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A combinatorial formula for Interpolation Macdonald polynomials[*]

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Abstract. In 1996, Knop and Sahi introduced a remarkable family of inhomogeneous symmetric polynomials, defined via vanishing conditions, whose top homogeneous parts are exactly the *Macdonald polynomials*. Like the Macdonald polynomials, these *interpolation Macdonald polynomials* are closely connected to the Hecke algebra, and admit nonsymmetric versions, which generalize the nonsymmetric Macdonald polynomials. In this paper we give a combinatorial formula for interpolation Macdonald polynomials in terms of *signed multiline queues*; this formula generalizes the combinatorial formula for Macdonald polynomials in terms of multiline queues given by Corteel–Mandelshtam–Williams.

Keywords: Interpolation Macdonald polynomials, ASEP polynomials, multiline queues, tableaux, Hecke operators.

1 Interpolation polynomials

Macdonald polynomials, introduced by Ian Macdonald in 1989 [18], are one of the most interesting families of polynomials in mathematics: they have connections to the geometry of the Hilbert scheme [15], and admit various beautiful combinatorial formulas in terms of tableaux [14], *multiline queues* [12], and *vertex models* [1]. There is a fascinating inhomogeneous generalization of Macdonald polynomials called *interpolation Macdonald polynomials*, introduced by Knop [17] and Sahi [24] around 1996, and further studied in [22, 23, 9]. These polynomials are related to gauge theories and vertex operators [8], the HOMFLY polynomial [16], and the theory of link invariants of $\mathfrak{gl}_n$ [3]. In the Jack limit, interpolation polynomials were recently proved to be monomial-positive [21] and shown to be closely related to the theory of non-orientable combinatorial maps [4].

The main result of this paper is a combinatorial formula for interpolation Macdonald polynomials. These polynomials can be defined via vanishing conditions as in Theorem 1.1.

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[*] This is an extended abstract of [5].

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We denote by $\mathbb{N}$ the set of nonnegative integers. Given a composition $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$, we define

$$k_i(\mu) := \#\{j : j < i \text{ and } \mu_j > \mu_i\} + \#\{j : j > i \text{ and } \mu_j \geqslant \mu_i\}, \text{ and} \tag{1.1}$$

$$\widetilde{\mu} := \left(q^{\mu_1} t^{-k_1(\mu)}, \dots, q^{\mu_n} t^{-k_n(\mu)}\right). \tag{1.2}$$

For example, when $\mu = (4, 2, 0, 1, 4)$ we have $\widetilde{\mu} = (q^4 t^{-1}, q^2 t^{-2}, t^{-4}, q t^{-3}, q^4)$.

Theorem 1.1. *[17, 24] For each partition $\lambda = (\lambda_1, \dots, \lambda_n)$, there is a unique inhomogeneous symmetric polynomial $P^*_\lambda = P^*_\lambda(\boldsymbol{x}; q, t) = P^*_\lambda(x_1, \dots, x_n; q, t)$ with degree at most $|\lambda|$, called the* interpolation Macdonald polynomial, *such that*

- *the coefficient $[m_\lambda]P^*_\lambda$ of the monomial symmetric polynomial in P^*_λ is 1,*
- *$P^*_\lambda(\tilde{\nu}) = 0$ for each partition $\nu \neq \lambda$ with $|\nu| \leqslant |\lambda|$.*

Moreover, the top homogeneous component of P^*_λ is the usual Macdonald polynomial P_λ.

There are also *nonsymmetric Macdonald polynomials E_μ*, introduced by Cherednik [11], associated to any composition $\mu \in \mathbb{N}^n$; these also have interpolation analogues E^*_μ due to Knop and Sahi.

More recently the so-called *ASEP polynomials f_μ* were introduced in connection to the *asymmetric simple exclusion process* (ASEP), see [7, 10]. The ASEP polynomials are in fact special cases of the *permuted-basement Macdonald polynomials* introduced in [13], as shown in [12].

In this article we define *interpolation ASEP polynomials* as in Definition 1.3 below; they have the property that their top homogeneous component recovers the usual ASEP polynomials. Our main result is a combinatorial formula for both the interpolation ASEP polynomials and the interpolation symmetric Macdonald polynomials, see Theorem 4.5.

Definition 1.2 (Hecke operators)**.** For $1 \leqslant i \leqslant n-1$, we let $s_i = (i, i+1)$ denote the transposition exchanging x_i and x_{i+1}. The *Hecke operator T_i* is the operator defined by

$$T_i := t - \frac{t x_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i). \tag{1.3}$$

These operators satisfy the relations of the Hecke algebra of type A_{n-1}.

Definition 1.3 (Interpolation ASEP polynomials)**.** Fix a partition λ. For $\mu \in S_n(\lambda)$, the *ASEP polynomial f_μ* is the homogeneous polynomial defined by

$$f_\mu = T_{\sigma_\mu} \cdot E_\lambda,$$

where σ_μ is the shortest permutation in S_n such that $\sigma_\mu(\lambda) = \mu$, with

$$\sigma \cdot (\mu_1, \mu_2, \dots, \mu_n) := (\mu_{\sigma^{-1}(1)}, \mu_{\sigma^{-1}(2)}, \dots, \mu_{\sigma^{-1}(n)}). \tag{1.4}$$

In particular, $f_\lambda = E_\lambda$. Similarly, we define the *interpolation ASEP polynomial f^** by $f^*_\mu := T_{\sigma_\mu} \cdot E^*_\lambda$. In particular, $f^*_\lambda = E^*_\lambda$.

Since the top homogeneous part of E^*_λ is E_λ, we get that the top homogeneous part of f^*_μ is then the ASEP polynomial f_μ. In particular, the degree of f^*_μ is $|\mu|$. We provide a characterization for interpolation ASEP polynomials with vanishing conditions, which justifies their name.

Theorem 1.4. *Fix $\mu \in S_n(\lambda)$ of size d. Then $f^*_\mu(x_1, \ldots, x_n)$ is the unique polynomial g with degree at most d such that:*

- *for any composition ν such that $|\nu| \leqslant |\mu|$ and $\nu \notin S_n(\lambda)$, we have $g(\widetilde{\nu}) = 0$.*
- *for $\tau \in S_n(\lambda)$, we have $[x^\tau]g = \delta_{\tau,\mu}$.*

2 Multiline queues and signed multiline queues

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$ be a partition. We can describe such a partition by its *content vector* $\mathbf{m} = (m_0, m_1, \ldots, m_L)$, where $m_i = \#\{j : \lambda_j = i\}$, and L is the largest part that occurs. Sometimes we denote our partition by $\lambda = \langle L^{m_L}, \ldots, 1^{m_1}, 0^{m_0} \rangle$. We have $\sum_{i=0}^{L} m_i = n$.

Definition 2.1. [12] Fix a partition $\lambda = \langle L^{m_L}, \ldots, 1^{m_1}, 0^{m_0} \rangle$ with $\sum_{i=0}^{L} m_i = n$. A *ball system* B of *content* λ is an $L \times n$ array, with rows labeled from bottom to top as $1, 2, \ldots, L$, and columns labeled from left to right from 1 to n, in which each of the Ln positions is either empty or occupied by a ball, and such that there are $m_L + m_{L-1} + \cdots + m_r$ balls in row r. We label each ball with an element of $\{1, \ldots, L\}$ (viewing empty spots as 0), such that in row r our configuration of balls gives a permutation of

$$\lambda^{(r)} := \langle L^{m_L}, (L-1)^{m_{L-1}}, \ldots, r^{m_r}, 0^{m_{r-1}+\cdots+m_0} \rangle.$$

Definition 2.2 is a slight variant of a definition from [20].

Definition 2.2. [12] A *multiline queue* (or MLQ) of *type* $\mu \in S_n(\lambda)$ is a ball system of content λ such that each ball in row $r > 1$ is paired with a ball of the same label in the row below it, and the configuration of balls on the bottom row is μ. We require that the set of pairings between row r and $r-1$ form a *classic layer*, i.e. satisfy the following rules:

- We pair two balls using a shortest strand that travels either straight down or from left to right, allowing the strand to wrap around the cylinder if necessary;
- In row r, each ball with label a has either an empty spot below it, or a ball with label a', where $a' \geqslant a$, and if $a = a'$, they must be trivially paired, i.e. paired to each other with a straight segment.

See Figure 1 for an example. Let MLQ(μ) denote the set of multiline queues of type μ.

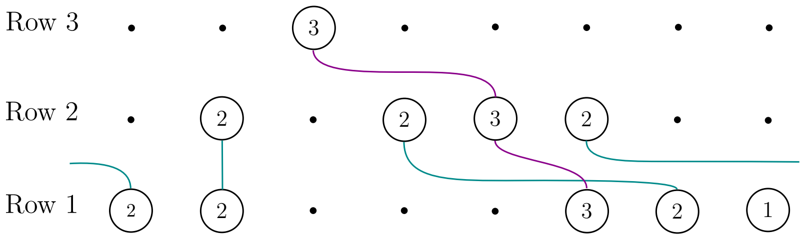


Figure 1: A multiline queue of type $(2,2,0,0,0,3,2,1)$.

Definition 2.3. An *enhanced ball system B* of content λ is a $2L \times n$ array, with rows labeled from bottom to top as $1, 1', 2, 2', \dots, L, L'$, and columns labeled from left to right from 1 to n, in which each of the $2Ln$ positions is either empty or occupied by a ball, and such that there are $m_L + m_{L-1} + \cdots + m_r$ balls in each of rows r and r'. Moreover:

(a) in row r our balls are labeled by $\{1, 2, \dots, L\}$ (we call them *regular balls*) and the configuration of balls gives a permutation of $\lambda^{(r)}$.

(b) in row r' our balls are labeled by $\{\pm 1, \dots, \pm L\}$ (we call them *signed balls*) and the configuration of balls gives a *signed* permutation of $\lambda^{(r)}$.

A signed ball with a positive (respectively negative) label will be called *a positive ball* (respectively *a negative ball*).

Definition 2.4. A *signed multiline queue $Q^{\pm}$* (or SMLQ) of type $\mu \in S_n(\lambda)$ is an enhanced ball system of content λ such that each ball in a row above row 1 is paired with a ball of the same absolute value in the row below it, and the configuration of balls on the bottom row is μ. We require that, if we consider only the absolute values of the ball labels, then the pairings between row r and row $(r-1)'$ form a classic layer, as in Definition 2.2, and we call them *classic pairings*. And we require that the pairings between row r' and row r form a *signed layer*, i.e. satisfy the following rules (and we call them *signed pairings*):

(a′) Each pairing connects two balls with a shortest strand that travels either straight down or from left to right, and does not wrap around;

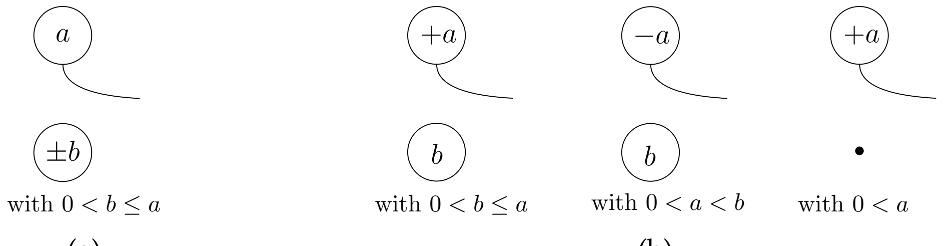
(b′) In row r', each positive ball with label $a \in \mathbb{Z}^+$ must always have a ball labeled a' underneath it, where $a' \geqslant a$, and if $a' = a$, the two balls must be trivially paired;

(c′) In row r', each negative ball with label $-a$ (for $a \in \mathbb{Z}^+$) has either an empty spot below it or a ball with label a', where $a \geqslant a'$.

Let $\mathrm{MLQ}^{\pm}(\mu)$ denote the set of signed multiline queues of type μ.

In Figure 2a and Figure 2b we illustrate the forbidden configurations in the classic and signed layers, respectively.

See Figure 3 for an example of a signed multiline queue.



(a) (b)

Figure 2: Figure 2a illustrates the forbidden configurations for the classic layers, while the three other figures (Figure 2b) show the forbidden configurations for the signed layers. The three figures on the left show two balls on top of each other, which are not trivially paired, whereas the rightmost figure features a regular ball with an empty position beneath it.

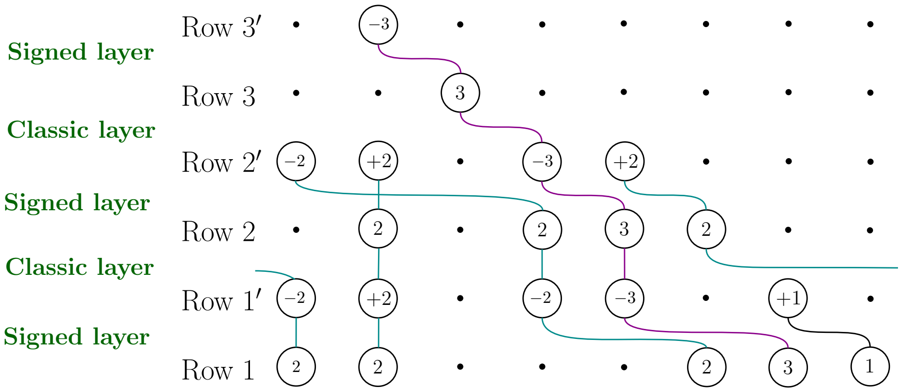


Figure 3: A signed multiline queue of type $(2, 2, 0, 0, 0, 2, 3, 1)$.

Remark 2.5. In [20], multiline queues were given an interpretation in terms of "priority queues," with balls at each level representing customers who are each seeking a service on the level below. We can also interpret our signed multiline queues as follows. The signed balls all represent customers, with the positive balls having "polite" and "attractive" characteristics and the negative balls having "needy" characteristics. The regular balls all represent services; they are always "polite". We will work our way from top to bottom of the multiline queue: in Row r' the customers seek a service from Row r below, and in Row r the services seek a customer from Row $(r-1)'$ below. When we construct pairings between two adjacent rows, we will start by examining the balls of largest absolute value in the higher row; if we need to break ties, we will work from right to left. The polite customers and services can choose any unused service/customer below, except that if there is an unused service immediately underneath them, politeness dictates that they must accept it; this explains the leftmost two diagrams in Figure 2. When we construct the pairings on a signed layer, because the negative balls/customers are so needy, no ball (positive or negative) dares to pair with an unused service that is immediately below a negative customer who has not yet accepted a service; this explains the third diagram in Figure 2. Since the positive balls/customers are attractive, there is always a service to be found just underneath them (though it may be taken already); this explains the fourth diagram in Figure 2. Finally, pairings initiated by services from Row r can wrap around, because servers "know the building"; however, pairings initiated by customers cannot.

3 Combinatorics of homogeneous ASEP polynomials

In this section we define weights for multiline and signed multiline queues. We then review the combinatorial formula for ASEP polynomials and give a combinatorial formula for interpolation ASEP polynomials.

Definition 3.1. Let Q be a multiline queue. If the balls in row r form the composition $\mu = (\mu_1, \dots, \mu_n)$, we define the *ball-weight* of row r and of Q to be

$$\mathrm{wt}_{\mathrm{ball}}(r) = \prod_{i:\mu_i>0} x_i \qquad \text{and} \qquad \mathrm{wt}_{\mathrm{ball}}(Q) = \prod_{r=1}^{L} \mathrm{wt}_{\mathrm{ball}}(r). \tag{3.1}$$

We also define the *pairing-weight* $\mathrm{wt}_{\mathrm{pair}}(Q)$ of Q by associating a weight to each nontrivial pairing p of balls. Consider the pairings in a connecting balls in row r and row $(r-1)$. Their weights are computed via the following *pairing order*. We read the balls in row r in decreasing order of their label; within a fixed label, we read the balls from right to left. As we read the balls in this order, we imagine placing the strands pairing the balls one by one. The balls in row $(r-1)$ that have not yet been matched right before we place p are considered *free* for p. If pairing p matches a ball labeled a in row r and column j to a ball in row $(r-1)$ and column j', then the free balls in row $(r-1)$ and columns $j+1, j+2, \dots, j'-1$ (indices considered modulo n) are considered *skipped*. (When pairing balls of label a between rows r and $(r-1)$, trivially paired balls of label a in row $(r-1)$ are not considered free.) We then associate to pairing p the weight

$$\mathrm{wt}_{\mathrm{pair}}(p) = \begin{cases} \frac{(1-t)t^{\mathrm{skip}(p)}}{1-q^{a-r+1}t^{\mathrm{free}(p)}} \cdot q^{a-r+1} & \text{if } j' < j \\ \frac{(1-t)t^{\mathrm{skip}(p)}}{1-q^{a-r+1}t^{\mathrm{free}(p)}} & \text{if } j' > j. \end{cases} \tag{3.2}$$

Note that the factor q^{a-r+1} appears precisely when the pairing wraps around the cylinder. Having associated a weight to each nontrivial pairing, we define

$$\mathrm{wt}_{\mathrm{pair}}(Q) = \prod_{p} \mathrm{wt}_{\mathrm{pair}}(p),$$

where the product is over all nontrivial pairings of balls in Q. Finally the *weight* of Q is defined to be $\mathrm{wt}(Q) = \mathrm{wt}_{\mathrm{ball}}(Q)\,\mathrm{wt}_{\mathrm{pair}}(Q)$.

Definition 3.2. Let $\mu = (\mu_1, \dots, \mu_n) \in \{0, 1, \dots, L\}^n$ be a composition. We set

$$F_\mu = F_\mu(x_1, \dots, x_n; q, t) = F_\mu(\boldsymbol{x}; q, t) = \sum_{Q \in \mathrm{MLQ}(\mu)} \mathrm{wt}(Q).$$

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition with n parts. We set

$$Z_\lambda = Z_\lambda(x_1, \dots, x_n; q, t) = Z_\lambda(\boldsymbol{x}; q, t) = \sum_{\mu \in S_n(\lambda)} F_\mu(x_1, \dots, x_n; q, t).$$

We call Z_λ the *combinatorial partition function* for multiline queues.

Theorem 3.3. *[12] Let $\mu \in \mathbb{N}^n$ be a composition, and let λ be a partition. Then the ASEP polynomial f_μ equals the weight-generating function F_μ for MLQ of type μ. And the Macdonald polynomial $P_\lambda(\boldsymbol{x}; q, t)$ is equal to the combinatorial partition function $Z_\lambda(\boldsymbol{x}; q, t)$ for MLQ.*

4 Combinatorics of interpolation ASEP polynomials

Our goal is now to give an analogue of Theorem 3.3 for interpolation polynomials.

Definition 4.1. Let $Q^\pm$ be a signed multiline queues. If the balls in row r' form the signed composition $\alpha = (\alpha_1, \dots, \alpha_n)$, we define the *shifted ball-weight* of row r' to be

$$\mathrm{wt}_{\mathrm{ball}}(r') = \left(\prod_{i:\alpha_i > 0} x_i\right)\left(\prod_{i:\alpha_i < 0} \frac{-q^{r-1}}{t^{n-1}}\right) \tag{4.1}$$

and we define the *shifted ball-weight* of $Q^\pm$ to be

$$\mathrm{wt}_{\mathrm{ball}}(Q^\pm) = \prod_{r=1}^{L} \mathrm{wt}_{\mathrm{ball}}(r'). \tag{4.2}$$

In other terms, we assign to a ball in column i and row r' the weight x_i if it is a positive ball and the weight $\frac{-q^{r-1}}{t^{n-1}}$ if it is a negative ball.

We define the *pairing-weight* $\mathrm{wt}_{\mathrm{pair}}(Q^\pm)$ of $Q^\pm$ by associating a weight to each nontrivial pairing p of balls. For the pairings in a classic layer connecting balls in row r and row $(r-1)'$, we use the weighting scheme from (3.2), where we ignore the signs on ball labels and only work with the absolute value.

For the pairings in a signed layer connecting balls in row r' and row r, we read the balls in row r' in decreasing order of the absolute value of their label; within a fixed absolute value, we read the balls from right to left. Reading the balls in this order, we place the strands pairing the balls one by one. The balls in row r that have not yet been matched are *free*. If pairing p matches a ball labeled $\pm a$ in row r' and column j to a ball labeled a in row r and column $k > j$, then the free balls (respectively, empty positions) in row r and columns $j+1, j+2, \dots, k-1$ are *skipped* (respectively, *empty*). We then set

$$\mathrm{wt}_{\mathrm{pair}}(p) = \begin{cases} (1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)} & \text{if } p \text{ connects a positive and regular ball,} \\ -(1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)} & \text{if } p \text{ connects a negative and regular ball.} \end{cases} \tag{4.3}$$

Having associated a weight to each nontrivial pairing, we define

$$\mathrm{wt}_{\mathrm{pair}}(Q^{\pm}) = \prod_{p} \mathrm{wt}_{\mathrm{pair}}(p),$$

where the product is over all nontrivial pairings of balls in $Q^{\pm}$. Finally the *weight* of $Q^{\pm}$ is defined to be

$$\mathrm{wt}(Q^{\pm}) = \mathrm{wt}_{\mathrm{ball}}(Q^{\pm})\,\mathrm{wt}_{\mathrm{pair}}(Q^{\pm}).$$

Remark 4.2. If our multiline queue does not have negative balls, i.e all labels are in $\mathbb{N}_+$, then it follows from items (a′) and (b′) of Definition 2.4 that all the signed pairings are trivial. As a consequence, the contribution of the signed layers to the pairing weight of the system is 1. We can then remove these layers and keep only the classic ones; the definition of signed multiline queue then reduces to Definition 2.2.

Example 4.3. In Figure 3, the ball-weight of $Q^{\pm}$ is $\left(\frac{-q^2}{t^7}\right) \cdot x_2 x_5 \left(\frac{-q}{t^7}\right)^2 \cdot x_2 x_7 \left(\frac{-1}{t^7}\right)^3$. Meanwhile, from left to right, the weights of the nontrivial pairings are as follows:

- From Row 3′ to Row 3: $-(1-t)$
- From Row 3 to Row 2′: $\frac{1-t}{1-qt^4}$
- From Row 2′ to Row 2: $-t(1-t)$, $-(1-t)$, and $(1-t)$
- From Row 2 to Row 1′: $\frac{(1-t)t}{1-qt^2} \cdot q$
- From Row 1′ to Row 1: $-t(1-t)$, $-t(1-t)$, and $(1-t)$.

Thus, multiplying all of these weights, we obtain

$$\mathrm{wt}(Q^{\pm}) = -x_2^2 x_5 x_7 \frac{q^5(1-t)^9}{t^{38}(1-qt^2)(1-qt^4)}.$$

We now define the weight-generating function for signed multiline queues of a given type, as well as the *combinatorial partition function* for signed multiline queues.

Definition 4.4. Let $\mu = (\mu_1, \ldots, \mu_n) \in \{0, 1, \ldots, L\}^n$ be a composition. We set

$$F^*_\mu = F^*_\mu(x_1, \ldots, x_n; q, t) = F^*_\mu(\boldsymbol{x}; q, t) = \sum_{Q^\pm \in \mathrm{MLQ}^\pm(\mu)} \mathrm{wt}(Q^\pm).$$

Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$ be a partition with n parts. We set

$$Z^*_\lambda = Z^*_\lambda(x_1, \ldots, x_n; q, t) = Z^*_\lambda(\boldsymbol{x}; q, t) = \sum_{\mu \in S_n(\lambda)} F^*_\mu(x_1, \ldots, x_n; q, t).$$

We call Z^*_λ the *combinatorial partition function* for signed multiline queues.

Theorem 4.5 (Main theorem)**.** *Let μ be a composition, and let λ be a partition. Then the interpolation ASEP polynomial f^*_μ equals the weight-generating function F^*_μ for signed multiline queues of type μ. And the interpolation Macdonald polynomial $P^*_\lambda(\boldsymbol{x}; q, t)$ is equal to the combinatorial partition function $Z^*_\lambda(\boldsymbol{x}; q, t)$ for signed multiline queues.*

Example 4.6. To use Theorem 4.5 to compute the interpolation ASEP polynomial $f^*_{(0,2)}$, we enumerate all signed multiline queues of type $(0, 2)$, see Figure 4, and then sum up their weights, obtaining

$$\begin{aligned} f^*_{(0,2)} =& \frac{1-t}{1-qt}(x_1 - q/t)(x_2 - 1/t) + \frac{1-t}{t}(x_1 - q/t) + (x_2 - q/t)(x_2 - 1/t) + \\ & (1-t)\frac{q}{t}(x_2 - 1/t) + \frac{q^2}{t^2}\frac{(1-t)^3}{1-qt} + \frac{q}{t}\frac{(1-t)^2}{1-qt}(x_2 - q/t). \end{aligned}$$

The usual ASEP polynomial is the top homogeneous part of the expression above,

$$f_{(0,2)} = \frac{1-t}{1-qt} x_1 x_2 + x_2^2.$$

This can be computed from the signed multiline queues which have no negative balls, and whose pairings from row r' to row r are all trivial.

When $q = 1$, the ASEP polynomials and the Macdonald polynomials have a probabilistic interpretation in terms of the t-Push TASEP [2]. We will give an interpolation analogue of this result in [6].

5 Strategy of the proof

While our combinatorial formulas for interpolation ASEP and Macdonald polynomials can be seen as generalizations of the combinatorial formulas in [12], our proof strategy is quite different. The proof in [12] utilized the circular symmetry

$$q^{\mu_n} f_\mu(\mathbf{x}; q, t) = f_{\mu_n, \mu_1, \ldots, \mu_{n-1}}(q x_n, x_1, \ldots, x_{n-1}; q, t)$$

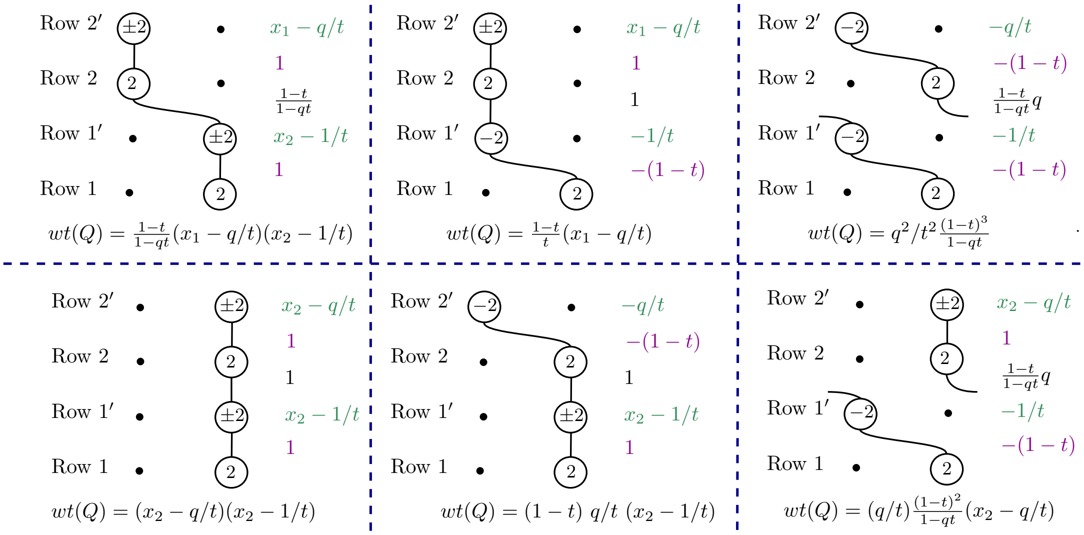


Figure 4: The signed multiline queues of type $(0, 2)$, with their weights superimposed. Note that a ball labeled ± 2 represents the fact that the corresponding ball can either be a positive or a negative ball. Thus, the six diagrams above actually represent 15 SMLQ.

for ASEP polynomials; however, interpolation ASEP polynomials lack this property.

The starting point of the proof is the fact that signed multiline queues can be recursively built by gluing the layers successively, alternating classic and signed. This is a generalization of the recursion given in [12, Section 2] for classic multiline queues. The proof consists then of three main steps.

- (Step 1) We start by studying the *packed case*. This is the case when the composition μ has all non-zero parts in adjacent positions at the left. In this case, the vanishing conditions of the interpolation ASEP polynomials f^*_μ allow us to write a recursion for this polynomial, obtained by "shifting" the recursive equation obtained in [12] for the homogeneous polynomial f_μ.
- (Step 2) We start from the recursive equation obtained in Item (Step 1) for packed compositions μ. By applying Hecke operators to this recursion, we generalize it to any composition μ. The recursive equations obtained involve a family of coefficients (b^α_μ), and are encoded by the action of the Hecke operators on a variant f_α of the ASEP polynomials, indexed by signed compositions.
- (Step 3) We show that the generating function of one single signed layer satisfies the same recursion as the coefficients (b^α_μ). Thus, we show that the *algebraic recursion* for the polynomials f^*_μ corresponds to a *combinatorial recursion* for SMLQs.

6 Integrality and Comparison with Okounkov's Formula

Knop and Sahi proved that the *integral form* of the interpolation Macdonald polynomials P^*_λ satisfy an integrality property (see e.g [17, Corollary 5.5]). This can be proved using our combinatorial formula for P^*_λ, and extended to interpolation ASEP polynomials f^*_μ.

In [22], Okounkov gave a formula for the interpolation symmetric Macdonald polynomials, which, to our knowledge, was the only formula prior to our work. This formula is a generalization of Macdonald's formula for the homogeneous symmetric Macdonald polynomials [19, Section VI.7]. It expresses the polynomial P^*_λ as a sum over tableaux of shape λ, counted with coefficients given by products of *Pieri coefficients*. These coefficients are quite complicated to compute, and in particular, the integrality property is not apparent from this tableau formula.

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