)} \left((-1)^{|\mathrm{wty}(P)| - \ell(w)} \prod_{(i,j) \in \mathrm{wty}(P)} x_i \right). \tag{2.1}$$

We define the *homogenized Grothendieck polynomial* $\widetilde{\mathfrak{G}}_w(\mathbf{x}, z)$ to be

$$\widetilde{\mathfrak{G}}_w(\mathbf{x}, z) := \sum_{P \in \mathrm{PD}(w)} \left(z^{\deg(\mathfrak{G}_w) - |\mathrm{wty}(P)|} \cdot (-1)^{|\mathrm{wty}(P)| - \ell(w)} \prod_{(i,j) \in \mathrm{wty}(P)} x_i \right), \tag{2.2}$$

so that $\widetilde{\mathfrak{G}}_w$ is homogeneous of degree $\deg(\mathfrak{G}_w)$.

A *diagram* is a subset of $[n] \times [n]$, which can be represented by placing a cell in row i column j of an $n \times n$ grid. We adopt the convention where row 1 is the highest and column 1 is the leftmost. For $w \in S_n$, its Rothe diagram $D(w)$ is the following diagram

$$D(w) := \{(i, w(j)) : 1 \le i < j \le n, w(i) > w(j)\}.$$

Definition 2.1. The upward closure of a diagram D, denoted as $\overline{D}$, is $\{(i,j) \in [n] \times [n] : (i', j) \in D \text{ for some } i' \ge i\}$.

3 Fireworks Grothendieck polynomials

Conjecture 1.1 for fireworks permutations

For any permutation w, Mészáros, Setiabrata and St. Dizier [17] showed that every monomial $\mathbf{x}^\alpha$ appearing in $\mathfrak{G}_w$ divides $\mathbf{x}^{\overline{D(w)}}$. We start by interpreting this result via pipe dreams. Consider the top-to-left trace of a pipe dream P and write $P_{\mathrm{TL}}^{(j)}$ for the pipe P entering in the j-th column. The *pipe-weight* of a pipe dream $P \in \mathrm{PD}(w)$ of $w \in S_n$ is the set

$$\mathrm{pwt}(P) := \{(i,j) \colon \text{pipe } P_{\mathrm{TL}}^{(j)} \text{ exits from the bottom edge of a } \boxplus \text{ in row } i\} \subseteq [n]^2. \tag{3.1}$$

The pipe weight $\mathrm{pwt}(P)$ of any pipe dream $P \in \mathrm{PD}(w)$ is always a subset of $\overline{D(w)}$. It follows that $\mathbf{x}^{\mathrm{wt}(P)}$ divides $\mathbf{x}^{\mathrm{wt}(\overline{D(w)})}$. In the fireworks case, when $\mathbf{x}^{\mathrm{wt}(\overline{D(w)})}$ appears in $\mathfrak{G}_w^{\mathrm{top}}$, top-degree pipe dreams have *inclusion-maximal* pipe-weights (rather than just *componentwise-maximal* weights).

The key ingredient in our proof of Theorem 1.6 is an explicit pipe dream construction guided by pipe-weights. Specifically, Theorem 1.6 follows from repeated application of the following claim.

Theorem 3.1 ([4]). *Let $w \in S_n$ be fireworks and suppose $P \in \mathrm{PD}(w)$ satisfies $\mathrm{wt}(P) < \mathrm{wt}(\overline{D(w)})$. For any a with $\mathrm{wt}(P)_a < \mathrm{wt}(\overline{D(w)})_a$, there exists a pipe dream $Q \in \mathrm{PD}(w)$ so that $\mathrm{wt}(Q)_b = \mathrm{wt}(P)_b$ for $b < a$, $\mathrm{wt}(Q)_a = \mathrm{wt}(P)_a + 1$, and $\mathrm{wt}(Q)_b \leq \mathrm{wt}(P)_b$ for $b > a$.*

Our proof is local in nature and involves finding configurations like the one in Figure 1, which are guaranteed to exist by pipe-weight considerations (cf. (3.1)).

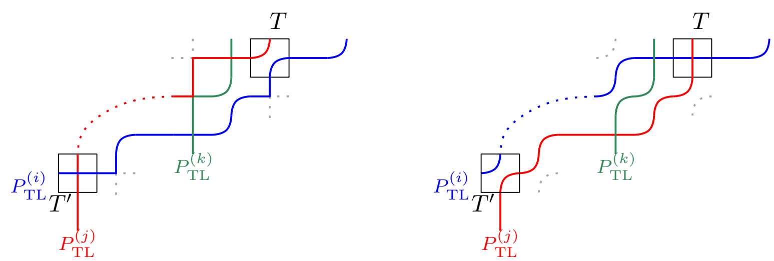


Figure 1: Left: One type of local configuration in $P \in \mathrm{PD}(w)$ of non-maximal degree. Right: One step of our algorithm constructing desired $Q \in \mathrm{PD}(w)$ for Theorem 3.1.

Remark 3.2. Our construction in the proof of Theorem 3.1 yields pipe dreams Q which are equal to P in rows $< a$.

Theorem 1.6. Let w be a fireworks permutation. If $\mathbf{x}^\alpha$ appears in $\mathfrak{G}_w$ and $x_i \cdot \mathbf{x}^\alpha$ divides $\mathbf{x}^{\mathrm{wt}(\overline{D(w)})}$, then $x_i \cdot \mathbf{x}^\alpha$ appears in $\mathfrak{G}_w$. In particular,

$$\mathrm{supp}(\mathfrak{G}_w) = \bigcup_{\alpha \in \mathrm{supp}(\mathfrak{G}_w)} [\alpha, \mathrm{wt}(\overline{D(w)})].$$

Proof of Theorem 1.6. Suppose that $\mathbf{x}^\alpha$ appears in $\mathfrak{G}_w$ and that $x_i \cdot \mathbf{x}^\alpha$ divides $\mathbf{x}^{\mathrm{wt}(\overline{D(w)})}$. Pick $P \in \mathrm{PD}(w)$ with $\mathrm{wt}(P) = \alpha$. By repeated application of Theorem 3.1, there exists a pipe dream $Q \in \mathrm{PD}(w)$ with $\mathbf{x}^{\mathrm{wt}(Q)} = x_i \cdot \mathbf{x}^\alpha$. The claim follows from Equation (2.1). $\square$

Schubitopes and M-convexity of $\mathrm{supp}(\mathfrak{G}_w)$

We prove Theorem 1.7 by decomposing the support of $\mathfrak{G}_w$ into a Minkowski sum of smaller M-convex sets (Proposition 3.9). We are motivated by the following analogous story for Schubert polynomials.

Definition 3.3. A set $S \subset \mathbb{Z}^n$ is *M-convex* if for any $\alpha, \beta \in S$ and any $i \in [n]$ for which $\alpha_i > \beta_i$, there is an index $j \in [n]$ for which $\alpha_j < \beta_j$ and $\alpha - \mathbf{e}_i + \mathbf{e}_j \in S$ and $\beta + \mathbf{e}_i - \mathbf{e}_j \in S$.

There is an interpretation of M-convexity in terms of certain polytopes called generalized permutahedra: A set is M-convex if and only if it is the set of integer points of an integral generalized permutahedron. In particular, a polynomial f has M-convex support if and only if f has SNP and its Newton polytope is a generalized permutahedron. We refer to [23] for background on generalized permutahedra.

Lemma 3.4. *The Minkowski sum of M-convex sets is M-convex.*

Lemma 3.5. *Let $S \subset \mathbb{Z}^n$ be an M-convex set, and let $I \subseteq [n]$. Then the sequence*

$$a_c := \left| S \cap \left\{ \sum_{i \in I} x_i = c \right\} \right|$$

of point counts on the hyperplanes $\{\sum_{i \in I} x_i = c\}$ is log-concave.

Definition 3.6. Given a subset $S \subseteq [n]$ with elements $s_1 < \cdots < s_r$, write $\mathcal{B}$ for the collection of sets $B \subseteq [s_r]$ with elements $b_1 < \cdots < b_r$ satisfying $b_i \leq s_i$ for all i. The indicator vectors $\{\zeta_B \colon B \in \mathcal{B}\} \subset \mathbb{Z}^n$ form an M-convex set, and their convex hull $P(\mathrm{SM}(S)) := \mathrm{conv}\{\zeta_B \colon B \in \mathcal{B}\}$ the *Schubert matroid polytope*.

Schubert matroid polytopes compute the support of Schubert polynomials.

Theorem 3.7 ([5, Thm 4, Thm 7])**.** *Let $w \in S_n$, and let $D_1, \ldots, D_n$ denote the columns of $D(w)$. Then $\mathfrak{S}_w$ has M-convex support and*

$$\mathrm{supp}(\mathfrak{S}_w) = \left(\sum_{i=1}^{n} P(\mathrm{SM}(D_i)) \right) \cap \mathbb{Z}^n.$$

Definition 3.8. Given a subset $S \subseteq [n]$ with elements $s_1 < \cdots < s_r$, write $\mathcal{B}_{\mathrm{sp}}$ for the *spanning sets* of the Schubert matroid; it is the collection of sets $B' \subseteq [s_r]$ with elements $b_1 < \cdots < b_{r'}$, with $r' \geq r$, satisfying $b_i \leq s_i$ for all $i \in [r]$. The convex hull $P_{\mathrm{sp}}(\mathrm{SM}(S)) := \mathrm{conv}(\{\zeta_{B'} \colon B' \in \mathcal{B}_{\mathrm{sp}}\})$ of the indicator vectors is called the *Schubert spanning set polytope*. The homogenized indicator vectors $\{\widetilde{\zeta}_{B'} \colon B' \in \mathcal{B}_{\mathrm{sp}}\}$ form an M-convex set, and their convex hull $\widetilde{P}_{\mathrm{sp}}(\mathrm{SM}(S)) := \mathrm{conv}(\{\widetilde{\zeta}_{B'} \colon B' \in \mathcal{B}_{\mathrm{sp}}\})$ is the *homogenized Schubert spanning set polytope*.

Proposition 3.9 (cf. [18, Thm 3.6])**.** *Fix subsets $D_1, \ldots, D_m \subseteq [n]$ and set $P := P(\mathrm{SM}(D_1)) + \cdots + P(\mathrm{SM}(D_m))$. Then*

$$\left(\sum_{j=1}^{n} P_{\mathrm{sp}}(\mathrm{SM}(D_j)) \right) \cap \mathbb{Z}^n = \bigcup_{\beta \in \mathrm{supp}(P)} [\beta, \mathrm{wt}(\overline{D})].$$

Theorem 1.7. For fireworks $w \in S_n$, the homogenized Grothendieck polynomial $\widetilde{\mathfrak{G}}_w$ has M-convex support, i.e., $\widetilde{\mathfrak{G}}_w$ has SNP and its Newton polytope is a generalized permutahedron. In particular, each degree component $\mathfrak{G}_w^{(c)}$ has M-convex support, and the sizes $|\mathrm{supp}(\mathfrak{G}_w^{(c)})|$ of their supports form a log-concave sequence.

Proof of Theorem 1.7. Let D_j denote the j-th column of $D(w)$ and write

$$\mathcal{G}_{D(w)} := \sum_{j=1}^{n} P_{\mathrm{sp}}(\mathrm{SM}(D_j)), \qquad \widetilde{\mathcal{G}}_{D(w)} := \sum_{j=1}^{n} \widetilde{P}_{\mathrm{sp}}(\mathrm{SM}(D_j)).$$

By Theorem 1.6, Theorem 3.7, and Proposition 3.9, we have $\mathrm{supp}(\mathfrak{G}_w) = \mathcal{G}_{D(w)} \cap \mathbb{Z}^n$. Homogenizing yields $\mathrm{supp}(\widetilde{\mathfrak{G}}_w) = \widetilde{\mathcal{G}}_{D(w)} \cap \mathbb{Z}^{n+1}$, and Lemma 3.4 implies that $\widetilde{\mathfrak{G}}_w$ has M-convex support. The final claim follows from Lemma 3.5. □

Inclusion-maximal support

We prove Corollary 1.8 using the support formula of Theorem 1.6 to reduce the problem to the following simple claim about inclusions among Schubert spanning set polytopes.

Lemma 3.10. *Let $A \subseteq B \subseteq [n]$ be two sets satisfying $\max(A) = \max(B)$. Then $P_{\mathrm{sp}}(\mathrm{SM}(B)) \subseteq P_{\mathrm{sp}}(\mathrm{SM}(A))$.*

Definition 3.11. For integers $b_1, \dots, b_m$, the *layered permutation* with *block sizes* $(b_1, \dots, b_m)$ is the longest element of the Young subgroup $S_{b_1} \times S_{b_2} \times \cdots \times S_{b_m}$; i.e., it is the permutation with one-line notation

$$c_1\,(c_1 - 1)\,\dots\,1\,c_2\,(c_2 - 1)\,\dots\,(c_1 + 1)\,\dots\,c_m\,(c_m - 1)\,\dots\,(c_{m-1} + 1),$$

where $c_i := b_1 + \cdots + b_i$.

Lemma 3.12. *Let w be a fireworks permutation and write $\pi(w)$ for the layered permutation whose block sizes are the lengths of descending runs of w. Then $D(w)_{w(i)} \supseteq D(\pi(w))_{(\pi(w))(i)}$.*

Corollary 1.8. Let w be a fireworks permutation and write $\pi(w)$ for the layered permutation whose block sizes are the lengths of descending runs of w. Then $\mathrm{supp}(\mathfrak{G}_{\pi(w)}) \supseteq \mathrm{supp}(\mathfrak{G}_w)$.

Proof of Corollary 1.8. Write $D_i := D(w)_{w(i)}$ and $D'_i := D(\pi(w))_{(\pi(w))(i)}$. Lemma 3.12 implies that $D_i \supseteq D'_i$ and $\max(D_i) = \max(D'_i)$, so Lemma 3.10 gives

$$\mathrm{supp}(\mathfrak{G}_w) = \left(\sum_{i=1}^{n} P_{\mathrm{sp}}(\mathrm{SM}(D_i))\right) \cap \mathbb{Z}^n \subseteq \left(\sum_{i=1}^{n} P_{\mathrm{sp}}(\mathrm{SM}(D'_i))\right) \cap \mathbb{Z}^n = \mathrm{supp}(\mathfrak{G}_{\pi(w)}). \quad \square$$

4 Inverse Fireworks Grothendieck polynomials

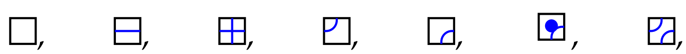
The main result of this section is Theorem 1.9 and a bumpless vertical-less pipe dream formula (Theorem 1.10). More precisely, we introduce a new set of combinatorial objects $\mathrm{MVPD}(w)$ for each $w \in S_n$ and define weight-preserving bijections

$$\Phi\colon \mathrm{PD}(w) \to \mathrm{MVPD}(w) \qquad \text{(Proposition 4.5)},$$
$$\Psi\colon \mathrm{BVPD}(\pi(w)) \to \widehat{\mathrm{MVPD}}(\pi(w)) \qquad \text{(Theorem 4.10)},$$

where $\widehat{\mathrm{MVPD}}(w)$ denotes the subset of $\mathrm{MVPD}(w)$ of highest degree and $\pi(w)$ is the inverse fireworks permutation guaranteed by Theorem 1.5.

Marked vertical-less pipe dreams

Definition 4.1. A *marked vertical-less pipe dream* (MVPD) is a filling of an $n \times n$ grid using the tiles



such that any pipe in a marked tile (call it T) has a in a lower row than T (see Figure 2). The adjective *vertical-less* reflects the absence of vertical tiles in a MVPD.

Similar to left-to-top traces of ordinary pipe dreams, we trace pipes of a MVPD from left to top, ignoring double crossings. The pipe entering in row i is denoted $P^{(i)}$.

Definition 4.2. The *column-to-row code* $\alpha(M)$ of an MVPD M is the sequence of n numbers defined as follows. If there is no pipe exiting at column c of M, then $\alpha(M)_c = 0$. Otherwise, if the pipe $P^{(r)}$ exits in column c, then $\alpha(M)_c = r$.

Definition 4.3. Let $w \in S_n$. Let $\alpha(w)$ denote the sequence obtained by writing the one-line notation $(w^{-1}(1), \ldots, w^{-1}(n))$ of w^{-1} and then replacing the left-to-right maxima with 0's. Then define $\mathrm{MVPD}(w)$ to be the set of MVPDs with column-to-row code $\alpha(w)$. Also define $\widehat{\mathrm{MVPD}}(w) \subseteq \mathrm{MVPD}(w)$ to be the set of MVPDs with the maximal number of weighty tiles among all $M \in \mathrm{MVPD}(w)$.

When drawing an MVPD, we omit blank rows and columns on the bottom and right.

Example 4.4. Let $w = 12643857$, so that $w^{-1} = 12547386$ and $\alpha(w) = (0, 0, 0, 4, 0, 3, 0, 6)$. Figure 2 consists of a pipe dream $P \in \mathrm{PD}(w)$ and an MVPD $M \in \mathrm{MVPD}(w)$.

For an MVPD M, let $\mathrm{wty}(M)$ be the set of tiles of M that are , or . We now present a $\mathrm{wty}(\cdot)$-preserving bijection $\Phi\colon \mathrm{PD}(w) \to \mathrm{MVPD}(w)$. Let $P^{(p_i)}_{\mathrm{LT}}$ denote the pipe that enters from the left in row i.

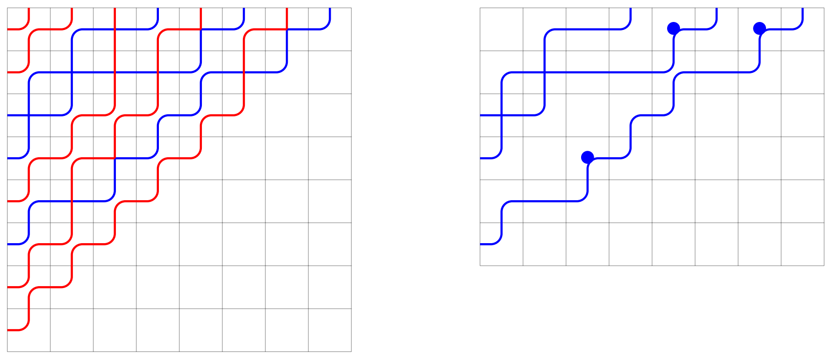


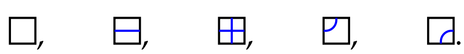
Figure 2: A pipe dream $P \in \mathrm{PD}(w)$ and a marked vertical-less pipe dream $M \in \mathrm{MVPD}(w)$, for $w = 12643857$. The MVPD M is equal to $\Phi(P)$.

Proposition 4.5. *Let $w \in S_n$ and let p_i denote the left-to-right maxima of w^{-1}. The assignment $\Phi\colon \mathrm{PD}(w) \to \mathrm{MVPD}(w)$ given by removing the pipes $P_{\mathrm{LT}}^{(p_i)}$ and replacing any resulting tiles with is a $\mathrm{wty}(\cdot)$-preserving bijection (see Figure 2).*

Therefore, MVPD is a formula for $\mathfrak{G}_w$ just like pipe dream. However, a key feature of MVPDs is the presence of local *droop moves*, inherited from bumpless pipe dreams [12, 13]. Thus, MVPDs serve as an interpolating model that allows certain phenomena from bumpless pipe dreams to be transported into the classical pipe dream setting.

Bumpless vertical-less pipe dreams

Definition 4.6. A *bumpless vertical-less pipe dream* (BVPD) is a tiling of an $n \times (n-1)$ grid using the tiles



As with MVPDs, we trace pipes from left to top, and write $P^{(i)}$ for the pipe entering in the i-th row.

Definition 4.7. The *BVPD column-to-row code* $\alpha'(M)$ of a BVPD B is the sequence of $n-1$ numbers defined (similarly to the MVPD setting) as follows. If there is no pipe exiting at column c of B, then $\alpha'(B)_c = 0$. Otherwise, if the pipe $P^{(r)}$ exits in column c, then $\alpha'(B)_c = r$.

Definition 4.8. Let w be an *inverse fireworks* permutation. Let $\alpha'(w)$ denote the sequence obtained by removing the first entry from $\alpha(w)$. Let $\mathrm{BVPD}(w)$ denote the set of BVPDs with BVPD column-to-row code $\alpha'(w)$.

When drawing a BVPD, we omit blank rows and columns on the bottom and right.

Example 4.9. See Figure 3 for a BVPD for $w = 165234$, so that $w^{-1} = 145632$ and $\alpha'(w) = (0, 0, 0, 3, 2)$.

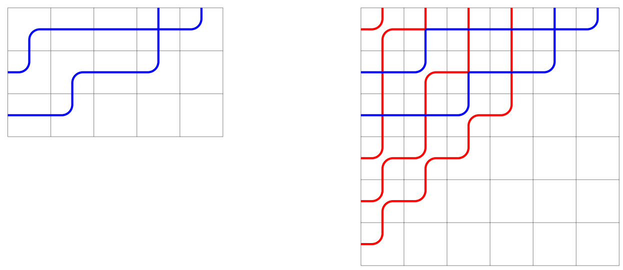


Figure 3: A BPVD $B \in \mathrm{BVPD}(w)$ and a pipe dream $P \in \mathrm{PD}(w)$, for $w = 165234$. The pipe dream P turns out to be of highest degree in $\mathrm{PD}(w)$.

For a BVPD B, let $\mathrm{wty}(B)$ be the set of tiles of B that are , or . We now present a $\mathrm{wty}(\cdot)$-preserving bijection $\Psi\colon \mathrm{BVPD}(w) \to \widehat{\mathrm{MVPD}}(w)$ for all inverse fireworks permutations w.

Theorem 4.10. *Let $w \in S_n$ be inverse fireworks. The assignment $\Psi\colon \mathrm{BVPD}(w) \to \widehat{\mathrm{MVPD}}(w)$ given by adding a column on the left of B consisting of and and then replacing all tiles with tiles is a $\mathrm{wty}(\cdot)$-preserving bijection.*

Figure 3 gives an example of the bijection $\Phi^{-1} \circ \Psi\colon \mathrm{BVPD}(w) \to \mathrm{PD}(w)$.

Our proof of Theorem 4.10 relies on the following structural lemma.

Lemma 4.11. *Let w be an inverse fireworks permutation. Then $\widehat{\mathrm{MVPD}}(w)$ consists of elements in $\mathrm{MVPD}(w)$ without and .*

Remark 4.12. The bijections Φ and Ψ preserve $\mathrm{wty}(\cdot)$, so the formula in Theorem 1.10 works for *double Grothendieck polynomials* of inverse fireworks permutations as well.

Droop moves and going-up

In [3, Defn 5.4], we define *droop moves* $\mathrm{droop}_{(i,j)}(M)$ and $\mathrm{droop}'_{(i,j)}(M)$ on MVPDs, which we use to construct the pipe dreams required to establish Theorem 1.9. We give two examples of the effect of $\mathrm{droop}_{(i,j)}$ and $\mathrm{droop}'_{(i,j)}$ in Figure 4.

Theorem 1.9. Let w be an inverse fireworks permutation. If $\mathbf{x}^\alpha$ appears in $\mathfrak{G}_w$ and $\deg(\mathbf{x}^\alpha) < \deg(\mathfrak{G}_w)$, then there exists i so that $x_i \cdot \mathbf{x}^\alpha$ appears in $\mathfrak{G}_w$.

Proof sketch. Pick $M \in \mathrm{MVPD}(w)$ of weight α; necessarily, $M \notin \widehat{\mathrm{MVPD}}(w)$. If we can turn a into a or a into a and still remain in $\mathrm{MVPD}(w)$, then we are done. Otherwise, we show that there exists or ; in this situation we find and perform a droop move and repeat the whole process. $\square$

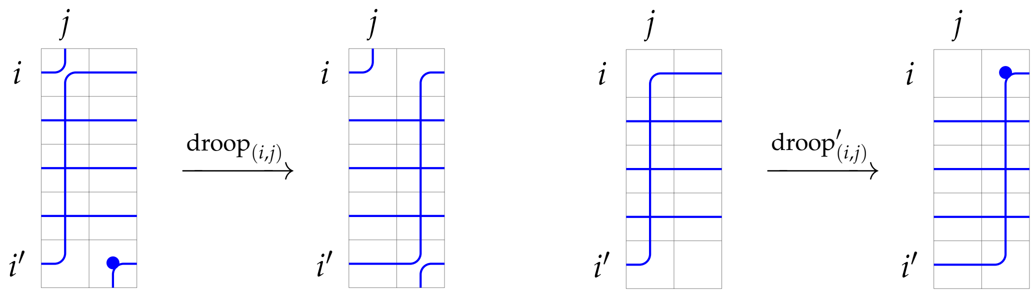


Figure 4: Examples of droop moves $\mathrm{droop}_{(i,j)}$ and $\mathrm{droop}'_{(i,j)}$.

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