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Excluding a line from positroids

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Abstract. For all positive integers ℓ and r, we determine the maximum number of elements of a simple rank-r positroid without the rank-2 uniform matroid $U_{2,\ell+2}$ as a minor, and characterize the positroids with the maximum number of elements. This result continues a long line of research into upper bounds on the number of elements of matroids from various classes that forbid $U_{2,\ell+2}$ as a minor. This is the first paper to study positroids in this context, and it suggests methods to study similar problems for other classes of matroids, such as gammoids or base-orderable matroids.

Keywords: positroids, extremal matroid theory, excluded minors

1 Introduction

Extremal matroid theory was formally introduced in 1993 with the following problem posed by Joseph Kung [11].

Problem 1.1. *Let $\mathcal{M}$ be an interesting class of matroids. Determine the maximum number of elements of a simple rank-r matroid in $\mathcal{M}$, and characterize the matroids for which equality holds.*

Since a rank-2 uniform matroid $U_{2,\ell}$ can have an arbitrary number of elements, this maximum does not exist for many natural classes of matroids. For instance, there is no finite answer to Problem 1.1 for the class of all matroids, or for any class that contains all uniform matroids. However, Kung [11, Theorem 4.3] proved that every simple rank-r matroid without $U_{2,\ell+2}$ as a minor has at most $\frac{\ell^r-1}{\ell-1}$ elements. If ℓ is a prime power, then this upper bound is tight for the rank-r projective geometry over the finite field of order ℓ. It is thus natural to study Problem 1.1 for classes of matroids without $U_{2,\ell}$-minors.

In this paper, we study the class of *positroids* without $U_{2,\ell}$-minors. Positroids, which are matroids representable by a matrix over $\mathbb{R}$ with all maximal subdeterminants nonnegative, occupy a central position at the intersection of combinatorics, algebraic geometry, and physics. They serve as combinatorial models for the totally nonnegative

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Grassmannian, which in turn underlies the construction of the amplituhedron in quantum field theory. The study of amplituhedra and related objects remains an active and influential area in theoretical physics [2]. Through these connections, positroids emerge as unifying objects that integrate discrete combinatorial frameworks with geometric and physical theory. Combinatorially, they are remarkably rich: Postnikov [18] showed that positroids are in bijection with decorated permutations, Grassmann necklaces, plabic graphs, and Ⅎ-diagrams, providing multiple equivalent viewpoints. Positroids also form a minor-closed class of matroids. It is thus natural to ask extremal questions about positroids and their minors. There has been work enumerating both positroids [22] and connected positroids [1], but as far as the authors are aware, no other extremal results about positroids are known. Our main result tightly resolves Problem 1.1 for the class of $U_{2,\ell+2}$-minor-free positroids.

Theorem 1.2. *For all integers $r, \ell \geq 1$, if M is a simple rank-r positroid with no $U_{2,\ell+2}$-minor, then $|M| \leq \ell(r-1)+1$. Moreover, if $r \geq 2$ then equality holds if and only if M can be obtained by taking parallel connections of copies of $U_{2,\ell+1}$.*

This result lies at the intersection of three areas of research: extremal matroid theory, structural properties of positroids, and structural properties of matroids with no $M(K_4)$-minor. We will explain how Theorem 1.2 continues lines of research in these three areas, beginning with extremal matroid theory. The following beautiful result of Geelen and Nelson [8, Theorem 1.2] generalizes the initial work of Kung and answers Problem 1.1 for the class of matroids with no $U_{2,\ell+2}$-minor.

Theorem 1.3 (Geelen, Nelson [8])**.** *Let $\ell \geq 2$ be an integer and let q be the largest prime power at most ℓ. For r sufficiently large, every simple rank-r matroid with no $U_{2,\ell+2}$-minor has at most $\frac{q^r-1}{q-1}$ elements, with equality only for the rank-r projective geometry over the order-q finite field.*

Since projective geometries are not representable over fields of characteristic zero (see [13, page 205, Exercise 8]), this leads to the following question: for a field $\mathbb{F}$ with characteristic zero, what is the maximum number of elements of a simple rank-r $\mathbb{F}$-representable matroid with no $U_{2,\ell+2}$-minor? This was answered by Geelen, Nelson, and Walsh [9, Theorem 1.0.5] in the case that $\mathbb{F} = \mathbb{C}$.

Theorem 1.4 (Geelen, Nelson, Walsh [9])**.** *Let $\ell \geq 2$ be an integer and let r be a sufficiently large integer. Then every simple rank-r $\mathbb{C}$-representable matroid with no $U_{2,\ell+2}$-minor has at most $(\ell-1)\binom{r}{2}+r$ elements, with equality only for the rank-r Dowling geometry over the cyclic group of order $\ell-1$.*

When $\mathbb{F} \in \{\mathbb{R}, \mathbb{Q}\}$ the best known lower bound to the maximization part of Problem 1.1 is $2\binom{r}{2}+r$ and the best known upper bound is $2\binom{r}{2}+f(\ell)\cdot r$ with f a function of ℓ, due to Geelen, Nelson, and Walsh [9, Theorem 1.0.6], but no exact solution is known.

Theorem 1.2 can be seen as a further specialization of Theorems 1.3 and 1.4 to the subclass of $\mathbb{R}$-representable matroids consisting of positroids.

Theorem 1.2 also continues a recent line of research that studies positroids from a structural matroid theory perspective. For example, Bonin [4] studied how positroids behave with respect to matroid amalgams, Quail [19] studied positroids representable over the finite fields of order 3 and of order 4, Quail and Rombach [20] studied graphic positroids, and Park [15] studied excluded minors for positroids.

Finally, Theorem 1.2 contributes to a line of research studying matroids with no $M(K_4)$-minor, where $M(K_4)$ is the graphic matroid of the complete graph K_4. For example, Sims [21] proved that the class of $M(K_4)$-minor-free matroids is complete, Brylawski [6] characterized the binary matroids with no $M(K_4)$-minor, and Oxley [14] characterized the 3-connected ternary matroids with no $M(K_4)$-minor. More recently, work of Pendavingh and van der Pol [16] and van der Pol [17] investigated the importance of $M(K_4)$ in matroid enumeration. Most notably for this paper, Kung studied Problem 1.1 for the class of matroids with no $M(K_4)$-minor and no $U_{2,\ell+2}$-minor, proving that every such simple rank-r matroid has at most $(6\ell^{\ell-1} + 8\ell) \cdot r$ elements [10, Theorem 1.1]. The graphic matroid $M(K_4)$ is not a positroid, so this also bounds from above the maximum number of elements of a simple rank-r positroid with no $U_{2,\ell+2}$-minor. To the best of our knowledge, this was in fact the previous best upper bound for this quantity. There are several other notable $M(K_4)$-minor-free classes of matroids that contain the class of positroids, such as gammoids, strongly base-orderable matroids, and k-base-orderable matroids. We will discuss these further in Section 6.

The rest of this paper is structured as follows. In Section 2, we will give more background on matroids and positroids. In Section 3, we will give a list of 10 excluded minors for the class of positroids which are particularly useful for our work. In Section 4, we will describe the matroids for which equality holds in Theorem 1.2 and prove that they are in fact positroids and are $U_{2,\ell+2}$-minor-free. We will prove Theorem 1.2 in Section 5, and conclude in Section 6 by discussing several directions for future work.

2 Preliminaries

2.1 Matroids

In this section, we recall relevant concepts from matroid theory. We assume basic knowledge of matroids and refer the reader to [13] for more details. We focus on the perspective of *flats*, which will be particularly useful for discussing restrictions, contractions, and geometric representations.

Definition 2.1 (Matroids in terms of flats)**.** Let E be a finite set. A *matroid* is a pair $M = (E, \mathcal{F})$ where $\mathcal{F} \subseteq 2^E$ satisfies the following axioms:

(F1) $E \in \mathcal{F}$.
(F2) For all $F_1, F_2 \in \mathcal{F}$, we have $F_1 \cap F_2 \in \mathcal{F}$.
(F3) For each $F \in \mathcal{F}$ and each $e \in E - F$, there exists a unique $F' \in \mathcal{F}$ that is minimal with respect to inclusion among elements of $\mathcal{F}$ containing $F \cup \{e\}$.

The set $\mathcal{F}$ is the set of *flats* of M. A matroid on ground set E is *simple* if $\emptyset \in \mathcal{F}$ and $\{i\} \in \mathcal{F}$ for each $i \in E$.

Remark 2.2. Flats are highly structured: ordered by inclusion, they form a lattice called the *matroid lattice*. Given a collection of flats $\mathcal{F} \subseteq 2^E$ satisfying the axioms above, one can recover other familiar data of the matroid $M = (E, \mathcal{F})$:

- The *rank* of the matroid, denoted $r(M)$, is the length of a longest chain of flats in the lattice.
- The *bases* are subsets $B \subseteq E$ of size $r(M)$ which are not contained in any flat other than E itself.
- The *closure* of a set X, denoted $\mathrm{cl}(X)$, is the join of all flats contained in X. Explicitly, this is the minimal flat containing X.

Example 2.3. An important matroid in this paper is the rank-2 uniform matroid on ℓ elements, denoted $U_{2,\ell}$. This is the matroid where the flats are the empty set, singletons, and the entire ground set. Since $\emptyset$ and all singletons are flats, this matroid is simple. The corresponding lattice is illustrated in Figure 1. The bases of this matroid are thus all the 2-element subsets of the ground set.

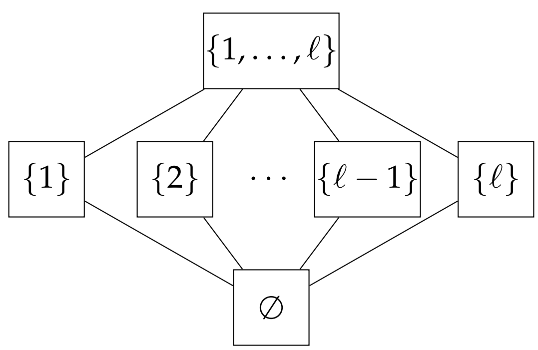

Figure 1: Lattice of flats of the uniform matroid $U_{2,n}$.

Perhaps the canonical examples of matroids come from matrices. If A is a matrix over a field $\mathbb{F}$ with column set E, then the set $\mathcal{F}$ of maximal subsets of columns spanning each subspace of the column span of A is the set of flats of a matroid on E. Such a matroid is *representable* over $\mathbb{F}$. In this case the bases of the matroid are the subsets of columns that are bases of the column span, and the rank of the matroid is the rank of the matrix.

Matroids representable over $\mathbb{R}$ can be visualized using geometric diagrams that represent their flats. This provides intuition about the lattice structure and the geometric relationships among elements.

Definition 2.4 (Geometric Representation)**.** Let M be a matroid. A *geometric representation* of M depicts flats as follows:

- rank-1 flats are represented as *points*.
- rank-2 flats are represented as *lines* containing the corresponding points.
- rank-3 flats are represented as *planes* containing the corresponding points.

We could in principle represent higher rank flats, but will not need to do so here. To avoid cluttered images, we will not explicitly draw lines containing exactly two points, nor planes containing exactly the union of a point and a line. We call the lines that contain at least three points *long lines*; these will play a distinguished role in what follows.

Example 2.5. The matroid in Figure 3 has a ground set of size 9 and is simple. It contains exactly 4 long lines. There are also many lines not drawn that connect pairs of points not on a long line. These are all the flats of rank 2. There are 3 planes drawn. Each such plane contains two long lines, as well as 4 other lines represented by pairs of points not connected by a drawn line. There are also many planes not drawn, which consist of the union of a line and a point. These are all the flats of rank 3.

We will need two matroid operations, *restriction* and *contraction*, in this paper, each of which we can define in terms of both the lattice of flats of the matroid and the geometric representation. We begin by working in terms of the lattice of flats.

Let M be a matroid on a ground set E with flat set $\mathcal{F}$, and let $A \subseteq E$. The *restriction* of M to A, denoted $M|A$, is the matroid on A whose flats are exactly the intersections $F \cap A$ for $F \in \mathcal{F}$. That is,

$$\mathcal{F}(M|A) = \{F \cap A \mid F \in \mathcal{F}\}.$$

If $A = F'$ happens to be a flat, then we have that

$$\mathcal{F}(M|F') = \{F \mid F \in \mathcal{F},\, F \subseteq F'\}.$$

Intuitively, restriction "removes" elements not in A while preserving the lattice structure among elements of A. We will sometimes also write $M\backslash(E - A)$ for $M|A$; this is the *deletion* of $E - A$ from M. Dually, the *contraction* of M by $F' \in \mathcal{F}$, denoted M/F', is a matroid on $E - F'$ whose flats are the flats of M containing F', with F' removed:

$$\mathcal{F}(M/F') = \{F - F' \mid F \in \mathcal{F},\, F' \subseteq F\}.$$

Contraction can be thought of as "collapsing" F' to the empty set, retaining the lattice structure among elements outside of A.

Both operations interact nicely with the geometric representation: restriction to A corresponds to deleting points not in A, while contraction of a flat F' corresponds to projecting the matroid along the subspace spanned by F', preserving incidences among the remaining points, lines, and planes. These perspectives will allow us to manipulate matroids in a way that is compatible with both their flats and their geometric structures.

This paper studies classes of matroids that are closed under deletion and contraction.

Definition 2.6. Let M be a matroid. A *minor* of M is any matroid obtained from M by a sequence of deletions and contractions. A class $\mathcal{C}$ of matroids is *minor-closed* if, whenever $M \in \mathcal{C}$ and N is a minor of M, we have $N \in \mathcal{C}$. Equivalently, $\mathcal{C}$ is closed under both restriction and contraction.

2.2 Positroids

Positroids are a special class of matroids that arise naturally in the study of the totally nonnegative Grassmannian and have remarkable combinatorial structure. They are $\mathbb{R}$-representable matroids whose representations satisfy additional positivity constraints.

Definition 2.7. A matroid M is a *positroid* if it can be represented by a full-rank real matrix such that all maximal subdeterminants of the matrix are nonnegative.

Example 2.8. The matroid on the ground set $\{1,2,3,4\}$ with bases $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, and $\{2,4\}$ is a positroid, represented by the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{pmatrix}$.

For a non-example, consider the matroid in the bottom right of Figure 2. This is the graphic matroid of the complete graph K_4, denoted $M(K_4)$. Indeed, one can verify that any $\mathbb{R}$-representation of $M(K_4)$ has at least one negative maximal subdeterminant.

There are many equivalent ways to think about positroids, many of which are described in [18]. In line with our earlier introduction to matroid flats, we will give a description of positroids in terms of flats, due to Bonin. While the details are not important for this abstract, we remark that matroids admit a natural notion of direct sum and we call a matroid *connected* if it does not admit a non-trivial direct sum decomposition.

Theorem 2.9 (Bonin [4])**.** *Let M be a matroid on a ground set E. Then M is a positroid if and only if its ground set E admits a total ordering $<$ such that, for every flat F with both $M|F$ and M/F connected, either F or its complement is an interval in $<$ order.*

Example 2.10. The only non-empty flats of $U_{2,\ell}$ are singletons and the entire ground set, which satisfy the conditions of Theorem 2.9 for any ordering. Thus, $U_{2,\ell}$ is a positroid.

3 Excluded minors

Positroids form a minor-closed class of matroids. This follows from the geometric description of positroids as the representable matroids corresponding to points in the totally nonnegative Grassmannian. Indeed, deleting or contracting an element corresponds respectively to projecting onto a smaller coordinate subspace or intersecting with a coordinate hyperplane – operations that preserve total nonnegativity [1, Prop. 3.5]. As a result, positroids can be characterized by their *excluded minors*, which are minor-minimal matroids that are not positroids. In this section we will discuss known results regarding excluded minors for the class of positroids.

The rank-3 excluded minors for the class of positroids have been completely classified by Blum [3, Corollary 4.12]. While there are infinitely many rank-3 excluded minors, we use the subset shown in Figure 2 to prove that rank-3 matroids satisfying a certain structural property are not positroids.

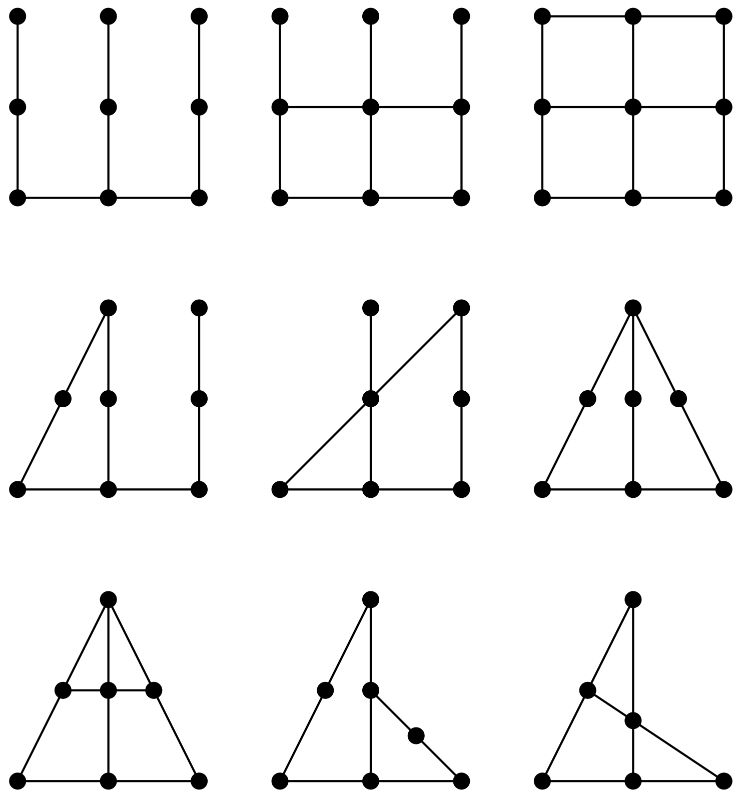

Figure 2: Some rank-3 excluded minors for the class of positroids.

Proposition 3.1. *If M is a rank-3 matroid with a line L and distinct points e_1, e_2, e_3 on L so that for $i = 1, 2, 3$ the point e_i is on at least two long lines, then M is not a positroid. In particular, M has a minor from Figure 2.*

Similarly, we can show that rank-4 matroids satisfying a certain structural property are not positroids.

Proposition 3.2. *If M is a rank-4 matroid with a line L, distinct points e_1, e_2, e_3 on L, and distinct planes P_1, P_2, P_3 through L so that for $i = 1, 2, 3$ the point e_i is on at least two long lines in P_i, then M is not a positroid. In particular, M has a minor from Figure 2 or Figure 3.*

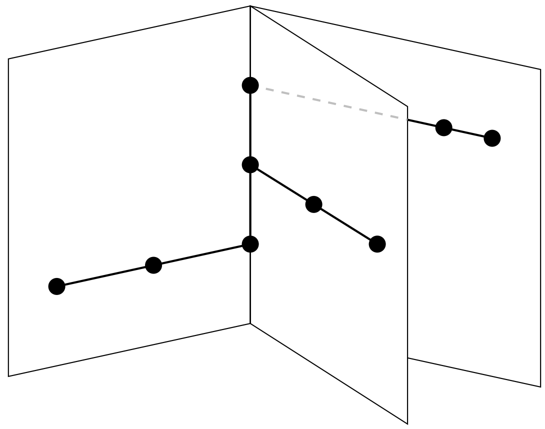

Figure 3: A rank-4 excluded minor for the class of positroids.

4 The extremal examples

In this section we will define the family of positroids for which equality holds in Theorem 1.2, and prove that they are in fact positroids and are $U_{2,\ell+2}$-minor-free. Given matroids M and N with $E(M) \cap E(N) = \{e\}$, the *parallel connection* of M and N with basepoint e is the matroid $P_e(M, N)$ with the following set of flats:

$$\{F \subseteq E(M) \cup E(N) \colon F \cap E(M) \text{ is a flat of } M \text{ and } F \cap E(N) \text{ is a flat of } N\}.$$

Definition 4.1. For each integer $\ell \geq 2$, let $\mathcal{M}_{2,\ell}$ be the class of matroids isomorphic to $U_{2,\ell+1}$, and then for each integer $r \geq 3$ let $\mathcal{M}_{r,\ell}$ be the class of matroids of the form $P_e(M, N)$ with $M \in \mathcal{M}_{r-1,\ell}$ and $N \cong U_{2,\ell+1}$.

Example 4.2. Let M and N both be isomorphic to the uniform matroid $U_{2,3}$, on ground sets $\{1, 2, 3\}$ and $\{3, 4, 5\}$, respectively. We construct $P_3(M, N) \in \mathcal{M}_{3,2}$. The parallel connection $P_3(M, N)$ is a matroid on $\{1, 2, 3, 4, 5\}$ for which:

- Singletons $\{i\}$ are flats of rank 1.
- The sets $\{1, 2, 3\}$ and $\{3, 4, 5\}$ are flats of rank 2.
- Any pair $\{i, j\}$ with $i \in \{1, 2\}$ and $j \in \{4, 5\}$ is also a flat of rank 2, since its intersections with both M and N are rank-1 flats.
- The full ground set $\{1, 2, 3, 4, 5\}$ is a flat of rank 3.

Geometrically, $P_3(M, N)$ can be viewed as two lines sharing a single common point 3. This configuration is illustrated in Figure 4.

Since each subsequent parallel connection with $U_{2,\ell+1}$ increases the rank by one [13, Proposition 7.1.15(i)], each matroid in $\mathcal{M}_{r,\ell}$ has rank r. And since each subsequent parallel connection increases the number of elements by ℓ, each matroid in $\mathcal{M}_{r,\ell}$ has $\ell(r-1)+1$ elements. Bonin proved that the class of positroids is closed under parallel

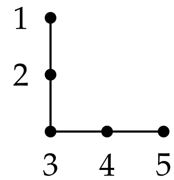

Figure 4: The parallel connection of two matroids M and N, each isomorphic to $U_{2,3}$.

connections [4, Proposition 4.18] and, by Example 2.10, uniform matroids are positroids. Thus, every matroid in $\mathcal{M}_{r,\ell}$ is a positroid. The following lemma implies that each matroid in $\mathcal{M}_{r,\ell}$ is $U_{2,\ell+2}$-minor-free.

Lemma 4.3. *For each integer $\ell \geq 2$, if a matroid M is $U_{2,\ell+2}$-minor-free, then $P_e(M, U_{2,\ell+1})$ is $U_{2,\ell+2}$-minor-free.*

Thus, every matroid in $\mathcal{M}_{r,\ell}$ is a simple, rank-r positroid with no $U_{2,\ell+2}$-minor and $\ell(r-1)+1$ elements, and is therefore a sharp example for Theorem 1.2.

5 The main result

In light of Section 3, we present a strengthening of Theorem 1.2 which considers a class of matroids avoiding only a finite subset of the infinitely many excluded minors for positroids.

Theorem 5.1. *For all integers $r, \ell \geq 1$, if M is a simple rank-r matroid with no minor from Figure 2 or Figure 3 and no $U_{2,\ell+2}$-minor, then $|M| \leq \ell(r-1)+1$. Moreover, if $r \geq 2$ then equality holds if and only if M can be obtained by taking parallel connections of copies of $U_{2,\ell+1}$.*

Proof sketch. The statement is true when $\ell = 1$ because every simple rank-r matroid with no $U_{2,3}$-minor has exactly r elements, so we may assume that $\ell \geq 2$. We will proceed by induction on r. The statement is clearly true when $r \leq 2$ so we may assume that $r \geq 3$. If M has an element e that is not on any long lines, then by induction we have

$$|M| = 1 + |M/e| \leq 1 + \ell(r-2) + 1 = \ell(r-1) + 2 - \ell < \ell(r-1) + 1,$$

as desired. So we may assume that every element of M is on a long line. Let L be a long line of M. We will now consider the connected components of M/L. Let $\mathcal{X}$ be the partition of $E(M) - L$ given by the ground sets of the connected components of M/L. Let $X \in \mathcal{X}$, and let $N = M|(L \cup X)$. We now bound the size of N. The following claim, whose proof is omitted, relies on Propositions 3.1 and 3.2.

Claim 5.1.1. *At most two elements in L are on a long line of N other than L, and $|N \backslash L| \leq \ell(r(N) - 2)$.*

With this claim, we can now show that $|M| \leq \ell(r-1)+1$. Let $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ for some positive integer k, and for each $i \in [k]$ let $N_i = M|(L \cup X_i)$. Since the sets in $\mathcal{X}$ are the ground sets of the connected components of M/L we have $\sum_{i=1}^{k} r(N_i/L) = r(M/L)$, so $\sum_{i=1}^{k}(r(N_i)-2) = r-2$. Using Claim 5.1.1 we have

$$|M \backslash L| = \sum_{i=1}^{k} |N_i \backslash L| \leq \sum_{i=1}^{k} \ell(r(N_i)-2) = \ell \sum_{i=1}^{k}(r(N_i)-2) = \ell(r-2).$$

Therefore, $|M| \leq \ell(r-2) + |L| \leq \ell(r-2) + \ell + 1 = \ell(r-1)+1$, as desired.

By carefully tracking when equality holds in the preceding computation, we can also show that the matroids in the extremal family of Definition 4.1 are the only ones to achieve the bound of Theorem 5.1. □

6 Future Work

6.1 3-connected positroids

Every simple rank-r positroid with no $U_{2,\ell+2}$-minor which achieves the bound of Theorem 1.2 is not 3-connected when $r \geq 3$ (see [13, page 293] for the definition). This leads to the following question: what is the maximum number of elements of a simple 3-connected rank-r positroid with no $U_{2,\ell+2}$-minor? Brylawski [6] proved that every 3-connected matroid which is both $M(K_4)$-minor-free and $U_{2,4}$-minor-free has at most 5 elements, so this question is not interesting when $\ell = 2$. However, for each $\ell \geq 3$ we can construct a family of 3-connected positroids, described below, that generalize the rank-r whirl and which we believe achieve the maximum number of elements.

Let r, ℓ be integers with $r \geq 2$ and $\ell \geq 3$. Let $W(r,\ell)$ be the matroid with a basis $B = \{b_1, b_2, \ldots, b_r\}$ and $\lfloor \frac{\ell-1}{2} \rfloor$ elements freely placed (see [13, page 270]) in the span of $\{b_i, b_{i+1}\}$ for all $i \in [r]$, reading indices modulo r. Construct $W(r,\ell)^+$ from $W(r,\ell)$ by freely placing one element in $\mathrm{cl}(\{b_1, b_2\})$. It is straightforward to show that $W(r,\ell)$ and $W(r,\ell)^+$ are 3-connected, $W(r,\ell)$ is $U_{2,\ell+2}$-minor-free, and $W(r,\ell)^+$ is $U_{2,\ell+2}$-minor-free when ℓ is even. Noting that $|W(r,\ell)| = r + r\lfloor \frac{\ell-1}{2} \rfloor$, we make the following conjecture.

Conjecture 6.1. *For integers r and ℓ with $r \geq 2$ and $\ell \geq 3$, every 3-connected rank-r positroid with no $U_{2,\ell+2}$-minor has at most $r + r\lfloor \frac{\ell-1}{2} \rfloor$ elements if ℓ is odd, and at most $r + r\lfloor \frac{\ell-1}{2} \rfloor + 1$ elements if ℓ is even.*

As evidence, these bounds hold when $r = 2$, and it is not difficult to prove that they hold when $r = 3$. By a result of Oxley [14, Corollary 4.3], they also hold for $\ell = 3$ in the special case that the positroids are ternary. Conjecture 6.1 may be approachable with techniques from [14] and [19].

6.2 Other $M(K_4)$-minor-free classes

There are several natural classes of $M(K_4)$-minor-free classes of matroids other than positroids. Most notably, *transversal matroids* encode matchable sets of vertices in bipartite graphs, and *gammoids* are minors of transversal matroids. *Strongly base-orderable* matroids [13, page 435, Exercise 10] and *k-base-orderable* matroids [5] are classes of matroids which satisfy strengthenings of the basis exchange axiom and contain all gammoids. The class of transversal matroids is incomparable with the class of positroids, but every positroid is a gammoid [7], and is thus both strongly base-orderable and k-base-orderable for all $k \geq 1$. These classes of matroids are indeed $M(K_4)$-minor-free by [13, page 435, Exercise 11]. It is thus natural to ask the following question:

Problem 6.2. *Does the bound of Theorem 1.2 hold for transversal matroids, gammoids, strongly base-orderable matroids, or k-base-orderable matroids?*

Problem 6.2 seems approachable for transversal matroids, because one can show that all of the matroids of Figures 2 and 3 are not transversal. However, the class of transversal matroids is not closed under contraction, and we do make use of contraction several times in the proof of Theorem 1.2. An important challenge in studying Problem 6.2 for the other classes of matroids suggested is that one can show via [12, Theorem 2.4] that all of the matroids of Figures 2 and 3 other than $M(K_4)$ are gammoids. Accordingly, while Problem 6.2 is interesting, solving it will require new ideas. We leave this an open direction for future work.

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