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On Orbit Realizations of Coxeter Permutahedra

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Abstract. The permutahedron is a fundamental polytope associated with the symmetric group and serves as a central example of the interaction between geometry, combinatorics, and algebra. More generally, for W a finite reflection group, one can define the W-permutahedron. In this work, we study the *orbit realizations* of the W-permutahedron, which are obtained by taking the convex hull of the W-orbit of any point in the fundamental chamber. Although all orbit realizations have the same normal fan, they do not have the same geometric structure. We show that the orbit realization space of the W-permutahedron is contractible. We also explore whether various geometric and combinatorial features persist among all orbit realizations. In particular, we show that certain subsets of vertices always form simplices, and some triangulations similarly persist. We experimentally explore the cases of S_3 and S_4 and we uncover some non-persistent phenomena. Our results suggest a rich internal structure within orbit realization spaces, motivating further study of their topology and combinatorial geometry.

Keywords: permutahedron, realization space, triangulation, oriented matroid

1 Introduction

The permutahedron is a classical polytope associated with the symmetric group, first studied by Schoute in 1911 [15]. Its vertices are labeled bijectively by the elements of S_n and its geometry reflects the combinatorics of S_n. Its one-skeleton is the Hasse diagram of the weak order. Its face lattice is isomorphic to the lattice of cosets of parabolic subgroups (ordered by inclusion), or equivalently, to ordered block partitions of $[n]$ (ordered by coarsening). The best known realization of the permutahedron is the convex hull of all permutation vectors $\{(w(1), \ldots, w(n)) : w \in S_n\}$, which we call the *regular permutahedron*. This polytope has a myriad of interpretations and connections to other objects in algebraic and geometric combinatorics. The regular permutahedron is the graphical zonotope associated to the complete graph, as well as the secondary polytope

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of a simplicial prism [4]. There is also a canonical surjection from the Birkhoff polytope (the convex hull of permutation matrices) to the regular permutahedron, reflecting the projection from matrix space to coordinate permutations. The normal fan of the regular permutahedron is the famous braid arrangement, providing a connection to Coxeter groups and algebraic geometry. For instance, [3] constructs the permutohedral variety – the toric variety whose fan is the braid arrangement.

There are many other realizations of the permutahedron besides the regular permutahedron. Recall that the *realization space* $\mathcal{R}(P)$ of a d-dimensional polytope P with vertex set V is a semi-algebraic set inside $(\mathbb{R}^d)^{|V|}$ defined by the face lattice of the polytope and describing all coordinate configurations of V that preserve the combinatorial incidence relations of P (see [2, 12] for formal definitions). The entire $\mathcal{R}(P)$ is a mysterious, difficult object even for familiar polytopes. However, as we know the face lattice of the permutahedron is invariant under the S_n-action so it is natural to restrict one's attention to realizations which are also preserved by the S_n-action. In this paper, we investigate such realizations of the permutahedron, called the *orbit permutahedra*, and we denote by $\mathcal{R}_{\mathcal{O}}$ the set of all orbit permutahedra, which we call the *orbit realization space*. It is a proper subset of the realization space $\mathcal{R}(P)$.

Concretely, for $\mathbf{a} \in \mathbb{R}^n$ with strictly increasing coordinates, the *orbit permutahedron* $P(\mathbf{a})$ is the convex hull of the $n!$ points obtained by permuting the coordinates of $\mathbf{a}$. The normal fan of an orbit permutahedron is again the braid arrangement. Their facet inequalities are described by a classical result of Rado [10]. The volume and Ehrhart polynomial of orbit permutahedra were studied in the beautiful work of Postnikov [9]. Other combinatorial properties are described in [8].

More generally, one may replace S_n by an arbitrary finite reflection group W, the braid arrangement by the hyperplane arrangement H^W for W, and define the *orbit W-permutahedron* in exactly the same way: for any base point $\mathbf{a}$ in the fundamental chamber B of H^W, $P^W(\mathbf{a}) := \text{conv}(W \cdot \mathbf{a})$. The *orbit realization space* $\mathcal{R}_{\mathcal{O}}^W$ of W-permutahedra consists exactly of $\{P^W(\mathbf{a}) : \mathbf{a} \in B\}$. The main focus of our work is $\mathcal{R}_{\mathcal{O}}^W$. For more details on H^W and $P^W(\mathbf{a})$ see [1, 6].

The orbit realization space $\mathcal{R}_{\mathcal{O}}^W$ is the key object of study of this paper and we list now our contributions: While we do not know the topology of $\mathcal{R}(P^W(\mathbf{a}))$, in Theorem 7 we show that the orbit realization space $\mathcal{R}_{\mathcal{O}}^W$ of the W-permutahedron is homeomorphic to an open ball. This opens up the possibility of studying orbit permutahedra via continuous deformation arguments. We also explore geometric properties of $P^W(\mathbf{a})$ that hold regardless of the choice of $\mathbf{a}$. We focus specifically on triangulations, oriented matroids, and related properties, with an eye towards determining structures which "persist" for all choices of base point $\mathbf{a}$. Our initial motivation was [5, Theorem 1.5], which provided many triangulations of $P^W(\mathbf{a})$ that persist for all choices of $\mathbf{a}$. This leads to a number of natural questions we explore: Which triangulations of $P^W(\mathbf{a})$ remain triangulations as we move the base point? Given $S \subset W$ of size $d+1$, when is $\text{conv}(S \cdot \mathbf{a})$ a simplex for all choices of $\mathbf{a}$? What can be said about the *oriented matroid* (see [2]) of $P^W(\mathbf{a})$ over different choices of $\mathbf{a}$?

The answers to questions of this flavor prove to be quite subtle; We begin with Section 2 where we explore the permutahedra for S_3 and S_4. This is an experimental investigation of what is preserved or persists under different choices of $\mathbf{a}$, we show a number of interesting phenomena, including a count on the number of oriented matroids. Next, in Section 3 we discuss the contractibility of the orbit realization space, showing that for orbit realizations the space is homeomorphic to an open cone. In Section 4 we give a sufficient and necessary condition for a triangulation to persist, as well as sufficient conditions on persistent simplices and persistently coplanar sets in relation to subsets that are related by transpositions. In Section 5 we show that for generic $\mathbf{a}$, the set of $\mathbf{a}$ simplices (and thus the associated matroid) is constant, however, we also demonstrate that the oriented matroid is not constant for generic $\mathbf{a}$. Finally, in Section 6 we propose open questions on several topics: properties of the realization spaces of the permutahedron, persistent simplices, persistently coplanar sets, persistent hyperplanes, and persistent triangulations.

2 Definitions and warmup

As in the introduction, W is a finite reflection group with d simple reflections and B is the fundamental chamber of the associated hyperplane arrangement. We always identify the vertices of $P^W(\mathbf{a})$ with W, so that the vertices of $P^W(\mathbf{a})$ and $P^W(\mathbf{a}')$ are in natural correspondence and it makes sense to talk about structure "persisting" for different choices of base point. To streamline notation, we introduce the following.

Definition 1. For $\mathcal{S} \subset W$ of size $d+1$ and $\mathbf{a} \in B$, $\mathcal{S}$ is an $\mathbf{a}$-*simplex* if $\operatorname{conv}(\mathcal{S} \cdot \mathbf{a})$ is a d-dimensional simplex and is $\mathbf{a}$-coplanar otherwise. A collection $\{\mathcal{S}_i\}_{i=1}^p$ of $\mathbf{a}$-simplices is called an $\mathbf{a}$-*triangulation* if $\{\operatorname{conv}(\mathcal{S}_i \cdot \mathbf{a})\}_{i=1}^p$ is a triangulation of $P^W(\mathbf{a})$. That is, $P^W(\mathbf{a}) = \cup_i \operatorname{conv}(\mathcal{S}_i \cdot \mathbf{a})$ and for all i, j, $\operatorname{conv}(\mathcal{S}_i \cdot \mathbf{a})$ and $\operatorname{conv}(\mathcal{S}_j \cdot \mathbf{a})$ intersect in a common face (possibly empty).

We note that the set of $\mathbf{a}$-simplices is exactly the set of bases of the *matroid* of $P^W(\mathbf{a})$. As we shall see in subsequent sections, the set of $\mathbf{a}$-simplices and of $\mathbf{a}$-triangulations is heavily dependent on the choice of $\mathbf{a}$. So we are more interested in the following notion.

Definition 2. We say that $\mathcal{S} \subset W$ of size $d+1$ is a *persistent simplex* (respectively, is *persistently coplanar*) if it is an $\mathbf{a}$-simplex for every choice of $\mathbf{a} \in B$ (respectively, is $\mathbf{a}$-coplanar for all $\mathbf{a} \in B$). A collection $\{\mathcal{S}_i\}$ of persistent simplices is a *persistent triangulation* if $\{\mathcal{S}_i\}$ is an $\mathbf{a}$-triangulation for each $\mathbf{a} \in B$.

Through several examples we illustrate these concepts in the next two subsections.

2.1 The S_3 case

For $W = S_3$ orbit permutahedra, the fundamental chamber B consists of vectors with strictly increasing coordinates. For $\mathbf{a} = (a_1, a_2, a_3) \in B$, $P(\mathbf{a})$ lies in the hyperplane $\{x_1 + x_2 + x_3 = a_1 + a_2 + a_3\}$ and is a hexagon that is invariant under permuting coordinates. All 3-element subsets $\mathcal{S} \subset S_3$ are persistent simplices. In addition, all triangulations are persistent.

The realization space of the S_3 permutahedron consists of all hexagons. All of the results above hold for combinatorial permutahedra. However, not all combinatorial permutahedra are orbit permutahedra, as one can easily find hexagons in $\mathbb{R}^3$ which are not invariant under permuting coordinates. For emphasis, we record this in a proposition.

Proposition 3. *The realization space of the W-permutahedron is strictly larger than the orbit realization space.*

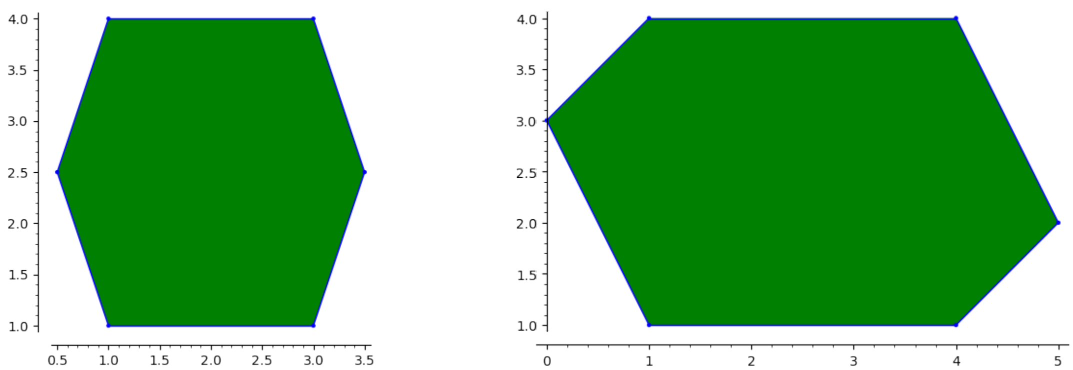


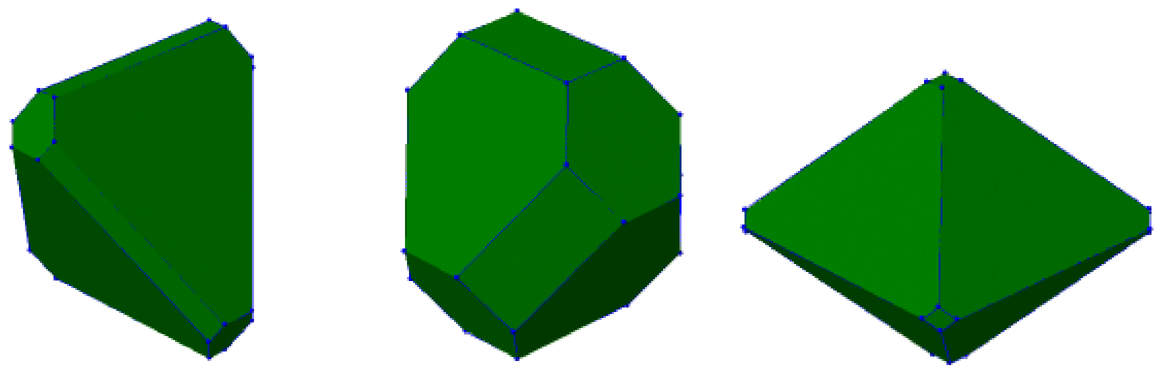
Figure 1: Hexagons which are not (projections of) an orbit S_3-permutahedron.

2.2 The S_4 case

Again $\mathbf{a}$ has increasing coordinates, and $P(\mathbf{a})$ is a 3-dimensional polytope in $\mathbb{R}^4$. From the work of Steinitz [16] [17], the realization space of the S_4 permutahedron (modulo affine equivalence) is contractible. Again, the orbit realization space is a proper subset of the realization space. The geometry of orbit realizations strongly depends on $\mathbf{a}$ (see Figure 2), as does the collection of $\mathbf{a}$-simplices. For example, the permutations $\{(1,2,3,4), (1,3,4,2), (2,3,1,4), (3,4,2,1)\}$ form an $\mathbf{a}$-simplex if and only if the polynomial

$$a_4^2 a_1 - a_4^2 a_2 - a_1^2 a_2 + a_4 a_2^2 + a_1 a_2^2 - a_2^3 - 2a_4 a_1 a_3 + a_1^2 a_3 + a_2^2 a_3 + a_4 a_3^2 - a_2 a_3^2$$

does not vanish; if the polynomial vanishes, these permutations are instead $\mathbf{a}$-coplanar. See Figure 3 for a different, concrete example of this phenomenon.



(a) (101714,753589,861628,925275) (b) $(1, 2, 3, 5)$ (c) $(32919, 67492, 433170, 464108)$

Figure 2: Various orbit S_4-permutahedra $P(\mathbf{a})$. Displayed beneath each polytope is the choice of base point $\mathbf{a}$.

Example 4. *Let* $\mathcal{S} = \{(1234), (2143), (3412), (4321)\}$. *Then as shown in Figure 3,* $\mathcal{S}$ *is an* $\mathbf{a}$*-simplex for* $\mathbf{a} = (101714, 753589, 861628, 925275)$ *but is* $\mathbf{a}'$*-coplanar for* $\mathbf{a}' = (1, 2, 3, 4)$. *In fact, there are* 846 $\mathbf{a}'$*-coplanar sets, while there are only* 474 $\mathbf{a}$*-coplanar sets.*

Proposition 5. *For subsets of size* 4 *in* S_4*, there are* 474 *persistently coplanar sets,* 8766 *persistent simplices, and the remaining* 1386 *take both forms depending on the choice of* $\mathbf{a}$. *The maximal number of coplanar sets is* 846 *which is achieved for the regular permutahedron.*

To investigate the triangulations of $P(\mathbf{a})$, one might turn to the oriented matroid, or chirotope, of the point configuration $S_4 \cdot \mathbf{a}$. Assume without loss of generality that the coordinates of $\mathbf{a}$ do not sum to 0. The *chirotope* of $P(\mathbf{a})$ is the function

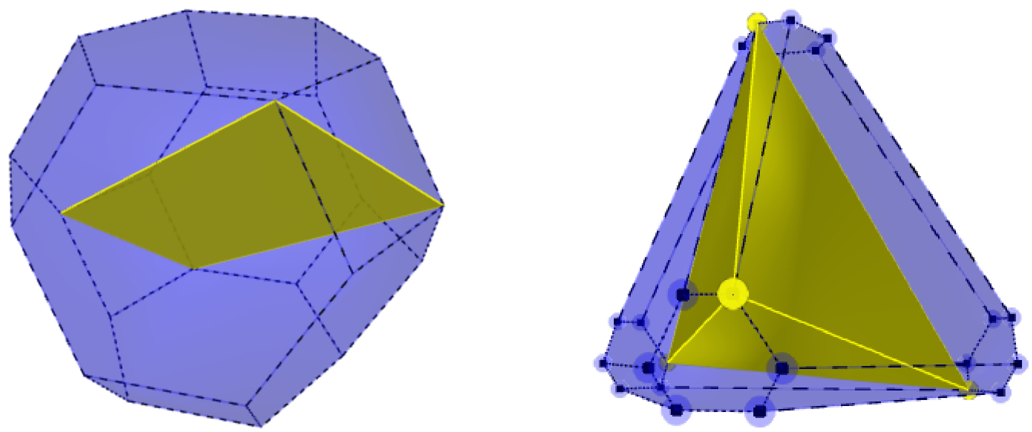
$$\begin{aligned} \chi : \{\mathcal{S} \subset S_4 : |\mathcal{S}| = 4\} &\to \{0, +, -\} \\ \{v_1, v_2, v_3, v_4\} &\mapsto \operatorname{sign}(\det[v_1 \cdot \mathbf{a}, v_2 \cdot \mathbf{a}, v_3 \cdot \mathbf{a}, v_4 \cdot \mathbf{a}]) \end{aligned}$$

where we list the elements of $\mathcal{S}$ according to a fixed arbitrary total order of the permutations in S_4.

We enumerated all generic chirotopes for S_4 and we counted 80 for a fixed ordering. The chirotope of the regular permutahedron had the most zeros (i.e., coplanar sets of vertices), and most number of nonzero values was 10152. The reader can find the computational result in github:Permutahedra.

By [4, Corollary 4.1.44], if $P(\mathbf{a})$ and $P(\mathbf{a}')$ have the same chirotope, the set of $\mathbf{a}$-triangulations is equal to the set of $\mathbf{a}'$-triangulations. We have already seen that the chirotope of $P(\mathbf{a})$ is strongly dependent on $\mathbf{a}$. Example 4 shows that even the number of sets which χ sends to 0 depends on $\mathbf{a}$.

Remark 6. *The oriented matroid of an orbit permutahedron is not determined (even up to isomorphism) by its Coxeter group and dimension.*



(a) $(1,2,3,4)$ (b) $(101714, 753589, 861628, 925275)$

Figure 3: The convex hull of $\{(1234), (2143), (3412), (4321)\} \cdot \mathbf{a}$ in two different orbit permutahedra.

The subdivision of B according to the chirotope of $P(\mathbf{a})$ is nonlinear and relatively intricate, as Figure 4 illustrates.

3 Topology of orbit realization space

Now we would like to give some context for our results. If P is a 2 or 3-dimensional polytope, $\mathcal{R}(P)/(\text{affine transformations})$ is contractible [16] [17]. But in general, $\mathcal{R}(P)$ can be extremely poorly behaved. Indeed, N. Mnëv [7] proved that realization spaces of convex polytopes (indeed, of oriented matroids) can be as complicated as *any* semialgebraic set. More precisely, for every primary semialgebraic set V defined over the integers, there exists a (sufficiently high-dimensional) convex polytope P whose realization space $\mathcal{R}(P)$ is "stably equivalent" (in homotopy type, and rationally equivalent) to V. In particular, questions about solutions to polynomial equations over $\mathbb{R}$ can be encoded as instances of polytope realization problems. Later work by J. Richter-Gebert [14, 12, 13] greatly strengthened the universality phenomenon by showing the complicated topology already occurs in 4-dimensional polytopes. (This is in contrast to the fact that 2 and 3-dimensional polytopes have contractible realization spaces.)

However, the situation for the orbit realization space is much simpler, as the following result shows.

Theorem 7. *Let B be a choice of fundamental chamber for some finite reflection group W and let $\mathcal{R}_{\mathcal{O}}^W = \{\operatorname{conv}(W \cdot \mathbf{a}) : \mathbf{a} \in B\}$ be the orbit realization space. Then $\mathcal{R}_{\mathcal{O}}^W$ is homeomorphic to B.*

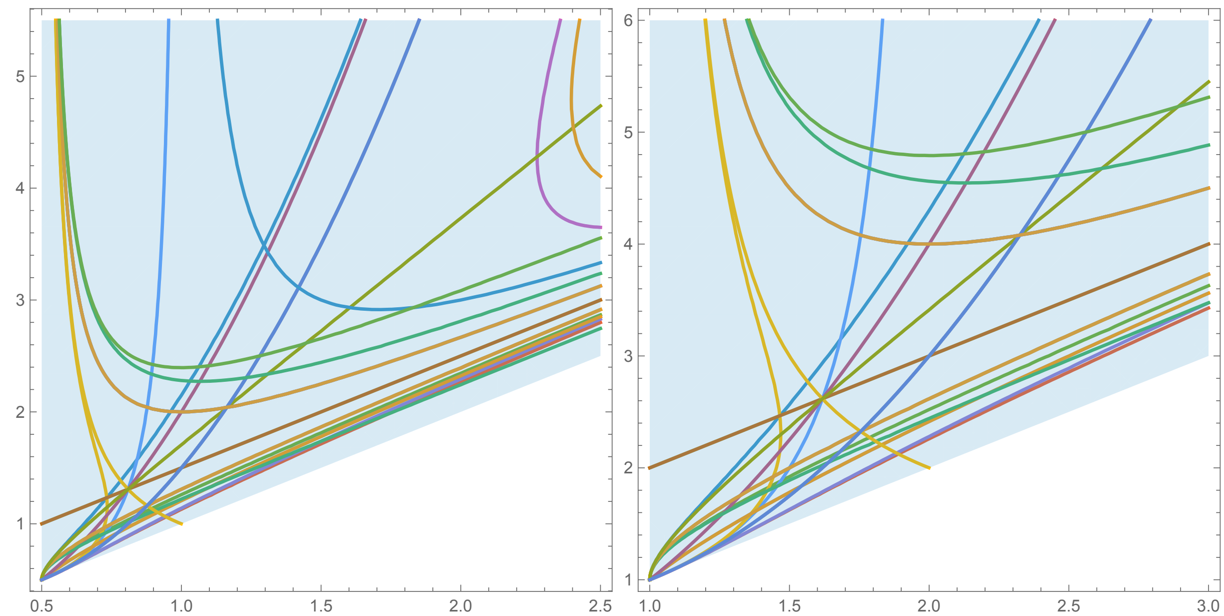


Figure 4: On the left, the plane $(0, 1/2, y, z)$, and on the right, the plane $(0, 1, y, z)$. The intersection B' with the fundamental chamber B shaded in blue. We show the subdivision of B' according to the chirotope of $P(\mathbf{a})$. That is, all $\mathbf{a}$ in the same region have the same chirotope. When you cross one of the curves shown, the chirotope changes sign on some subsets of S_4.

We note that $\mathcal{R}_{\mathcal{O}}^{W}$ is clearly in bijection with B; the content of the above theorem is that this bijection is topologically well-behaved.

Corollary 8. *The orbit realization space $\mathcal{R}_{\mathcal{O}}^{W}$ is contractible and thus path-connected.*

4 Results on persistence of triangulations and simplices

The starting point for our investigation of persistence is the following result.

Theorem 9. *An $\mathbf{a}$-triangulation is a persistent triangulation if and only if it consists of persistent simplices.*

By definition, a persistent triangulation consists of persistent simplices, so one direction of the above theorem is clear. The converse direction is the more interesting one. It tells us that we just have to check that a collection of persistent simplices forms a triangulation for some $P^W(\mathbf{a})$, and then we know that they form a triangulation for all orbit permutahedra.

Proof sketch of Theorem 9. Suppose $\{\mathcal{S}_i\}$ is an $\mathbf{a}$-triangulation consisting of persistent simplices. Choose $\mathbf{a}' \in B$. By Corollary 8, there is a path in $\mathcal{R}^W_\mathcal{O}$ from $P^W(\mathbf{a})$ to $P^W(\mathbf{a}')$. By assumption, each $\mathcal{S}_i$ is an $\mathbf{a}''$-simplex for all $P^W(\mathbf{a}'')$ along this path. This means that the orbit permutahedra along this path, together with the collection $\{\mathcal{S}_i\}$, satisfy the assumptions of [5, Proposition 3.3], so that result implies that $\{\mathcal{S}_i\}$ is an $\mathbf{a}'$-triangulation as well. □

Theorem 9 shows that the first step to understanding persistent triangulations is to understand persistent simplices. While we do not have a complete characterization, we have the following sufficient condition.

Proposition 10. *Suppose $\mathcal{S} \subset S_n$ has size n. Define*

$$T := \{\{i, j\} : \text{ for some } v, w \in \mathcal{S},\ vw^{-1} = (i\ j)\}.$$

and consider T as a graph on $[n]$. If T is connected, then $\mathcal{S}$ is a persistent simplex.

Proof sketch. If $v = (i\ j)w$, then the line segment $v \cdot \mathbf{a} - w \cdot \mathbf{a}$ is parallel to $e_i - e_j$, regardless of the choice of point $\mathbf{a} \in B$. Using the fact that the matroid of the roots $\{e_i - e_j\}_{i<j}$ is the graphical matroid of the complete graph, one can show that any spanning tree of T corresponds to $n - 1$ linearly independent differences $v \cdot \mathbf{a} - w \cdot \mathbf{a}$ where $v, w \in \mathcal{S}$. This shows that $\operatorname{conv}(\mathcal{S} \cdot \mathbf{a})$ is a simplex. □

The converse of Proposition 10 does not hold in general—some persistent simplices contain relatively few permutations which differ by transpositions. For instance, in S_4 there are only 3396 persistent simplices of this type, which is only about 40 percent of the total 8766 persistent simplices.

We can similarly give a partial description of the persistently coplanar sets. Let $G_\mathcal{S}$ be the graph with vertices $\mathcal{S}$ and edge set

$$\{\{v, w\} : v, w \in \mathcal{S} \text{ and } vw^{-1} \text{ is a transposition}\}.$$

Proposition 11. *Suppose $\mathcal{S} \subset S_n$ has size n. Suppose there is a forest $F \subset G_\mathcal{S}$ such that the graph T_F on $[n]$ with edges*

$$\{\{i, j\} : vw^{-1} = (i\ j) \text{ for some edge } \{v, w\} \in F\}$$

is not a forest, then $\mathcal{S}$ is persistently coplanar.

Proof sketch. Again, the edges of T' correspond to roots which are parallel to $v \cdot \mathbf{a} - w \cdot \mathbf{a}$, for certain $v, w \in \mathcal{S}$. If T_F is not a forest, then the roots $\{e_i - e_j : \{i, j\} \in T_F\}$ are linearly dependent. This implies that the affine hull of $\mathcal{S} \cdot \mathbf{a}$ is not full-dimensional. □

5 Complications for non-persistent triangulations

In this section, we explore some of the non-persistent behavior of triangulations and simplices in orbit permutahedra. One natural strategy is to study the behavior holding for a dense subset of the possible base points. Along these lines, we characterize $\mathbf{a}$-simplices for “most" $\mathbf{a}$.

Proposition 12. *If $\mathbf{a} \in B$ is sufficiently generic, then the set of $\mathbf{a}$-simplices is equal to*

$$\{\mathcal{S} \subset W : |\mathcal{S}| = d+1, \mathcal{S} \textit{ not persistently coplanar}\}$$

and in particular does not depend on $\mathbf{a}$.

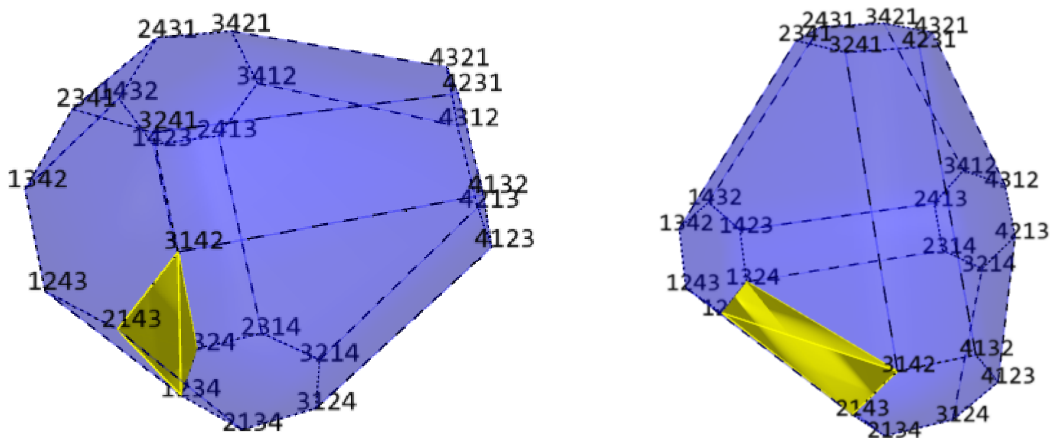
This follows from the fact that the set of $\mathbf{a}$ such that $\mathcal{S}$ is *not* an $\mathbf{a}$-simplex is Zariski-closed. Thus, if $\mathcal{S}$ is not persistently coplanar, then $\mathcal{S}$ is an $\mathbf{a}$-simplex for $\mathbf{a}$ in a Euclidean-dense subset of B.

Proposition 12 may inspire hope that the set of $\mathbf{a}$-triangulations does not depend on $\mathbf{a}$ as long as $\mathbf{a}$ is sufficiently generic; that is, if $\{\mathcal{S}_i\}$ is an $\mathbf{a}$-triangulation for one generic $\mathbf{a}$, it will be an $\mathbf{a}$-triangulation for a (Euclidean) dense subset of B. However, this is far from being true, as we now explain. The key point is that, though the matroid of $P^W(\mathbf{a})$ is constant for generic $\mathbf{a}$, the oriented matroid is not, so triangulations do not “generically persist."

Consider $\mathcal{S} = \{(1234), (1324), (2143), (3142)\} \subset S_4$. Then the signed volume of $\mathrm{conv}(\mathcal{S} \cdot \mathbf{a})$ is positive for $\mathbf{a} = (1, 2, 3, 6)$ and is negative for $\mathbf{a} = (1, 5, 6, 7)$ (see Figure 5). In other words, imagine moving along the path $\mathbf{a}(t) = (1, 2, 3, 6)(1-t) + t(1, 5, 6, 7)$. At the beginning, say for $t < t_0$, one vertex of $\Delta(t) := \mathrm{conv}(\mathcal{S} \cdot \mathbf{a}(t))$ is on one side of the hyperplane H_t spanned by the other vertices. For a while, this vertex stays on this side. Then, for some t_0, this vertex lands on the hyperplane H_{t_0}, and for $t > t_0$, the vertex lies on the other side of the hyperplane H_t. Perturbing our path slightly if necessary, we can assume that for $t \in [t_0 - \epsilon, t_0 + \epsilon]$, $\mathbf{a}(t)$ is generic in the sense of Proposition 12, and that the only simplex which collapses for $\mathbf{a}(t_0)$ is $\Delta(t_0)$. For $t \in [t_0 - \epsilon, t_0)$, $\mathcal{S}$ is part of some $\mathbf{a}(t)$-triangulation $\{\mathcal{S}_i\}$ (for example, there is a regular triangulation of $P^W(\mathbf{a}(t))$ that includes $\Delta(t)$). As such, the hyperplane H_t intersects $\Delta(t)$ and another simplex, say $\mathrm{conv}(\mathcal{S}_1 \cdot \mathbf{a}(t))$ in a common facet. This means that $\{\mathcal{S}_i\}$ cannot be an $\mathbf{a}(t)$-triangulation for $t \in (t_0, t_0 + \epsilon]$, because for those t, $\Delta(t)$ and $\mathrm{conv}(\mathcal{S}_1 \cdot \mathbf{a}(t))$ have intersecting interiors.

6 Future Work and Conjectures

There are a couple of further questions that we plan to explore next:



(a) $(1,2,3,6)$ (b) $(1,5,6,7)$

Figure 5: The convex hull of $\{(1234),(1324),(2143),(3142)\}\cdot \mathbf{a}$ in two different orbit permutahedra. Note how the simplex has negative volume in (a) and positive volume in (b).

6.1 More on the Realization Spaces

One can weaken the restriction of the realizations from coming from just orbit realizations to those of a combinatorial permutahedron. Not much work has been done on the contractibility or connectivity of these spaces, nor has much work been done on different ways to realize the permutahedron through other methods. This leads to several questions:

- Orbit realization of permutahedra are parametrized by elements of the fundamental chamber of finite reflection groups. What objects, if any, parametrize combinatorial permutahedra?
- Is there a uniform method to produce all realizations of the permutahedron up to some kind of equivalence (affine, projective, normal, etc.)?
- Is the realization space of the permutahedron contractible or connected? More generally, what is its topology?

6.2 Persistence

As we described in Proposition 10, one class of persistent simplices are those determined by their transpositions. These represent a fraction of total persistent simplices as we explicitly compute for S_4 in Proposition 5. This leads to the following questions: Is there a classification of all the persistent simplices in type A? Is there one that extends naturally to all types?

The dual question of which sets are persistently coplanar is also of interest. Proposition 11 gives a sufficient condition for $\mathcal{S} \subset S_n$ to be persistently nonplanar. This condition is also necessary for $n = 4$, but is not necessary for $n = 5$. Adding in a couple more conditions gives us the following conjectural characterization of persistently coplanar sets.

Conjecture 13. *Let $\mathcal{S} \subset S_n$ have size n, T be as in Proposition 10, and $G_{\mathcal{S}}$ be as in Proposition 11. Then $\mathcal{S}$ is a persistently coplanar set if and only if it satisfies one of the following:*

- *There is a forest $F \subset G_{\mathcal{S}}$ such that the associated transposition graph T_F is not a forest;*
- *There are d elements of $\mathcal{S}$ contained in a parabolic subgroup generated by $d-2$ simple transpositions*
- *Viewing the elements of T as transpositions, for some subset $T' \subseteq T$ of size at least 2 consisting of commuting transpositions, there is some $v, w \in \mathcal{S}$ such that $vw^{-1} = \prod_{\sigma \in T'} \sigma$*
- *There are four distinct elements, $v, w, x, y \in \mathcal{S}$, such that $vw^{-1} = \sigma$, $vx^{-1} = \tau$, and $vy^{-1} = \sigma\tau$ where σ and τ are two commuting permutations.*

A similar style of problem one can ask is which hyperplanes persist under different choices of orbits? One obvious example of a persistent family of hyperplanes are the facet-defining hyperplanes of orbit permutahedra (they only depend on the combinatorics). But finding persistent families seems tricky. For example, we discovered (inner) hyperplanes parallel to facets are not necessarily preserved under varying choice of basepoint. This family is interesting as it contains the Duistermaat–Heckman hyperplanes (see [11] and references there).

Note that just like the study of persistent simplices ties in with the bases of the oriented matroid defined by the vertices of the permutahedron, the persistent hyperplanes are related to the covectors and cocircuits of the same oriented matroid. It is known that two point configurations with the same oriented matroid will have the same set of triangulations [4], and some classes of triangulations are known to depend only on the combinatorial type of a polytope, the main class being lexicographic triangulations. We have the following conjecture:

Conjecture 14. *The persistent triangulations of the permutahedron are exactly its lexicographic triangulations.*

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