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Classification of Zamolodchikov periodic cluster algebras

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Abstract. Zamolodchikov periodicity is a property of certain discrete dynamical systems and was one of the primary motivations for the creation of cluster algebras. It was proved by Keller to hold for tensor products of two Dynkin diagrams, and further shown by Galashin and Pylyavskyy to hold for pairs of commuting simply-laced Cartan matrices of finite type, which Stembridge classified in his study of admissible W-cells. We prove that the Zamolodchikov periodic cluster algebras are in natural bijection with pairs of commuting (not necessarily reduced or simply-laced) Cartan matrices of finite type. We fully classify all such pairs into 29 infinite families and 14 exceptional types in addition to the 6 infinite families and 11 exceptional types in Stembridge's classification, and show that all of these families can be derived from simply-laced types through two operations preserving Zamolodchikov periodicity, folding and taking transpose.

Our work holds connections to Kazhdan–Lusztig theory, and our main theorem also helps classify all nonnegative W-cells for products of two dihedral groups $W = I_2(p) \times I_2(q)$.

Keywords: Cluster algebras, Zamolodchikov periodicity, commuting Cartan matrices, T-systems, Coxeter groups, W-graphs.

1 Introduction

Cluster algebras were first introduced in [2] by Fomin and Zelevinsky with applications to dual canonical bases, total positivity in semisimple Lie groups, and Zamolodchikov periodicity for Y-systems of finite root systems. For a generic cluster algebra, repeatedly applying the same nontrivial mutation sequence $\mu_1\mu_2\cdots\mu_k$ will yield infinitely many cluster variables. However, if $\mathcal{A}$ is a cluster algebra of finite type, any mutation sequence will result in finitely many cluster variables. In 2003, Fomin and Zelevinsky proved that the finite type cluster algebras are in bijection with finite type Cartan matrices [3]. In particular, the finite type quiver cluster algebras are in bijection with the simply-laced (type A, D, or E) Cartan matrices, while the finite type cluster algebras with skew-symmetrizable exchange matrices correspond to the (not necessarily ADE) Cartan matrices.

For our problem, we focus on a specific bipartite mutation sequence and analyze which cluster algebras yield finitely many cluster variables under this sequence.

Let $[n] := \{1, 2, \dots, n\}$. An $n \times n$ matrix B is *bipartite* if there exists a coloring $\epsilon : [n] \to \{\circ, \bullet\}$ of $[n]$ such that for all i, j of the same color, $b_{ij} = b_{ji} = 0$. It is well known that mutation is a local move, so mutations μ_i and μ_j commute if $b_{ij} = b_{ji} = 0$. Thus in a bipartite matrix, we may define the *bipartite mutations* $\mu_\circ$ and $\mu_\bullet$ as the combined mutation at all white vertices and all black vertices, respectively. We say a bipartite matrix B is *recurrent* if $\mu_\circ(B) = \mu_\bullet(B) = -B$. In particular, $\mu_\circ\mu_\bullet(B) = B$. A bipartite recurrent matrix B is *Zamolodchikov periodic* if the sequence of mutations $\mu_\circ\mu_\bullet\mu_\circ\mu_\bullet \dots$ acting on the seed is periodic.

Zamolodchikov periodicity was first observed by Zamolodchikov in his study [16] of the thermodynamic Bethe ansatz, initially focusing on the simply-laced Dynkin diagrams A_n, D_n, and E_n. His conjecture was generalized by Ravanini–Valeriani–Tateo [13], Kuniba–Nakanishi [11], and Kuniba–Nakanishi–Suzuki [12] in directions different from the framework considered in this abstract. The conjecture was proved for finite type cluster algebras (Dynkin diagrams) by Fomin–Zelevinsky [4]. It was also proved for tensor products $\Gamma \otimes \Delta$ of two Dynkin diagrams by Keller [10] and in a different way by Inoue–Iyama–Keller–Kuniba–Nakanishi [7, 8]. More recently, Zamolodchikov periodic quivers were fully classified by Galashin–Pylyavskyy in 2019 [5].

The following theorem is the main result.

Theorem 1. *The Zamolodchikov periodic B-matrices are in natural bijection with pairs of commuting Cartan matrices.*

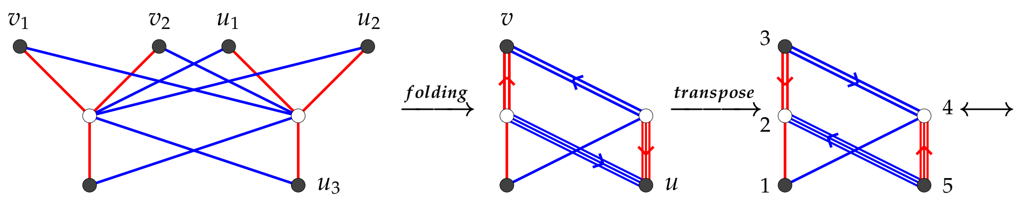
Remark 2. *It is worth noting that the different Zamolodchikov periodic B-matrices do not necessarily generate distinct cluster algebras. For example, there are two distinct Zamolodchikov periodic quivers, D_4 and $A_2 \otimes A_2$, that produce the same cluster algebra. D_4 has a period of length 8, while $A_2 \otimes A_2$ has a period of length 6.*

The proof of Theorem 1 is broken into two parts. First, we classify all pairs of commuting Cartan matrices as objects called admissible Dynkin biagrams.

Theorem 3. *Pairs of commuting Cartan matrices are in natural bijection with admissible Dynkin biagrams, which are made up of 35 infinite families and 25 exceptional types.*

Remark 4. *Notably, the pairs of commuting ADE Cartan matrices were classified in [15] into 6 infinite families and 11 exceptional types.*

Then, we prove that these objects are in bijection with Zamolodchikov periodic B-matrices using two operations: folding and taking transpose ($B \mapsto B^T$); see Figure 1. It is a standard fact that folding the B-matrix commutes with cluster mutations and thus preserves Zamolodchikov periodicity. On the other hand, taking transpose does not commute with cluster mutations, and thus it is surprising that it preserves the class of Zamolodchikov periodic B-matrices.



	1	2	3	4	5
1	0	−1	0	1	0
2	1	0	2	0	−3
3	0	−1	0	1	0
4	−1	0	−2	0	3
5	0	1	0	−1	0

Figure 1: A sequence of folds and global flips from $D_4 \times D_4$ to $B_3 \bowtie_1 G_2$ to $C_3 \bowtie_2 G_2$ with its corresponding B-matrix. The vertices v_1, v_2 and u_1, u_2, u_3 indicate the nontrivial orbits in the bicolored automorphism.

2 Background

2.1 Cluster Algebras

An $n \times n$ matrix B is *skew-symmetrizable* if there exists some nonzero scaling of the rows $c \in \mathbb{R}^n_{>0}$ that makes the matrix skew-symmetric. In the context of cluster algebras, we call such matrices *B-matrices*. A *quiver* is a directed graph without loops and cycles of length 2. Each quiver has an associated adjacency matrix which gives a natural bijection between quivers and skew-symmetric B-matrices. Given a *seed* $(\boldsymbol{x}, B)$ made up of a *cluster* $\boldsymbol{x} = (x_1, \ldots, x_n)$ and a B-matrix B, one can *mutate* at $k \in [n]$ to generate a new seed $(\boldsymbol{x'}, B')$ whose *cluster variables* $x'_1, \ldots, x'_n$ are elements of $\mathbb{C}(x_1, \ldots, x_n)$.

The *cluster algebra* $\mathcal{A}$ of geometric type over $\mathbb{C}$ generated by the seed $(\boldsymbol{x}, B)$ is the $\mathbb{C}$-subalgebra of $\mathbb{C}(x_1, \ldots, x_n)$ generated by all cluster variables across all mutations.

2.2 Dynkin biagrams

Given a Dynkin diagram Λ with Cartan matrix C_Λ, we let $A_\Lambda := 2I - C_\Lambda$ denote the *Coxeter adjacency matrix* of Λ. A *Dynkin biagram* is a tuple (Γ, Δ) of two (not necessarily irreducible) $n \times n$ Coxeter adjacency matrices such that their sum $B = \Gamma + \Delta$ is bipartite and Γ, Δ share no nonzero entries. The *dual* of a Dynkin biagram (Γ, Δ), denoted $(\Gamma, \Delta)^*$, is defined to be (Δ, Γ). Finally, a Dynkin biagram (Γ, Δ) is *admissible* if the associated Cartan matrices $C_\Gamma = 2I - \Gamma$, $C_\Delta = 2I - \Delta$ commute, or equivalently if Γ and Δ commute.

Each Dynkin biagram has a corresponding B-matrix (see Figure 1). We depict Dynkin biagrams as a union of red Dynkin diagram components associated to Γ connected by blue Dynkin diagram components associated to Δ.

Remark 5. *If (Γ, Δ) is an admissible Dynkin biagram, then all the components of Γ (resp., Δ) have the same Coxeter number h_Γ (resp., h_Δ).*

In the case of simply-laced Γ, Δ, these objects are known as *ADE bigraphs*, and have corresponding skew-symmetric B-matrices. In [15], Stembridge classifies all connected

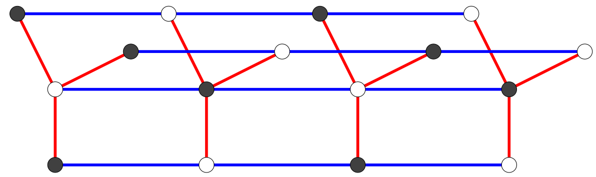


Figure 2: The tensor product $D_4 \otimes A_4$.

admissible ADE bigraphs into 6 infinite families and 11 exceptional types. Below, we list two of these families whose definition extends to Dynkin biagrams.

Tensor products. Let Λ and Λ' be two Dynkin diagrams of rank n and m, respectively. The *tensor product*, or *box product*, of Λ and Λ' is denoted $\Lambda \otimes \Lambda'$. It consists of m red copies of Λ connected to each other with blue edges. The red and blue Coxeter adjacency matrices are Kronecker products of the form $A_\Lambda \otimes I_m$ and $I_n \otimes A_{\Lambda'}$, respectively. From this, it is straightforward to see that tensor products are admissible. An example of the tensor product $D_4 \otimes A_4$ is given in Figure 2.

Twists. Twists are denoted $\Lambda \times \Lambda$, and consist of two red copies of Λ connected to each other with blue edges. The red and blue Coxeter adjacency matrices are of the form

$$\Gamma = \begin{bmatrix} A_\Lambda & 0 \\ 0 & A_\Lambda \end{bmatrix}, \qquad \Delta = \begin{bmatrix} 0 & A_\Lambda \\ A_\Lambda & 0 \end{bmatrix}, \tag{2.1}$$

where A_Λ is the Coxeter adjacency matrix of Λ. An example of the twist $D_4 \times D_4$ is given in Figure 1.

Remark 6. *One can also visualize admissibility of an ADE bigraph. The number of length two red-blue paths from i to j is the same as the number of length two blue-red paths from i to j for every pair of vertices i, j if and only if this ADE bigraph is admissible.*

2.3 Bindings and Folding

Definition 7. *Let (Γ, Δ) be a Dynkin biagram. If $\Gamma = \Gamma_1 \sqcup \Gamma_2$ and every edge of Δ connects Γ_1 to Γ_2, then (Γ, Δ) is a binding of Γ_1 and Γ_2.*

For instance, the tensor product of Γ with A_2, B_2, or G_2 is a binding of two copies of Γ. We use the notation

$$(\Gamma \equiv \Gamma) := \Gamma \otimes A_2, \quad (\Gamma \equiv_2 \Gamma) := \Gamma \otimes B_2, \quad (\Gamma \equiv_3 \Gamma) := \Gamma \otimes G_2,$$

and refer to these as *parallel bindings*.

In the special case that (Γ, Δ) and (Δ, Γ) are both bindings, we refer to this as a *double binding*. For example, the twist $\Gamma \times \Gamma$ is a double binding of two copies of Γ.

Definition 8. *A bicolored automorphism $f : [n] \to [n]$ is an automorphism that satisfies the following.*

(i) $\epsilon_{f(i)} = \epsilon_i$ *(preserves bipartite coloring of vertices).*

(ii) For all i_1, i_2 *in the same* f*-orbit* I*, and for all* $j \in [n]$*,* $b_{i_1 j} b_{i_2 j} \geq 0$ *(preserves edge colors).*

(iii) For all $i, j \in [n]$*,* $b_{f(i)f(j)} = b_{ij}$*.*

(iv) For all i, j *in the same orbit* I*,* $b_{ij} = b_{ji} = 0$*.*

Definition 9. *Let B be a bipartite skew-symmetrizable matrix, and let f be a bicolored automorphism on B. Then, one may fold B to get $f(B)$, another bipartite skew-symmetrizable matrix with rows and columns indexed by f-orbits, defined as*

$$f(B)_{IJ} := \sum_{i \in I} b_{ij}, \text{ for any } j \in J.$$

In particular, the result does not depend on $j \in J$.

With a bicolored automorphism, one can fold bindings to obtain other bindings. For instance, there exists a bicolored automorphism that folds the exceptional binding $D_5 \boxtimes A_7$ into an exceptional binding of B_4 and C_4, denoted $B_4 \boxtimes C_4$ (see Figure 3).

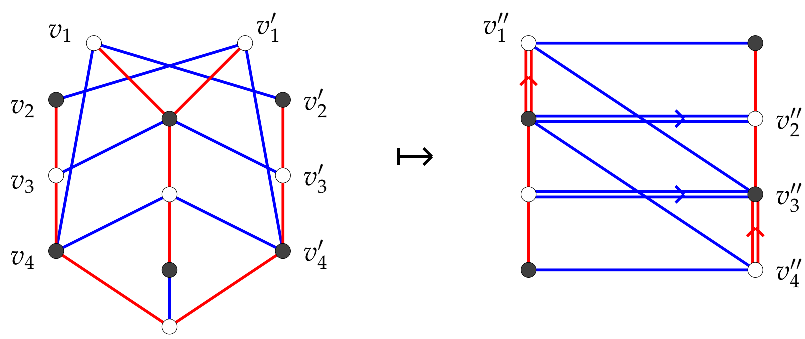


Figure 3: The folding of exceptional type $D_5 \boxtimes A_7$ to $B_4 \boxtimes C_4$. The vertices v_i and v_i' form the f-orbit v_i'' for each $i \in \{1, 2, 3, 4\}$. This folds D_5 to B_4 and A_7 to C_4.

In addition, with a valid bicolored automorphism that folds Dynkin diagram Γ to Dynkin diagram Λ, one can fold the twist $\Gamma \times \Gamma$ into another binding. We use the notation $\Lambda \ltimes \Gamma$ and $\Gamma \rtimes \Lambda$ to denote a binding of Λ and Γ obtained by folding one side of a twist, and $\Lambda_1 \bowtie \Lambda_2$ to indicate a binding of Λ_1 and Λ_2 obtained by folding both sides of the twist $\Gamma \times \Gamma$. Two examples of folded twists are depicted in Figures 1 and 4.

Remark 10. *In general, C_n is only defined for $n \geq 3$. However, we sometimes use C_2 to denote the opposite orientation of B_2. This notation is useful in distinguishing between bindings $B_n \ltimes D_{n+1}$ and $C_n \ltimes D_{n+1}$ when $n = 2$ (see Figure 4).*

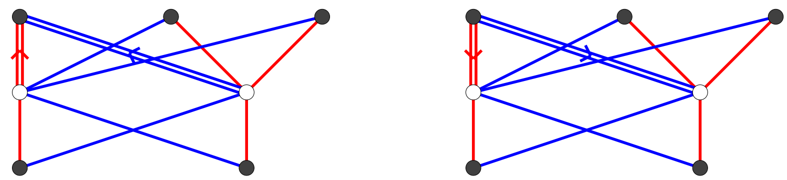


Figure 4: The double bindings $B_3 \ltimes D_4$ and $C_3 \ltimes D_4$.

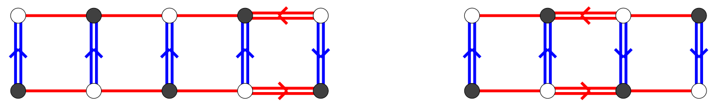


Figure 5: The bindings $B_5 * C_5$ and $F_4 * F_4$.

Proposition 11 ([15])**.** *A Dynkin biagram is admissible if and only if all of its bindings are admissible.*

By definition, all parallel bindings are tensor products, which are admissible. It is also clear from the form of twists (2.2) that Γ and Δ commute. Using Proposition 11, we can string together various sequences of parallel bindings, twists, and folded twists to form admissible Dynkin biagrams. For example,

$$\begin{aligned} B_n * C_n &:= (B_2 \equiv B_2 \equiv \cdots \equiv B_2 \times B_2)^*, \\ F_4 * F_4 &:= (B_2 \equiv B_2 \times B_2 \equiv B_2)^*, \\ E_6 *_1 F_4 &:= (B_2 \equiv B_2 \ltimes D_3 \equiv D_3)^*, \text{ and} \\ E_6 *_2 F_4 &:= (C_2 \equiv C_2 \ltimes D_3 \equiv D_3)^* \end{aligned}$$

are all admissible bindings (see Figures 5 and 6).

Lemma 12 (Folding biagrams)**.** *Let (Γ, Δ) be an admissible Dynkin biagram and let f be a bicolored automorphism. Then folding along f produces another admissible Dynkin biagram (Γ', Δ').*

The following operation plays an important role in the proof of the bijection between admissible Dynkin biagrams and Zamolodchikov periodic cluster algebras in Section 4.

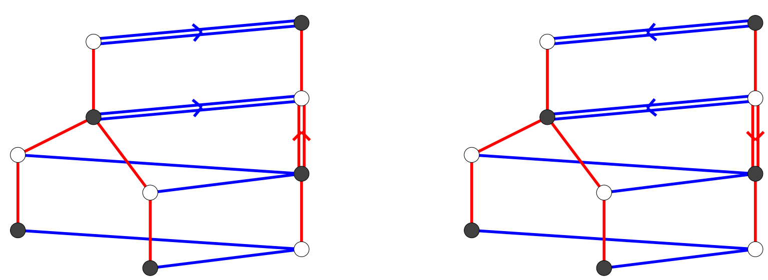


Figure 6: The bindings $E_6 *_1 F_4$ and $E_6 *_2 F_4$.

Definition 13 (Global flip). *Given a Dynkin biagram (Γ, Δ), we can produce another Dynkin biagram by taking the transpose $(\Gamma^\top, \Delta^\top)$. We call this operation a global flip of a Dynkin biagram.*

Notice that if (Γ, Δ) is a pair of commuting Cartan matrices, then $(\Gamma^\top, \Delta^\top)$ is also a pair of commuting Cartan matrices. In other words, admissibility is preserved under global flips. Figures 4 and 6 are examples of global flips.

Remark 14. *This operation is equivalent to taking the Langlands dual of a group. In particular, if G is a reductive algebraic group with Dynkin diagram D, its Langlands dual ${}^LG$ has Dynkin diagram D', where D' is the global flip of D.*

3 Classification of commuting Cartan matrices as admissible Dynkin biagrams

Much of the work of classification was in classifying the non-ADE double bindings by analyzing potential pairings of Dynkin diagrams and dominant eigenvectors of their corresponding Cartan matrices.

Theorem 15. *The admissible non-ADE double bindings are*

(a) parallel bindings $\Gamma \equiv \Gamma$, $\Gamma \equiv_2 \Gamma$, and $\Gamma \equiv_3 \Gamma$ for $\Gamma \in \{A_2, B_2, G_2\}$,

(b) twists $\Gamma \times \Gamma$ for non-ADE Γ,

(c) $B_n \ltimes D_{n+1}$, $C_n \ltimes D_{n+1}$, $B_n \bowtie C_n$, $G_2 \ltimes_{1,2} D_4$, $B_3 \bowtie_{1,2} G_2$, $C_3 \bowtie_{1,2} G_2$, and

(d) the exceptional binding $B_4 \boxtimes C_4$.

The idea of the proof of Theorem 3 uses Proposition 11 and is based on a basic argument adapted from [15]. Let Γ have k components, $\Gamma_1, \ldots, \Gamma_k$. We define $\mathcal{C}$ to be the component graph on the vertex set $[k]$ that has an edge $\{i, j\}$ if there exists an edge of Δ that connects vertices of Γ_i and Γ_j. Such an edge then denotes the binding of Γ_i and Γ_j.

Since every admissible Dynkin biagram can be formed through a series of taking duals and gluing together bindings, the class of connected admissible Dynkin biagrams is the smallest class $\mathcal{G}$ that contains

(i) $A_1 \otimes A_1$, $A_1 \otimes A_2$, $A_1 \otimes B_2$, $A_1 \otimes G_2$ and all admissible Dynkin double bindings,

(ii) all admissible Dynkin biagrams obtained by gluing together bindings in $\mathcal{G}$ along some connected component graph $\mathcal{C}$, and

(iii) the dual of every biagram (Γ, Δ) in $\mathcal{G}$ such that Δ has at most 2 components.

Thus, we need only show that the list of families of non-ADE Dynkin biagrams satisfies these three conditions. This leads us to the following classification.

Theorem 16. *The connected admissible non-ADE Dynkin biagrams are made up of 29 infinite families and 14 exceptional types, including all double bindings (see [1] for the full list). Moreover, any admissible Dynkin biagram can be obtained from an ADE bigraph through a sequence of global flips and folds along bicolored automorphisms.*

4 Admissible Dynkin biagrams satisfy Z-periodicity

Let B be a bipartite recurrent B-matrix with $|B| = \Gamma + \Delta$, and let $\lambda \in \mathbb{R}^n$ be a labeling of the n indices. The *tropical T-system* $\boldsymbol{t}^\lambda$ associated with B is a family $\boldsymbol{t}_k^\lambda(t) \in \mathbb{R}$ of real numbers with *tropical mutation* defined as follows:

$$\mu_k(\boldsymbol{t}_k^\lambda(t-1)) := \boldsymbol{t}_k^\lambda(t+1) = \max\left(\sum_i \Gamma_{ik}\boldsymbol{t}_i^\lambda(t), \sum_j \Delta_{jk}\boldsymbol{t}_j^\lambda(t)\right) - \boldsymbol{t}_k^\lambda(t-1), \tag{4.1}$$

where the system is defined only for $k \in [n]$ and $t \in \mathbb{Z}$ such that

$$\epsilon_k = \circ \quad \text{and} \quad t \equiv 0 \pmod 2 \qquad \text{or} \qquad \epsilon_k = \bullet \quad \text{and} \quad t \equiv 1 \pmod 2. \tag{4.2}$$

The tropical T-system is set to the following initial conditions:

$$\boldsymbol{t}_k^\lambda(0) = \lambda_k \quad \text{and} \quad \boldsymbol{t}_k^\lambda(1) = \lambda_k, \quad \text{for all } k \text{ satisfying (4.2)}.$$

We say the associated tropical T-system $\boldsymbol{t}^\lambda$ is *periodic* if there exists a positive integer N such that $\boldsymbol{t}_k^\lambda(t+2N) = \boldsymbol{t}_k^\lambda(t)$ for all $k \in [n], t \in \mathbb{Z}$.

Proposition 17. *Let B be a bipartite recurrent B-matrix. Then, B is Zamolodchikov periodic if and only if its associated tropical T-system $\boldsymbol{t}^\lambda$ is periodic for all $\lambda \in \mathbb{R}^n$.*

In this section, we outline one direction of the proof of Theorem 1. Galashin and Pylyavskyy proved in [5] that admissible ADE bigraphs have periodic tropical T-systems. Recall from Theorem 16 that every admissible Dynkin biagram can be obtained from an admissible ADE bigraph through a sequence of global flips and folds along bicolored automorphisms.

The following two lemmas show that both of these operations work well with tropical mutations.

Lemma 18. *Let $|B| = \Gamma + \Delta$, where B is a bipartite recurrent B-matrix. Let C be a diagonal matrix with positive diagonal entries $c \in \mathbb{R}^n_{>0}$ such that CB is a skew-symmetric matrix. Let $\rho \in \mathbb{R}^n$ be a labeling of B. Define $\lambda \in \mathbb{R}^n$ as $\lambda_i := \frac{1}{c_i}\rho_i$, for all $i \in [n]$. Let*

$$\rho'_k + \rho_k = \max\left(\sum_i \Gamma_{ik}\rho_i, \sum_j \Delta_{jk}\rho_j\right) \quad \textit{and} \quad \lambda'_k + \lambda_k = \max\left(\sum_i \Gamma_{ki}\lambda_i, \sum_j \Delta_{kj}\lambda_j\right)$$

for $k \in [n]$*. Then,* $\lambda'_k = \frac{1}{c_k}\rho'_k$*.*

Lemma 19. *Let* (B, f, ρ) *be a B-matrix, a bicolored automorphism, and a labeling such that* $\rho_{f(i)} = \rho_i$*, for all* $i \in [n]$*. We may apply* f *to any such labeling to create a new labeling where* $f(\rho)_I = \rho_i$ *for any element* i *in the orbit* I*. Then,* $\mu_I \circ f(\rho) = f \circ \prod_{i \in I} \mu_i(\rho)$*.*

From these lemmas, it is clear that if we start with a periodic tropical T-system, applying folding or global flips yields another system that is still periodic. With the help of Theorem 16 and Proposition 17, we have the following proposition.

Proposition 20. *The tropical T-system associated to an admissible Dynkin biagram* (Γ, Δ) *is periodic, with period dividing* $h_\Gamma + h_\Delta$*.*

5 Colored Mutations

Theorem 1 has further implications that hint at a strong connection between root systems and bipartite cluster algebras. Let B be a Zamolodchikov periodic B-matrix, and let $\boldsymbol{t}_i^\lambda$ be its corresponding tropical T-system. Recall the form (4.1) for tropical mutation. For a generic initial labeling $\lambda \in \mathbb{R}^n$, we can color this mutation either red or blue, depending on the evaluation of the max function. If the maximum is $\sum_i \Gamma_{ik} \boldsymbol{t}_i^\lambda(t)$, then we call this a Γ-mutation, and color it red. Otherwise it is a Δ-mutation, and color it blue.

Recall that the number of roots in a root system is $h \cdot r$, where h is the Coxeter number and r is the rank. If we color mutations for one period $h_\Gamma + h_\Delta$, we observe an interesting pattern:

Conjecture 21. *Let* (Γ, Δ) *be an admissible Dynkin biagram with r vertices. The number of* Γ*-mutations in one period is the number of roots in the root systems associated to each* Γ*-component, totaling to* $h_\Gamma \cdot r$*. Similarly, the number of* Δ*-mutations in one period is the number of roots in the root systems associated to each* Δ*-component,* $h_\Delta \cdot r$*.*

This conjecture has been verified by SageMath on all non-ADE families in Theorem 16, as well as all infinite ADE families for $n \leq 10$.

Example 22. *Let us consider the Dynkin biagram given by* $A_2 \otimes A_3$*, with* Γ *consisting of two* A_3 *components, and* Δ *consisting of three* A_2 *components. The Coxeter numbers of each are* $h_\Gamma = 4$ *and* $h_\Delta = 3$*, so the length of one period is* $h_\Gamma + h_\Delta = 7$*. In the tropical T-system, this corresponds to iterating t by 14 time steps, as each time step only mutates half of the vertices.*

Suppose our initial labeling is $\lambda = [2, 0, -0.9, 1, -1, 4] \in \mathbb{R}^6$*. In Table 1, we can see the evolution of the T-system* $\boldsymbol{t}^\lambda(t)$ *over half a period of mutations. In this case, the second half of the period behaves identically. Indeed, we see that the total number of* Γ*-mutations in one period is* $6h_\Gamma = 24$*, and the total number of* Δ*-mutations in one period is* $6h_\Delta = 18$*.*

t	0	1	2	3	4	5	6	7
$t^\lambda(t)$	2 0 −0.9 1 −1 4	**-1** **4.9** **6**	**6** **5** **2**	**7** **1.1** **1**	**2.1** **2** **-0.9**	**-4.9** **1** **1.1**	**-1** **-0.9** **2**	**4** **1** **0**

Table 1: The evolution of the tropical T-system of type $A_2 \otimes A_3$ for one half period. The red boldface numbers are the Γ-mutations, while the blue boldface numbers are the Δ-mutations.

In fact, one can visualize the space of all initial labelings $\lambda \in \mathbb{R}^r$ as partitioned into chambers based on the sequence of colored mutations λ produces in one period. Going from one chamber to another involves crossing a linear wall which swaps some number of red and blue mutations in the period. For the small case of single Dynkin diagrams, we observed the following, implying a potential bijection between chambers and clusters.

Conjecture 23. *Let $\Lambda \in \{A_{2n}, B_{2n}, C_{2n}, E_n, F_4, G_2\}$ be a Dynkin diagram of rank r. Consider the Dynkin biagram $\Lambda \otimes A_1$. The corresponding fan in the space of initial labelings $\mathbb{R}^r$ has c_Λ distinct chambers, where c_Λ is the number of seeds in the cluster algebra of type Λ.*

The following conjecture generalizes Conjecture 21, and applies to affine $\boxtimes$ finite ADE bigraphs as defined in [6].

Conjecture 24. *Let (Γ, Δ) be an admissible biagram with r vertices where the components of Γ are affine Dynkin diagrams and the components of Δ are Dynkin diagrams. The number of Δ-mutations is bounded above by the number of Δ-roots, $h_\Delta \cdot r$.*

This conjecture has also been verified by SageMath on all 19 families in the classification from [6] of affine $\boxtimes$ finite ADE bigraphs for $n \leq 10$.

6 Connections to W-graphs

In 2010, Stembridge studied admissible ADE bigraphs, as they helped classify the admissible $I_2(p) \times I_2(q)$-cells, which encode the action of the standard generators inherited from $I_2(p) \times I_2(q)$ on the Kazhdan–Lusztig basis of its associated Iwahori–Hecke algebra $\mathcal{H}(W)$. In this section, we explore which W-graphs are associated with admissible Dynkin biagrams. We adopt most of our notation from [14] and [15].

Let I be a finite index set. An *I-labeled graph* is a triple $\Lambda = (V, m, \tau)$ such that

(i) V is a vertex set of size $n \in \mathbb{N}$,

(ii) $m \in \mathrm{Mat}_{n\times n}(\mathbb{Z}[q^{\pm 1/2}])$ is a weighted adjacency matrix, and

(iii) $\tau : V \to \{\text{subsets of } I\}$ is a labeling of the vertices, called the *τ-invariant*.

Now, let W be a Coxeter group relative to a generating set $S = \{s_i \mid i \in I\}$. Let $\{T_i \mid i \in I\}$ be the corresponding set of generators of the associated Iwahori–Hecke algebra $\mathcal{H}$ over the ground ring $\mathbb{Z}[q^{\pm 1/2}]$. Recall that the defining relations for $\mathcal{H}$ are the quadratic relations $(T_i - q)(T_i + 1) = 0$ for all $i \in I$, and the braid relations inherited from (W, S).

Definition 25 (W-graph)**.** *An I-labeled graph Λ is a W-graph if the $\mathbb{Z}[q^{\pm 1/2}]$-module M_Λ freely generated by vertex set V may be given the $\mathcal{H}$-module structure such that for all $u \in V$,*

$$T_i(u) = \begin{cases} qu & \text{if } i \notin \tau(u), \\ -u + q^{1/2} \sum_{v:\, i \notin \tau(v)} m_{uv} v & \text{if } i \in \tau(u). \end{cases}$$

In particular, these operators always satisfy the quadratic relations $(T_i - q)(T_i + 1) = 0$, so the only nontrivial condition to satisfy is maintaining the braid relations.

In addition, if $m_{uv} = 0$ whenever $\tau(u) \subseteq \tau(v)$, then we say the I-labeled graph is *reduced*. Notice that when $\tau(u) \subseteq \tau(v)$, the value of m_{uv} never plays a role in the operators defined above. So without loss of generality, we assume our W-graphs are always reduced. A W-graph is a *W-cell* if its matrix of edge weights m is strongly connected. Since the vertex set of any directed graph can be uniquely partitioned into strongly connected components, it can be shown that a W-graph Λ can be broken into W-cells. If Λ is a reduced W-graph, then a vertex v with $\tau(v) = \emptyset$ or $\tau(v) = [n]$ necessarily has outdegree 0 or indegree 0. So in this case, $\{v\}$ forms its own cell of Λ. Such cells are called *trivial*.

Given a Coxeter group W, one can construct a W-graph (V, m, τ) by taking the matrices representing the action of the standard generators T_i on the Kazhdan–Lusztig basis of $\mathcal{H}$. In particular, the matrix m of edge weights is exactly a function of the coefficients in the Kazhdan–Lusztig polynomial $P_{u,v}(q)$. For more details on the construction, we refer the reader to [9].

Remark 26. *In this section, we focus on classifying W-cells that are bipartite with nonnegative edge weights. This is a generalization of admissible W-cells as defined in [14], which captures several combinatorial features of Kazhdan–Lusztig W-graphs. Indeed, since $P_{u,v}(q)$ only has integral powers of q, it follows that m is bipartite. Additionally, for a finite Coxeter group W, its associated Kazhdan–Lusztig polynomial $P_{u,v}(q)$ has nonnegative integer coefficients, so these W-graphs have nonnegative edge weights.*

Lemma 27. *A bipartite strongly connected multigraph with vertex color labels $\{1\}, \{2\}$ is a nontrivial $I_2(p)$-cell with nonnegative edge weights m if and only if it is a Dynkin diagram of finite type whose Coxeter number divides p.*

Theorem 28. *The $I_2(p) \times I_2(q)$-cells with nonnegative edge weights are in correspondence with certain $\{1, 2, 3, 4\}$-labeled connected admissible Dynkin biagrams (Γ, Δ) such that h_Γ divides p and h_Δ divides q. In particular, each such Dynkin biagram has four distinct labelings that produce an $I_2(p) \times I_2(q)$-cell.*

References

[1] A. Chin. “Classification of Zamolodchikov periodic cluster algebras”. *arXiv:2510.18031* (2025). Link.

[2] S. Fomin and A. Zelevinsky. “Cluster Algebras I: Foundations”. *J. Amer. Math. Soc.* **15**.2 (2002), pp. 497–529. DOI.

[3] S. Fomin and A. Zelevinsky. “Cluster Algebras II: Finite type classification”. *Invent. math.* **154** (2003), pp. 63–121. DOI.

[4] S. Fomin and A. Zelevinsky. “Y -systems and generalized associahedra”. *Ann. of Math.* **158**.3 (2003), pp. 977–1018. DOI.

[5] P. Galashin and P. Pylyavskyy. “The classification of Zamolodchikov periodic quivers”. *Amer. J. Math.* **141**.2 (2019), pp. 447–484. DOI.

[6] P. Galashin and P. Pylyavskyy. “Quivers with subadditive labelings: classification and integrability”. *Math. Z.* **295** (2020), pp. 945–992. DOI.

[7] R. Inoue, O. Iyama, B. Keller, A. Kuniba, and T. Nakanishi. “Periodicities of T-systems and Y-systems, Dilogarithm Identities, and Cluster Algebras I: Type Br”. *Publ. Res. Inst. Math. Sci.* **49**.1 (2013), pp. 1–42. DOI.

[8] R. Inoue, O. Iyama, B. Keller, A. Kuniba, and T. Nakanishi. “Periodicities of T-systems and Y-systems, dilogarithm identities, and cluster algebras II: types Cr, F4, and G2”. *Publ. Res. Inst. Math. Sci.* **49**.1 (2013), pp. 43–85. DOI.

[9] D. Kazhdan and G. Lusztig. “Representations of Coxeter groups and Hecke algebras”. *Invent. math.* **53** (1979), pp. 165–184. DOI.

[10] B. Keller. “The periodicity conjecture for pairs of Dynkin diagrams”. *Ann. of Math. (2)* **177**.1 (2013), pp. 111–170. DOI.

[11] A. Kuniba and T. Nakanishi. “Spectra in conformal field theories from the Rogers dilogarithm”. *Modern Phys. Lett. A* **7**.37 (1992), pp. 3487–3494. DOI.

[12] A. Kuniba, T. Nakanishi, and J. Suzuki. “Functional relations in solvable lattice models. I. Functional relations and representation theory”. *Internat. J. Modern Phys. A* **9**.30 (1994), pp. 5215–5266. DOI.

[13] F. Ravanini, A. Valleriani, and R. Tateo. “Dynkin TBAs”. *Internat. J. Modern Phys. A* **8**.10 (1993), pp. 1707–1727. DOI.

[14] J. R. Stembridge. “Admissible W-graphs”. *Represent. Theory* **12** (2008), pp. 346–368. DOI.

[15] J. R. Stembridge. “Admissible W-graphs and commuting Cartan matrices”. *Adv. in Appl. Math.* **44**.3 (2010), pp. 203–224. DOI.

[16] A. B. Zamolodchikov. “On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories”. *Phys. Lett. B* **253**.3-4 (1991), pp. 391–394. DOI.