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Totally positive skew-symmetric matrices

Jonathan Boretsky*[1], Veronica Calvo Cortes†[2] and Yassine El Maazouz‡[3]

[1]*McGill University, Montreal, Canada*
[2] *Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany*
[3] *California Institute of Technology, Pasadena, U.S.A.*

Abstract. A matrix is totally positive if all of its minors are positive. The classical notion of total positivity coincides with the type A version of Lusztig's more general total positivity in reductive real-split algebraic groups. Since skew-symmetric matrices always have nonpositive entries, they are not totally positive in the classical sense. The space of skew-symmetric matrices is an affine chart of the orthogonal Grassmannian $\mathrm{OGr}(n, 2n)$. Thus, we define a skew-symmetric matrix to be *totally positive* if it lies in the *totally positive orthogonal Grassmannian*. We provide a positivity criterion for these matrices in terms of a fixed collection of minors, and show that their Pfaffians have a remarkable sign pattern. The totally positive orthogonal Grassmannian is a CW cell complex and is subdivided into *Richardson cells*. We introduce a method to determine which cell a given point belongs to in terms of its associated matroid.

Keywords: Orthogonal Grassmannian, Total positivity, Pfaffians, Skew-symmetric matrices, Spinors

1. Introduction

Let $n \geq 1$ be a positive integer and denote by $\mathbb{S}_n = \mathbb{S}_n(\mathbb{R})$ the $\binom{n}{2}$-dimensional real vector space of skew-symmetric $n \times n$ matrices with real entries. We consider a bilinear form q in $\mathbb{R}^{2n}$ given in the standard basis by the following matrix where Id_n is the $n \times n$ identity

$$Q = \begin{bmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{bmatrix}.$$

In Section 2.1 we will realize $\mathbb{S}_n$ as a subset of the orthogonal flag variety by considering the rowspan of the $n \times 2n$ matrix $\begin{bmatrix}\mathrm{Id}_n | A\end{bmatrix}$. Hence, we can use Lusztig's notion of total positivity in flag varieties to define a *totally positive skew-symmetric matrix*.

Given a point in any partial flag variety, it is difficult to determine directly from the definition whether it is totally positive. Accordingly, positivity tests for certain partial

*jonathan.boretsky@mcgill.ca
†veronica.calvo@mis.mpg.de
‡maazouz@caltech.edu

flag varieties have been developed, for example [3, 8]. However, these are sometimes not very explicit. Explicit tests for positivity have been described in type A [4, 7, 14] and for certain flag varieties of types B and C [2]. Here we present an explicit and minimal positivity test for a skew symmetric matrix A in terms of its minors, which mirrors the fact that total positivity on $\mathrm{SL}(n)$ is determined by the positivity of minors.

Theorem 1.1. *There exists a collection $S \subset \binom{[2n]}{n}$ such that the positivity of a skew-symmetric matrix A is determined by the positivity of the maximal minors of $[\mathrm{Id}_n|A]$ indexed by elements of S. Concretely, $A \in \mathbb{S}_n$ is totally positive if and only if for all $I \in S$*

$$(-1)^{\epsilon_I}\Delta_I(X) > 0, \quad \text{where } \epsilon_I \text{ depends only on } I \text{ and } n.$$

Total positivity for reductive groups was introduced by Lusztig in the context of (dual) canonical bases in representation theory [18]. A key example is the case of $\mathrm{SL}(n)$. A parabolic quotient of $\mathrm{SL}(n)$ is a *flag variety* whose points are flags of linear subspaces. Such flags can be represented as row spans of matrices in $\mathrm{SL}(n)$. Lusztig's total positivity then matches the classical notion of total positivity: a flag is totally positive (resp. nonnegative) if it can be represented by a totally positive (resp. nonnegative) matrix, that is, one whose minors are all positive (resp. nonnegative). In general, the totally nonnegative part of a flag variety admits a nice topological structure that interplays well with matroid theory [11, 22]. Total positivity is ubiquitous in a wide range of areas in pure and applied mathematics: particle systems [13], matrix theory and splines [23], analysis [10] and many more. In particular, we point out the increasing role of positroid combinatorics in integrable systems [16] and scattering amplitudes [1].

Outline: We begin by collecting some necessary background in Section 2. In Section 3, we give our explicit positivity criterion for skew-symmetric matrices. Section 4 focuses on nonnegative skew-symmetric matrices $\mathbb{S}_n^{\geq 0}$: we provide a nonnegativity test and a combinatorial rule to determine the cell of a point in $\mathbb{S}_n^{\geq 0}$. In Section 5 we explain that the Pfaffians of a skew-symmetric matrix $A \in \mathbb{S}_n^{\geq 0}$ have a remarkable sign pattern.

2. Preliminaries

2.1 A pinch of Lie theory

The flag variety we will work with is the *orthogonal Grassmannian* $\mathrm{OGr}(n, 2n)$. This is the variety of n-dimensional vector subspaces V of $\mathbb{R}^{2n}$ that are *q-isotropic*, meaning that $q(v, w) = 0$ for any $v, w \in V$. The Zariski open set in $\mathrm{OGr}(n, 2n)$ where the Plücker coordinate $\Delta^{1,\ldots,n}$ does not vanish is isomorphic to the affine space $\mathbb{S}_n$. This identifies $A \in \mathbb{S}_n$ with the rowspan of the $n \times 2n$ matrix $[\mathrm{Id}_n|A]$.

We may also view $\mathrm{OGr}(n,2n)$ as the connected component of the identity in a parabolic quotient of the real special orthogonal group $\mathrm{SO}(2n)$. This is a connected reductive $\mathbb{R}$-split algebraic group and therefore admits a *totally positive part* in the sense of Lusztig [19].

Definition 2.1. A skew-symmetric matrix $A \in \mathbb{S}_n$ is *totally positive* (resp. *totally nonnegative*) if the rowspan of $\left[\mathrm{Id}_n | A\right]$ is in the totally positive region $\mathrm{OGr}^{>0}(n,2n)$ (resp. totally nonnegative region $\mathrm{OGr}^{\geq 0}(n,2n)$) of $\mathrm{OGr}(n,2n)$.

In [21, Theorem 11.3] Marsh and Rietsch give a parametrization of the totally positive part of the complete flag variety for any split reductive group G, with a choice of pinning, in terms of a reduced expression for the longest element in the Weyl group. Lusztig's characterization of positivity for flag varieties [20] guarantees that we can project this parametrization from the complete flag variety of G down to any partial flag variety. We describe this parametrization in our case of interest and refer the reader to [19, 20, 21] for the detailed construction.

We consider the Weyl group of $\mathrm{SO}(2n)$. This is the Coxeter group W generated by

$$s_i = (i,i+1)(n+i,n+i+1) \quad \text{for } i = 1,\dots n-1 \quad \text{and} \quad s_n = (n,2n-1)(n-1,2n), \tag{2.1}$$

We fix the following matrices representing the natural one parameter (positive) root subgroups of $\mathrm{SO}(2n)$ with our choice of quadratic form.

$$x_i(t) = \begin{array}{c} \\ \\ i \\ i+1 \\ \\ n+i \\ n+i+1 \\ \\ \\ \end{array} \begin{pmatrix} 1 &&&&&&&& \\ &\ddots&&&&&&& \\ &&1&t&&&&& \\ &&&1&&&&& \\ &&&&\ddots&&&& \\ &&&&&1&&& \\ &&&&&-t&1&& \\ &&&&&&&\ddots& \\ &&&&&&&&1 \end{pmatrix} \quad \text{for } 1 \leq i \leq n-1, \qquad x_n(t) = \begin{array}{cc} & \begin{array}{cccccccc} &&n-1&n&&&2n-1&2n \end{array} \\ \begin{array}{c} \\ \\ n-1 \\ n \\ \\ \\ 2n-1 \\ 2n \end{array} & \begin{pmatrix} 1 &&&&&&& \\ &\ddots&&&&&& \\ &&1&&&&&t \\ &&&1&&&-t& \\ &&&&1&&& \\ &&&&&\ddots&& \\ &&&&&&1& \\ &&&&&&&1 \end{pmatrix} \end{array}.$$

Note that instead of using an expression for w_0 in the Marsh–Rietsch parametrization of $\mathrm{OGr}(n,2n)^{>0}$, it suffices to use a representative of the coset $w_0 W_{[n-1]}$, where $W_{[n-1]} := \langle s_1,\dots,s_{n-1}\rangle$. Such a representative should have length $N := \binom{n}{2}$. We will find it useful to fix a reduced expression $\underline{w}_0^{[n-1]}$ for a minimal coset representative, that is.

$$\begin{aligned} \underline{w}_0^{[n-1]} &= s_n(s_{n-2}\cdots s_1)(s_{n-1}\cdots s_2)s_n(s_{n-2}\cdots s_3)(s_{n-1}\cdots s_4)\cdots s_n && \text{for } n \text{ even}, \\ \underline{w}_0^{[n-1]} &= s_n(s_{n-2}\cdots s_1)(s_{n-1}\cdots s_2)s_n(s_{n-2}\cdots s_3)(s_{n-1}\cdots s_4)\cdots s_n s_{n-2} s_{n-1} && \text{for } n \text{ odd}, \end{aligned} \tag{2.2}$$

Projecting the Marsh–Rietsch parametrization [21, Section 11] for $G = \mathrm{SO}(2n)$ yields

$$\mathrm{OGr}(n,2n)^{>0} = \left\{\pi_n\big(x_{i_1}(t_1)\cdots x_{i_N}(t_N)\big) : t_1,\dots,t_N \in \mathbb{R}_{>0}\right\}, \tag{2.3}$$

where $\pi_n : \mathrm{SO}(2n) \to \mathrm{OGr}(n, 2n)$ is the map that takes a $2n \times 2n$ orthogonal matrix to the row span of its first n rows, and $\underline{w}_0^{[n-1]} = s_{i_1} s_{i_2} \cdots s_{i_N}$. We denote by $X = X(t_1, \ldots, t_N)$ the $n \times 2n$ matrix in the first n rows of $x_{i_1}(t_1) \cdots x_{i_N}(t_N)$. Since the matrices x_i are all unipotent, we may row reduce X to the form $[\mathrm{Id}_n | A]$. As row-reducing does not change the row span of a matrix, $A = A(t_1, \ldots, t_N)$ is a unique skew-symmetric matrix with polynomial entries in the t_i. Thus, Definition 2.1 is equivalent to the following parametrized version of total positivity for skew-symmetric matrices.

Definition 2.2. A real skew-symmetric $n \times n$ matrix is *totally positive* if it is of the form $A(t_1, \ldots, t_N)$ with $t_1, \ldots, t_N \in \mathbb{R}_{>0}$.

2.2 The Lindström–Gessel–Viennot diagram

In this section, we introduce the Lindström–Gessel–Viennot (LGV) diagram which will be useful in the remainder of this article. Loosely, this diagram encodes the combinatorics that govern the multiplication of matrices $x_{i_1}(t_1) \cdots x_{i_N}(t_N)$ and hence also of $X = X(t_1, \ldots, t_N)$ and $A = A(t_1, \ldots, t_N)$. This construction is similar to [5, Section 3.2], in which the matrices $x_i(t)$ are seen as adjacency matrices of a weighted directed graph.

Definition 2.3. We define the *LGV diagram* as a directed graph consisting of $2n$ horizontal edges called *strands*, with $2\binom{n}{2}$ weighted vertical arrows connecting them in the following way. Let $i_1, \ldots, i_N$ be such that $\underline{w}_0^{[n-1]} = s_{i_1} s_{i_2} \cdots s_{i_N}$.

- Strands are always directed rightwards. We call the ends of the strands *vertices*. Label the vertices on both sides in order $1, \ldots, n, 2n, \ldots, n+1$ starting from the bottom. Vertices on the left and right are called resp. *source vertices* and *sink vertices*.
- For each $1 \le l \le N$, we add arrows to the strands, with arrows corresponding to smaller values of l always appearing to the left of those corresponding to larger values of l. For every non-zero entry of $x_{i_l}(t_l)$ in position (j, k) we draw an arrow from strand j to strand k with weight given by the value of the entry (t_l or $-t_l$).

We stress that the order of the arrows from different $x_{i_l}(t_l)$ matters, reflecting the non-commutativity of these matrices. However, the multiple arrows corresponding to a single matrix $x_{i_l}(t_l)$ can be drawn in any order. See Figure 1 for an example.

Definition 2.4. A *path* in the LGV diagram is a path in the directed graph from a source vertex to a sink vertex. We will identify a path $\mathcal{P}$ with the set of vertical arrows it uses.

Given $I, J \subset [2n]$ of same cardinality, we define a *path collection* from I to J in the LGV diagram as $|I|$ paths from the source vertices labeled by I to the sink vertices labeled by J. We say a path collection is *non-intersecting* if the paths do not intersect when drawn in

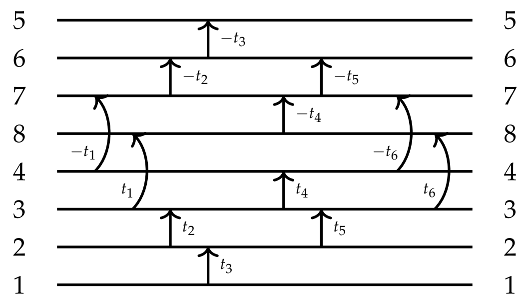


Figure 1: LGV diagram for $n = 4$

the LGV diagram. We say that a non-intersecting path collection from I to J is *unique* if it is the only non-intersecting path collection with source vertices I and sink vertices J.

A *left greedy path* in the LGV diagram originating from v is a path that starts from source vertex v and uses each vertical arrow it encounters until it reaches a sink vertex. These are paths that turn left whenever possible. We denote this path by $\mathrm{LGP}(v)$. A *left greedy path collection* is a path collection where all the paths are left greedy.

We say that a path $\mathcal{P}$ in a non-intersecting path collection P is *right greedy* if it never uses vertical arrows unless that is the only way to avoid intersecting paths originating below it in P. These are paths that turn right whenever possible, subject to P being a non-intersecting path collection.

Proposition 2.5. *Let $I \in \binom{[2n]}{n}$ and $\mathcal{P}_I$ be the set of non-intersecting path collections from $\{1, \dots, n\}$ to I in the LGV diagram. Then, the maximal minor of X indexed by I is given by*

$$\Delta^I(X) := \det(X_I) = \sum_{P=(P_1,\dots,P_n)\in\mathcal{P}_I} \mathrm{sgn}(\pi_P) \prod_{i=1}^{n} \prod_{e\in P_i} w(e),$$

where π_P is the permutation that sends i to the end point of P_i, and $w(e) \in \{\pm t_1, \dots, \pm t_N\}$ is the weight of the edge e.

Proof idea. The result follows from a simple corollary [5, Corollary 2.21] of the LGV lemma [15, 17]. $\square$

3. Positivity criterion

Definition 3.1. For any $n \times n$ matrix A we denote by $\Delta_I^J(A)$ the determinant of the submatrix of A in rows I and columns J. We denote by $M_{j,k}(A)$ the signed minor

$$M_{j,k}(A) := (-1)^{jk} \Delta^{\{1,2,\dots,n-j\}}_{\{1,\dots,n-k-1,n-k+j,\dots,n\}}(A) \qquad \text{for any } 1 \le j \le k \le n-1. \tag{3.1}$$

Note that the minor $M_{j,k}$ is a polynomial of degree $n-j$ in the entries of A. It corresponds up to a sign to a left justified minor where the rows are indexed by the complement of an interval, as illustrated by the shaded region in Figure 2.

Theorem 3.2. *[6, Thm. 1.4] A skew-symmetric matrix $A \in \mathbb{S}_n$ is totally positive if and only if*

$$M_{j,k}(A) > 0 \quad \text{for any } 1 \le j \le k \le n-1.$$

This test is minimal in the sense that it uses the fewest possible number of inequalities.

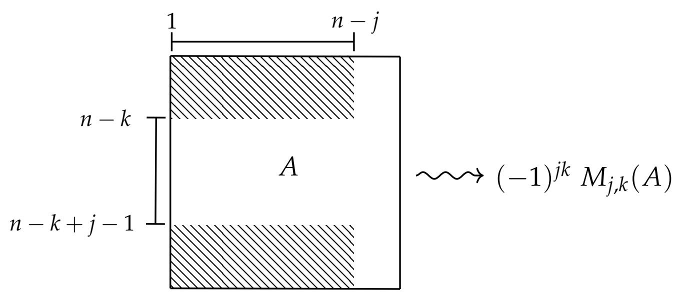


Figure 2: The shading indicates which minor of the matrix A is used to compute $M_{j,k}$.

We now introduce the main ingredient of the proof. The minors in Definition 3.1 correspond to particular path collections in the LGV diagram. First, note that the maximal minors of $X = X(t_1, \dots, t_N)$ are, up to sign, equal to the minors of $A = A(t_1, \dots, t_N)$:

$$M_{j,k} := M_{j,k}(A) = \pm\, \Delta^{n-k,\dots,n-k+j-1,n+1,n+2,\dots,2n-j}(X), \quad \text{for } 1 \le j \le k \le n-1.$$

By virtue of Proposition 2.5, to compute $M_{j,k}$, we must look at non-intersecting path collections from $\{1, \dots, n\}$ to $\{n-k, \dots, n-k+j-1, n+1, n+2, \dots, 2n-j\}$.

Proposition 3.3. *For any $1 \le j \le k \le n-1$, there is a unique non-intersecting path collection from the source vertices $\{1, \dots, n\}$ to the sink vertices $\{n-k, \dots, n-k+j-1, n+1, n+2, \dots, 2n-j\}$. Explicitly it is given by the following n paths:*

$$1 \to n-k, \quad \cdots, \quad j \to n-k+j-1, \quad j+1 \to 2n-j, \quad \cdots, \quad n \to n+1.$$

In particular, the minors $M_{j,k}$ are monomials in the variables t_i.

Let us give some intuition on the path collections in Proposition 3.3. For $1 \le j \le k \le n-1$, the corresponding path collection can be obtained as follows:

- Start with the path collection consisting of the path from 1 to k and the left greedy paths starting at $2, \dots, n$.
- Make the path starting at 2 right greedy, and then the one starting at 3, and so on until you make the path originating at $n-k+j-1$ right greedy.

Then, the index k indicates where the path starting at 1 ends while the index j tells us which paths are right greedy and which are left greedy.

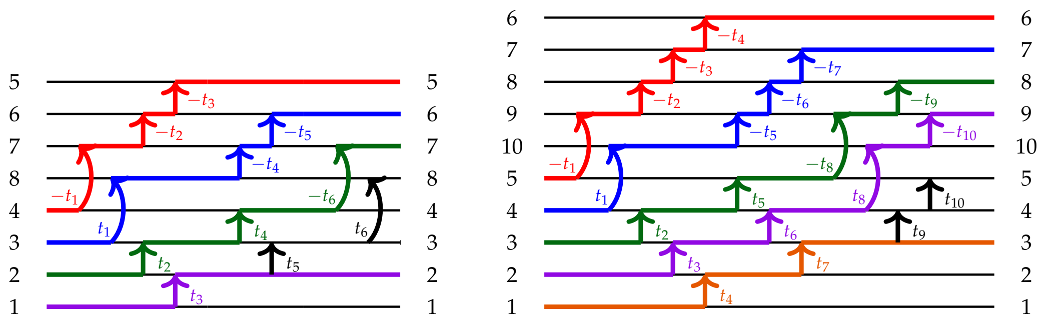


Figure 3: Illustration of the path collection corresponding to $M_{1,2}$ for $n = 4$ and $n = 5$.

4. Nonnegativity test and boundary cells

4.1 Nonnegativity test

Recall that the totally nonnegative orthogonal Grassmannian $\mathrm{OGr}^{\geq 0}(n, 2n)$ is the Euclidean closure of $\mathrm{OGr}^{>0}(n, 2n)$. We now state a nonnegativity criterion for skew-symmetric matrices, the details can be found in [6, Section 4].

Theorem 4.1. *Fix $X \in \mathrm{OGr}(n, 2n)$. Then, for any smooth 1-parameter family $Z(\epsilon)$ in $\mathrm{SO}^{>0}(2n)$ so that $Z(\epsilon) \xrightarrow[\epsilon \to 0]{} \mathrm{Id}_{2n}$ and $X(\epsilon) := X \cdot Z(\epsilon)$, the following are equivalent.*

(1) X is totally nonnegative.

(2) $X(\epsilon)$ is totally positive for all $\epsilon > 0$ sufficiently small.

(3) For all $1 \leq j \leq k \leq n-1$, the leading coefficient in the Taylor expansion of $M_{j,k}(B(\epsilon))$ is positive, where $B(\epsilon)$ is defined by $X(\epsilon) = \mathrm{rowspan}\big([\mathrm{Id}_n | B(\epsilon)]\big)$.

Moreover, the family $Z(\epsilon)$ can be chosen so that $M_{j,k}(B(\epsilon))$ are polynomials in ϵ.

4.2 Richardson and Deodhar decompositions

The boundary structure of the totally nonnegative part of partial flag varieties is combinatorially rich and closely related to the theory of positroids in type A. In this section we briefly describe this boundary structure for arbitrary flag varieties and then specialize to the case of the orthogonal Grassmannian. Our presentation follows [16].

Let G be a split reductive algebraic group over $\mathbb{R}$ with a fixed pinning. The reader unfamiliar with the general theory can continue to work with $G = \mathrm{SO}(2n)$, and the x_i matrices introduced in Section 2. The complete flag variety G/B can be identified with the variety of Borel subgroups $\mathcal{B}$ via conjugation: $gB \in G/B \leftrightarrow g \cdot B := gBg^{-1} \in \mathcal{B}$.

We can also lift the Weyl group elements $w \in W = N_G(T)/T$ to G. For each simple reflection $s_i \in W$, we denote a lift by $\dot{s}_i$. For a reduced expression $\underline{w} = s_{i_1} \cdots s_{i_k}$, define $\dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_k}$. Then, the Bruhat decomposition of G descends to the complete flag variety. The *Richardson cells* of G/B are intersections of opposite Bruhat cells. Concretely, given $v, w \in W$ we define

$$\mathcal{R}_{v,w} := (B\dot{w} \cdot B) \cap (B^- \dot{v} \cdot B).$$

For $\underline{v}, \underline{w}$ expressions of $v, w \in W$, Deodhar [9] defined the *Deodhar component* $\mathcal{R}_{\underline{v},\underline{w}}$. They refine the Richardson cells; explicitly, for a fixed reduced expression $\underline{w}$ we have

$$\mathcal{R}_{v,w} = \bigsqcup_{\underline{v} \prec \underline{w}} \mathcal{R}_{\underline{v},\underline{w}},$$

where the disjoint union is over all distinguished subexpressions $\underline{v}$ of $\underline{w}$. We recall that for a reduced expression $\underline{v} = s_{i_1} \cdots s_{i_p}$ of $v \in W$, a *subexpression* $\underline{u}$ of $\underline{v}$ is a choice of 1 or s_{i_j} for each $j \in [p]$. Let $\underline{v}$ be an expression for $v \in W$ and $\underline{u}$ be a subexpression of $\underline{v}$. For $k \geq 0$, let $\underline{u}_{(k)}$ be the subexpression of $\underline{v}$ which is identical to $\underline{u}$ in its last k entries and is all 1s beforehand. A subexpression $\underline{u}$ of $\underline{v}$ is *distinguished*, denoted $\underline{u} \preceq \underline{v}$, if $u_{(j)} \leq s_{i_{n-j+1}} u_{(j-1)}$ for all $j \in [p]$. Here, $<$ is the Bruhat order in W.

For each $\mathcal{R}_{v,w}$ there is a unique Deodhar component of maximal dimension, namely when we take $\underline{v}$ to be the unique reduced distinguished subexpression for v in $\underline{w}$, which we denote $\underline{v}^+$. Let $\mathcal{R}^{>0}_{v,w}$ be the positive part $\mathcal{R}_{v,w} \cap (G/B)^{\geq 0}$ of the Richardson cell and $\mathcal{R}^{>0}_{\underline{v},\underline{w}}$ the positive part $\mathcal{R}_{\underline{v},\underline{w}} \cap (G/B)^{\geq 0}$ of the Deodhar component. By [21, Theorem 11.3],

$$\mathcal{R}^{>0}_{v,w} = \mathcal{R}^{>0}_{\underline{v}^+,\underline{w}}.$$

When $w = w_0$ and $v = \text{id}$, the component $\mathcal{R}_{id,w_0} = (G/B)^{>0}$, which is dense in $(G/B)^{\geq 0}$. As explained in [16, Section 3], partial flag varieties also admit Richardson (resp. Deodhar) decompositions. Moreover, restricting Richardson cells and Deodhar components to $(G/B)^{\geq 0}$, we obtain a cell decomposition of the nonnegative part of the flag variety into positive Richardson cells $\mathcal{R}^{>0}_{v,w}$.

We now focus on the concrete case of $\text{OGr}(n, 2n)$ and its corresponding parabolic quotient $W^{[n-1]}$. We have a Richardson cell for each pair $v, w^{[n-1]} \in W$, where $w^{[n-1]}$ is a minimal coset representative and $v \leq w^{[n-1]}$. Slightly abusing notation, we also denote by $\mathcal{R}^{>0}_{v,w^{[n-1]}} = \mathcal{R}^{>0}_{\underline{v}^+,\underline{w}^{[n-1]}}$ the positive Richardson cells in $\text{OGr}(n, 2n)$.

Definition 4.2. For a subexpression $\underline{u}$ of a length p expression $\underline{v}$, we define subsets of $[p]$

$$J^+_{\underline{u}} := \{k \mid u_{(k)} > u_{(k-1)}\}, \quad J^\circ_{\underline{u}} := \{k \mid u_{(k)} = u_{(k-1)}\}, \quad J^-_{\underline{u}} := \{k \mid u_{(k)} < u_{(k-1)}\}.$$

As we did for the totally positive part in Section 2, we can use Marsh and Rietsch's work to parametrize any Deodhar component in terms of the corresponding pair of subexpressions [21, Section 5]. Let $\underline{w}^{[n-1]} = s_{i_1} \cdots s_{i_k}$ be a reduced expression. Then, for the reduced distinguished subexpression $\underline{v}^+ \preceq \underline{w}^{[n-1]}$, we have

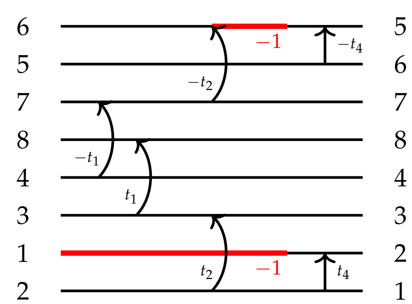
$$\mathcal{R}^{>0}_{\underline{v}^+,\,\underline{w}^{[n-1]}} = \left\{ \pi_n(g_1 \cdots g_p) : \quad \begin{array}{ll} g_k = x_{i_k}(t_k) \text{ with } t_k \in \mathbb{R}_{>0} & \text{if } k \in J^+_{\underline{v}^+} \\ g_k = s^T_{i_k} & \text{if } k \in J^\circ_{\underline{v}^+} \end{array} \right\}. \tag{4.1}$$

where $\pi_n : \mathrm{SO}(2n) \to \mathrm{OGr}(n, 2n)$ is the same projection as in (2.3).

We have worked extensively with the LGV diagram to view Plücker coordinates in terms of non-intersecting weighted path collections. The LGV diagram we constructed in Definition 2.3 is tailored to the top cell $\mathcal{R}_{\mathrm{id}, w_0^{[n-1]}} = \mathrm{OGr}^{>0}(n, 2n)$. This construction can be extended to arbitrary Richardson cells in the boundary of $\mathrm{OGr}^{\geq 0}(n, 2n)$. We now give some intuition on how this is done.

Remark 4.3. We define a weighted directed graph $G_{v,w}$ for $v \leq w$ and w a minimal coset representative. Let $\underline{w}_0^{[n-1]} = s_{i_1} \cdots s_{i_N}$ be as in (2.2), with $N = \binom{n}{2}$, and $\underline{w}^{[n-1]} = \rho_1 \cdots \rho_N$ be the reduced distinguished expression for $w^{[n-1]}$ in $\underline{w}_0^{[n-1]}$. Let $\underline{v} = r_1 \cdots r_l$ be the reduced distinguished subexpression for v in $\underline{w}^{[n-1]}$. We start from the LGV diagram. For each $j \in [N]$ we sequentially modify it by deleting arrows, permuting strands or adding a -1 weight to a portion of some strands. The specific step depends on ρ_j and r_j.

Example 4.4. The following is $G_{v,w}$ for $w = s_4 s_2 s_1 s_3$, $v = s_1$. Note that the reduced distinguished subexpression $\underline{w}^+ \prec \underline{w}_0^{[n-1]}$ is $s_4 s_2 s_1 s_3 11$ and the reduced distinguished subexpression $\underline{v}^+ \prec \underline{w}$ is $11 s_1 111$.



4.3 Identification of cells

Our goal in this section is to explain how we can identify the Richardson cell $\mathcal{R}^{>0}_{v,w}$ containing a given point $X \in \mathrm{OGr}^{\geq 0}(n, 2n)$. Let $\mathscr{M}_X$ denote the matroid associated to X, that is, the rank n matroid on $[2n]$ with bases $\left\{ I \in \binom{[2n]}{n} : \quad \Delta^I(X) \neq 0 \right\}$. In the nonnegative Grassmannian, Richardson cells are in bijection with *positroids*. The Richardson cell that contains a given point $X \in \mathrm{OGr}^{\geq 0}(n, 2n)$ is also determined by its matroid $\mathscr{M}_X$. Our proofs of the results in this section rely on the diagrams $G_{v,w}$ introduced previously.

Definition 4.5. Let I be a basis of a rank k matroid $\mathscr{M}$ on a ground set E. Let $<$ be a total order on E. We inductively define a sequence of bases of $\mathscr{M}$ as follows. Let $I^{(0)} = I$. For $1 \leq j \leq k$, suppose $I^{(j-1)} = \left\{ i_1^{(j-1)} < \cdots < i_k^{(j-1)} \right\}$. Then, $I^{(j)}$ is obtained by replacing $i_j^{(j-1)}$ in $I^{(j-1)}$ by the element $\min_<\{t \in E\colon \left(I^{(j-1)} \setminus \{i_j^{(j-1)}\} \right) \cup \{t\} \text{ is a basis of } \mathscr{M}\}$. For $0 \leq j \leq k$, we say $I^{(j)}$ is a *lowering* of I with respect to $<$.

Recall that permutations σ in the Weyl group W of $\mathrm{SO}(2n)$ satisfy $\sigma(n+j) = n + \sigma(j)$ modulo $2n$ for all $j \in [n]$. Thus, to determine v and w, it suffices to specify $v(j)$ and $w(j)$ for all $j \in [n]$. We consider a total order $\prec$ on $[2n]$ given by:

$$1 \prec 2 \prec \cdots \prec n \prec 2n \prec 2n-1 \prec \cdots \prec n+1.$$

We consider the corresponding *Gale order* on $\binom{[2n]}{n}$, also denoted $\prec$. That is, we have $\{a_1 \prec a_2 \prec \cdots \prec a_n\} < \{b_1 \prec b_2 \prec \cdots \prec b_n\}$ if and only if $a_i < b_i$ for all $i \in [n]$.

We make use of the following fact: the set of bases of a matroid on a ground set E has a unique Gale-maximal element with respect to any total order on E [12, Theorem 4]. We denote by I_X the unique maximal element of the matroid $\mathscr{M}_X$ with respect to $\prec$.

Theorem 4.6. *Let $X \in \mathrm{OGr}^{\geq 0}(n, 2n)$ and $v, w \in W$ such that $X \in \mathcal{R}_{v,w}^{>0}$. Let*

$$t_j := \min_{\prec} \left\{ t\colon \left(I_X^{(j-1)} \setminus \{i_j^{(j-1)}\} \right) \cup \{t\} \text{ is a basis of } \mathscr{M}_X \right\},$$

where $I_X^{(l)} := \{i_1^{(l)} \prec \cdots \prec i_n^{(l)}\}$ is the l-th lowering of $I_X = \{i_1 \prec i_2 \prec \cdots \prec i_n\}$ with respect to $\prec$ as defined in Definition 4.5. Then

(1) $w \in W$ is given by $w^{-1}(j) = i_j$ for $j \in [n]$, and

(2) $v \in W$ is given by $v(j) = t_j$ for $j \in [n]$.

In particular, the Richardson cell containing X is fully determined from $\mathscr{M}_X$.

Proposition 4.7. *The following holds: $\mathbb{S}_n^{\geq 0} = \bigsqcup_{v \in W_{[n-1]}} \mathcal{R}_{v,w}^{>0}$.*

5. Pfaffian signs

We recall that the determinant $\det(A)$ of a skew-symmetric matrix $A = (a_{ij})_{i,j=1}^{2m}$ of size $2m \times 2m$ is a homogeneous polynomial of degree $2m$ in the entries of A. Moreover, it is the square of a degree m homogeneous polynomial $\mathrm{Pf}(A)$ known as the *Pfaffian*,

$$\mathrm{Pf}(A) := \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \mathrm{sgn}(\sigma) \prod_{i=1}^{m} a_{\sigma(2i-1),\sigma(2i)}.$$

For any skew-symmetric matrix $A \in \mathbb{S}_n$ and any set $I \subset [n]$ of even size we denote by $\mathrm{Pf}_I(A)$ the Pfaffian of the submatrix A_I of A whose row and column indices are in I. By convention, we set $\mathrm{Pf}_\emptyset(A) = 1$. We denote the semi-algebraic set of *Pfaffian-positive* skew-symmetric matrices by

$$\mathbb{S}_n^{\mathrm{Pf}>0} := \Big\{ A \in \mathbb{S}_n : \mathrm{sgn}(I, [n])\, \mathrm{Pf}_I(A) > 0 \quad \text{for any } I \subset [n] \text{ of even size} \Big\}, \tag{5.1}$$

where $\mathrm{sgn}(I, [n]) := (-1)^{\sum_{i \in I} i - \binom{|I|+1}{2}}$. We denote by $\mathbb{S}_n^{\mathrm{Pf}\geq 0}$ its Euclidean closure in $\mathbb{R}^{n\times n}$.

Remark 5.1. Let $g = \mathrm{diag}(1, -1, 1 \ldots, (-1)^{n+1})$. We note that a skew-symmetric matrix A is in $\mathbb{S}_n^{\mathrm{Pf}>0}$ if and only if $\mathrm{Pf}_I(A') > 0$ for any subset $I \subset [n]$ of even size where $A' = gAg^T$. So the set $\mathbb{S}_n^{\mathrm{Pf}>0}$ is conjugate to the set of matrices with positive Pfaffians.

A priori, there is no reason why the notion of total positivity in $\mathbb{S}_n$ we have discussed so far should be compatible with $\mathbb{S}_n^{\mathrm{Pf}>0}$. The following theorem states their relation.

Theorem 5.2. *For any $A \in \mathbb{S}_n^{>0}$, and $I \subset [n]$ of even size, we have* $\mathrm{sgn}(\mathrm{Pf}_I(A)) = \mathrm{sgn}(I, [n])$. *If $I = \{i_1 < \cdots < i_{|I|}\}$ and $[n] \setminus I = \{j_1 < \cdots < j_{n-|I|}\}$, this is the sign of the permutation $i_1, \ldots, i_{|I|}, j_1 \ldots, j_{n-|I|}$ in one-line notation.*

Example 5.3. The inclusion $\mathbb{S}_n^{\geq 0} \subset \mathbb{S}_n^{\mathrm{Pf}\geq 0}$ is strict for $n \geq 4$. For example

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 \\ -2 & 0 & -2 & 0 \end{bmatrix} \in \mathbb{S}_4^{\mathrm{Pf}\geq 0} \setminus \mathbb{S}_4^{\geq 0}.$$

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