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Saturation for Non-symmetric Macdonald Polynomials

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Abstract. We prove that supports of non-symmetric Macdonald polynomials are M-convex. As a consequence, we resolve a 2019 conjecture of Monical, Tokcan, and Yong that they have the saturated Newton polytope property. As a further consequence, we prove that the moment polytopes of affine Schubert varieties in the affine Grassmannian of type GL are generalized permutahedra.

Keywords: Newton Polytopes, M-Convexity, Generalized Permutahedra

1 Introduction

Given a polynomial defined implicitly, it is not necessarily clear how to determine which of its monomials has a nonzero coefficient. For some families of polynomials, there is a polyhedral characterization of the monomials that arise. Namely, given a polynomial $f \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ with $f(x) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha$, one may compute its Newton polytope given by $\mathrm{Newton}(f) = \mathrm{conv}(\{\alpha \in \mathbb{Z}^n : c_\alpha \neq 0\})$. Every monomial is a lattice point in that polytope. We say that f has a **saturated Newton polytope** if the converse is true:

$$\mathrm{Newton}(f) \cap \mathbb{Z}^n = \{\alpha \in \mathbb{Z}^n : c_\alpha \neq 0\}.$$

A first example of this phenomenon is the determinant. Its Newton polytope is the Birkhoff polytope, and the only integer points it contains are precisely its vertices. Hence, the determinant has saturated Newton polytope. Any square free polynomial is more generally saturated, but the phenomenon becomes rarer and harder to verify in higher degree.

Saturation gained attention recently as it holds for Lorentzian polynomials [8]. These polynomials appear in connection to the Kahler package of Adiprasito, Huh, and Katz [1] used to resolve longstanding conjectures in matroid theory related to log-concavity, as well as in the theory of real stability of polynomials used to establish fast mixing for random walks on basis exchange graphs of matroids [3]. Lorentzian polynomials

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have a stronger property than saturation. Their supports are **M-convex sets**, meaning that they have saturated Newton polytopes, and their Newton polytope is a generalized permutahedron [33]. In fact, it was shown in [8] that a polynomial has M-convex support if and only if it has the same support set as some Lorentzian polynomial. These results motivate the recent study of M-convexity and saturation of supports of polynomials.

In [29], Monical, Tokcan, and Yong observed that many polynomials appearing in algebraic combinatorics have saturated Newton polytopes, either via proof or conjecturally after computations. They proved this for Schur polynomials, symmetric Macdonald polynomials, elementary symmetric functions, and resultants among several other examples. They also provided several nonexamples, such as discriminants and modified Macdonald polynomials. Based on computational evidence from small examples, they conjectured Schubert polynomials, double Schubert polynomials, key polynomials, Grothendieck polynomials, Kronecker products of Schur polynomials, Lascoux polynomials, Demazure atoms, and non-symmetric Macdonald polynomials were all saturated. Later, Fink, Mészáros, and St. Dizier proved that Schubert polynomials and key polynomials had the saturated Newton polytope property via an exact computation of their Newton polytopes as a Minkowski sum of matroid base polytopes [15]. Besson, Jeralds, and Kiers later extended the results for key polynomials with an alternative proof for more general Coxeter type [5], and Castillo, Cid Ruiz, Mohammadi, and Montaño showed saturation for double Schubert polynomials using the theory of multidegrees from commutative algebra [9]. For Grothendieck polynomials and Kronecker products of Schur polynomials, several special cases of the conjecture are known to hold [16, 10, 28, 13, 32]. Beyond those conjectures, in connection to the theory of Lorentzian polynomials, several other families of polynomials from algebraic combinatorics have been shown to have saturated Newton polytopes [26, 22, 17, 35, 14, 27, 2, 12, 31, 21]. For Lascoux polynomials, Demazure atoms, and non-symmetric Macdonald polynomials, to our knowledge, nothing was known prior to this work even in special cases. Here we resolve the final case equivalent to Conjecture 3.8 of [29]:

Theorem 1. *The supports of non-symmetric Macdonald polynomials are M-convex. In particular, they have the saturated Newton polytope property.*

The **non-symmetric Macdonald polynomials** $\{E_\mu(x_1, \ldots, x_n) | \mu \in \mathbb{Z}^n\}$ form an exceptional basis for the space of Laurent polynomials $\mathbb{Q}(q,t)[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. These polynomials are the non-symmetric counterparts of the symmetric Macdonald polynomials $P_\lambda(X; q, t)$ defined originally by Macdonald [24]. Both types of Macdonald polynomials occur across many areas of modern mathematics in the geometry of Hilbert schemes [20], the combinatorics of LLT polynomials [6, 18, 19], the representation theory of double affine Hecke algebras [11], and probability theory [7]. Upon performing various q, t parameter specializations, the $E_\mu(x_1, \ldots, x_n)$ recover many other important families of polynomials including the (affine) Demazure characters, (affine) Demazure atoms, non-symmetric Hall–Littlewood polynomials, and the non-symmetric Jack polynomials [**Ion_2003??**, 25, 19].

In Section 1.2 of [4], Besson, Jeralds, and Kiers recently asked whether affine Demazure characters have saturated Newton polytopes. By Theorem 8.2.2 of [23] non-symmetric Macdonald polynomials have the same support as affine Demazure characters of type GL. Hence, we resolve the type GL case here:

Corollary 1. *For each $\mu \in \mathbb{Z}^n_{\geq 0}$ the type GL_n affine Demazure character $E_\mu(x; q, 0)$ has M-convex support for generic q. In particular, they have the saturated Newton polytope property.*

Also in [4], the authors note that the Newton polytope of the affine Demazure character of type GL is the moment polytope for a type GL affine Schubert variety. Even the weaker observation that the moment polytope is a generalized permutahedron is new to our knowledge and follows also as a consequence of our results.

Our proof relies on the combinatorial formula for non-symmetric Macdonald polynomials of Haglund, Haiman, and Loehr [19] together with some polyhedral geometric observations. We will prove our result by induction by noting that the trivial non-symmetric Macdonald polynomial $E_{(0,\ldots,0)} = 1$ has M-convex support and any non-symmetric Macdonald polynomial may be built from smaller ones starting with $E_{(0,\ldots,0)} = 1$ by applying a sequence of applications of two operations that both preserve M-convexity of the support sets of the polynomials.

2 Non-symmetric Macdonald polynomials and their combinatorics

2.1 Non-symmetric Macdonald polynomials

The non-symmetric Macdonald polynomials E_μ may be defined in multiple equivalent ways. We refer the reader to [19] and [11] for an overview of these various definitions. We use the explicit combinatorial description of non-symmetric Macdonald polynomials due to Haglund, Haiman, and Loehr [19, Theorem 3.5.1] throughout this paper and will thus take their combinatorial formula as a definition. However, before we may define the non-symmetric Macdonald polynomials $E_\mu(x_1, \ldots, x_n; q, t)$, we must first define some required combinatorial objects.

Definition 1. *[19] For a (weak) composition $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$, define the column diagram of μ as*

$$\mathrm{dg}'(\mu) := \{(i, j) \in \mathbb{Z}^2_{\geq 0} : 1 \leq i \leq n, 1 \leq j \leq \mu_i\}.$$

This is represented by a collection of boxes in positions given by $\mathrm{dg}'(\mu)$. The augmented diagram of μ is given by

$$\widehat{\mathrm{dg}}(\mu) := \mathrm{dg}'(\mu) \sqcup \{(i, 0) : 1 \leq i \leq n\}.$$

*Visually, to get $\widehat{\mathrm{dg}}(\mu)$ we are adding a bottom row of boxes on length n below the diagram $\mathrm{dg}'(\mu)$ which we refer to as the basement of the diagram. A filling of μ is a function $\sigma : \mathrm{dg}'(\mu) \to \{1,\ldots,n\}$ and given a filling there is an associated augmented filling $\widehat{\sigma} : \widehat{\mathrm{dg}}(\mu) \to \{1,\ldots,n\}$ extending σ with the additional bottom row boxes filled according to $\widehat{\sigma}((j,0)) = j$ for $j = 1,\ldots,n$. Distinct lattice squares $u, v \in \mathbb{Z}^2_{\geq 0}$ are said to **attack** each other if one of the following is true:*

- *u and v are in the same row, or*
- *u and v are in consecutive rows and the box in the lower row is strictly to the right of the box in the upper row.*

*A filling $\sigma : \mathrm{dg}'(\mu) \to \{1,\ldots,n\}$ is **non-attacking** if $\widehat{\sigma}(u) \neq \widehat{\sigma}(v)$ for every pair of attacking boxes $u, v \in \widehat{\mathrm{dg}}(\mu)$. Define $\mathcal{L}(\mu)$ to be the set of all non-attacking labelings of μ. For a box $u = (i,j)$ let $d(u) = (i, j-1)$ denote the box just below u. Lastly, for a filling $\sigma : \mathrm{dg}'(\mu) \to \{1,\ldots,n\}$ set*

$$x^{\sigma} := x_1^{|\sigma^{-1}(1)|} \cdots x_n^{|\sigma^{-1}(n)|}.$$

Example 1. *Consider $\mu = (1,0,2)$. The diagram $\widehat{\mathrm{dg}}(\mu)$ is given by*

$$\begin{array}{ccc} & & \square \\ \square & & \square \\ 1 & 2 & 3 \end{array}$$

where the boxes labeled $1,2,3$ are the basement of $\widehat{\mathrm{dg}}(\mu)$. By definition, these basement boxes always carry the labels $1,2,3$ in any non-attacking labeling of $\widehat{\mathrm{dg}}(\mu)$. The following are valid non-attacking labelings $\widehat{\sigma}$ of $\widehat{\mathrm{dg}}(\mu)$:

$$\begin{array}{ccc} & & 1 \\ 1 & & 2 \\ 1 & 2 & 3 \end{array} \qquad \begin{array}{ccc} & & 3 \\ 1 & & 2 \\ 1 & 2 & 3 \end{array} \qquad \begin{array}{ccc} & & 3 \\ 1 & & 3 \\ 1 & 2 & 3 \end{array}.$$

For example, the following labelings are invalid as they all have a pair of attacking boxes with the same label, which we will mark with $*$'s:

$$\begin{array}{ccc} & & 1 \\ 2* & & 1 \\ 1 & 2* & 3 \end{array} \qquad \begin{array}{ccc} & & 2 \\ 1* & & 1* \\ 1 & 2 & 3 \end{array} \qquad \begin{array}{ccc} & & 2 \\ 3* & & 1 \\ 1 & 2 & 3* \end{array}.$$

Associated to any non-attacking diagram labeling σ and box $\square$ of the diagram $\mathrm{dg}'(\mu)$ are statistics $\mathrm{coinv}(\widehat{\sigma}), \mathrm{maj}(\widehat{\sigma}), \mathrm{leg}(\square)$, and $\mathrm{arm}(\square)$ which are required for defining E_μ.

However, as we will be considering **generic** q, t it will not be important for us to know these statistics explicitly outside of knowing that these statistics are finite non-negative integers. We refer the reader to [19] for their definitions. The combinatorial formula for non-symmetric Macdonald polynomials can now be stated.

Definition 2. *[19] For $\mu \in \mathbb{Z}^n_{\geq 0}$ define the non-symmetric Macdonald polynomial $E_\mu \in \mathbb{Q}(q,t)[x_1, \ldots, x_n]$ as*

$$E_\mu := \sum_{\substack{\sigma:\mu\to\{1,\ldots,n\} \\ \text{non-attacking}}} x^\sigma q^{\mathrm{maj}(\widehat{\sigma})} t^{\mathrm{coinv}(\widehat{\sigma})} \prod_{\substack{\square \in \mathrm{dg}'(\mu) \\ \widehat{\sigma}(\square) \neq \widehat{\sigma}(d(\square))}} \left(\frac{1-t}{1-q^{\mathrm{leg}(\square)+1} t^{\mathrm{arm}(\square)+1}} \right).$$

Here $\sigma : \mu \to \{1, \ldots, n\}$ is shorthand for diagram labelings of $\mathrm{dg}'(\mu)$. For general $\mu \in \mathbb{Z}^n$, say $\mu = (m, \ldots, m) + \mu'$ with $\mu' \in \mathbb{Z}^n_{\geq 0}$ and $m \in \mathbb{Z}$, define $E_\mu := (x_1 \cdots x_n)^m E_{\mu'}$.

Example 2. *For $\mu = (0, 2, 0)$, the non-symmetric Macdonald polynomial $E_{(0,2,0)}(x_1, x_2, x_3; q, t)$ is given by*

$$\begin{aligned} E_{(0,2,0)} = x_2^2 &+ \frac{1-t}{1-q^2t^2} x_1^2 + \frac{q(1-t)}{1-qt} x_2 x_3 + \frac{q(1-t)^2}{(1-qt)(1-q^2t^2)} x_1 x_3 \\ &+ \frac{1-t}{1-qt} x_1 x_2 + \frac{q(1-t)^2}{(1-qt)(1-q^2t^2)} x_1 x_2. \end{aligned}$$

Each of the terms above corresponds to exactly one of the following non-attacking labelings:

$$\begin{array}{ccc} & \boxed{2} & \\ & \boxed{2} & \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array} \qquad \begin{array}{ccc} & \boxed{1} & \\ & \boxed{1} & \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array} \qquad \begin{array}{ccc} & \boxed{3} & \\ & \boxed{2} & \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array} \qquad \begin{array}{ccc} & \boxed{3} & \\ & \boxed{1} & \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array} \qquad \begin{array}{ccc} & \boxed{1} & \\ & \boxed{2} & \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array} \qquad \begin{array}{ccc} & \boxed{2} & \\ & \boxed{1} & \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array}.$$

Since whenever $\mu = (m, \ldots, m) + \mu'$ with $\mu' \in \mathbb{Z}^n_{\geq 0}$, $\mathrm{Newton}(E_\mu) = (m, \ldots, m) + \mathrm{Newton}(E_{\mu'})$, it will suffice for our purposes throughout the remainder of this paper to only consider those $E_{\mu'}$ for $\mu' \in \mathbb{Z}^n_{\geq 0}$. Given a composition $\mu = (\mu_1, \ldots, \mu_n)$ write $|\mu| := \mu_1 + \cdots + \mu_n$.

The following operations will be important in this paper.

Definition 3. *For $1 \leq i \leq n-1$ define $s_i : \mathbb{Z}^n \to \mathbb{Z}^n$ as the simple transposition $s_i(\ldots, a_i, a_{i+1}, \ldots) := (\ldots, a_{i+1}, a_i, \ldots)$. Define the map $\pi : \mathbb{Z}^n \to \mathbb{Z}^n$ as $\pi(a_1, \ldots, a_n) := (a_n + 1, a_1, \ldots, a_{n-1})$. Define $\psi : \mathbb{Q}(q,t)[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to \mathbb{Q}(q,t)[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ to be the $\mathbb{Q}(q,t)$-algebra automorphism determined by $\psi(x_1^{a_1} \cdots x_n^{a_n}) := q^{-a_n} x_1^{a_n} x_2^{a_1} \cdots x_n^{a_{n-1}}$.*

Notice that the actions of $s_1, \ldots, s_{n-1}, \pi$ on $\mathbb{Z}^n$ generate the standard action of the extended affine symmetric group $\widehat{\mathfrak{S}}_n$ on $\mathbb{Z}^n$. Furthermore, $s_1, \ldots, s_{n-1}, \pi$ restrict to maps

$\mathbb{Z}^n_{\geq 0} \to \mathbb{Z}^n_{\geq 0}$, and $s_1, \ldots, s_{n-1}$ are invertible whereas $\pi \mid_{\mathbb{Z}^n_{\geq 0}}$ is now only injective and no longer invertible. We write $e_1, \ldots, e_n$ for the standard coordinate basis vectors of $\mathbb{Z}^n$.

The following is one of the Knop–Sahi recurrence relations for the non-symmetric Macdonald polynomials, which we will require later:

Proposition 1. *[19] For all $\mu \in \mathbb{Z}^n$, $E_{\pi(\mu)} = q^{\mu_n} x_1 \psi(E_\mu)$.*

2.2 Newton polytopes

Definition 4. *Given a Laurent polynomial $f(x_1, \ldots, x_n) \in \mathbb{R}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $\alpha \in \mathbb{Z}^n$, let $\langle x^\alpha \rangle f$ denote the coefficient of x^α in the monomial expansion of f. The set of all $\alpha \in \mathbb{Z}^n$ such that $\langle x^\alpha \rangle f \neq 0$ is called the **support** set of f and denoted $\mathrm{supp}(f)$. The **Newton polytope** $\mathrm{Newton}(f)$ is the polytope $\mathrm{conv}(\mathrm{supp}(f))$. The polynomial f is said to have the **saturated Newton polytope property** (SNP) if $\mathrm{Newton}(f) \cap \mathbb{Z}^n = \mathrm{supp}(f)$.*

We will be interested in studying the Newton polytopes of the non-symmetric Macdonald polynomials $\mathrm{Newton}(E_\mu) = \mathrm{Newton}(E_\mu(x_1, \ldots, x_n; q, t))$ with q, t *generic*. For the remainder of the paper, if we refer to E_μ without specifying q, t explicitly we are taking q, t generic. If the reader wishes to avoid the term generic, then one may replace the condition that q, t be generic with any choice $0 < q, t < 1$. In particular, from the explicit formula in Definition 2 it is clear that $q = t = 1/2$ is always sufficiently generic.

2.3 Labelings lemmas

To prove that the non-symmetric Macdonald polynomials have M-convex supports, we work directly with the combinatorics of non-attacking diagram labelings.

Definition 5. *Given any permutation $\gamma \in \mathfrak{S}_n$ and any (possibly attacking) diagram labeling $\sigma : \mu \to \{1, \ldots, n\}$, we will denote by $\gamma(\sigma)$ the diagram labeling given by $\gamma(\sigma)(\square) := \gamma(\sigma(\square))$. For $\mu \in \mathbb{Z}^n_{\geq 0}$, define the multiplicity map $\mathrm{mult} : \mathcal{L}(\mu) \to \mathbb{Z}^n_{\geq 0}$ as $\mathrm{mult}(\sigma) = (\mathrm{mult}_1(\sigma), \ldots, \mathrm{mult}_n(\sigma))$ where $\mathrm{mult}_j(\sigma) := \#(\square \in \mathrm{dg}'(\mu) \mid \sigma(\square) = j)$.*

From Definition 2 we see:

Proposition 2. *For all $\mu \in \mathbb{Z}^n_{\geq 0}$, $\mathrm{mult}(\mathcal{L}(\mu)) = \mathrm{supp}(E_\mu)$.*

For the remainder of this section, fix some composition $\mu \in \mathbb{Z}^n_{\geq 0}$, $a \geq 1$, and $m \geq 2$. Let $N := m + n$. We will be interested in understanding the relationship between the sets of non-attacking labelings corresponding to the compositions $\nu_1, \ldots, \nu_m$ defined by $\nu_i := 0^{i-1} * a * 0^{m-i+1} * \mu$ for $1 \leq i \leq m$, as the left-most nonzero column corresponding to $a \geq 1$ is shifted from left to right. Here the $*$s denote concatenations of compositions. We may identify the boxes of each of the diagrams $\widehat{\mathrm{dg}}(\nu_1), \ldots, \widehat{\mathrm{dg}}(\nu_m)$ in the obvious way: identify the basement boxes, identify the boxes corresponding to μ which do not shift, and identify those boxes in the moving column corresponding to a. As such, any (possibly attacking) labeling of some ν_i may be thought of as a (possibly attacking) labeling of any ν_j for all $1 \leq i, j \leq m$.

Example 3. *For instance, when $\mu = (2, 1)$, $a = 2$, and $m = 3$, we are considering the compositions $\nu_1 = (2, 0, 0, 2, 1)$, $\nu_2 = (0, 2, 0, 2, 1)$, and $\nu_3 = (0, 0, 2, 2, 1)$ with corresponding diagrams:*

$$\begin{array}{|c|c|c|c|c|} \hline b & & & x & \\ \hline c & & & y & z \\ \hline 1 & 2 & 3 & 4 & 5 \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|c|} \hline & b & & x & \\ \hline & c & & y & z \\ \hline 1 & 2 & 3 & 4 & 5 \\ \hline \end{array} \Rightarrow \begin{array}{|c|c|c|c|c|} \hline & & b & x & \\ \hline & & c & y & z \\ \hline 1 & 2 & 3 & 4 & 5 \\ \hline \end{array}$$

For $1 \leq i < j \leq n$, we will write $\sigma_{i,j}$ for the transposition swapping the i and j entries of vectors in $\mathbb{Z}^n$. Use $\sqcup$ to denote disjoint unions. We have the following fundamental observation.

Lemma 1. *For all $1 \leq i \leq m - 1$, $\mathcal{L}(\nu_{i+1}) = \mathcal{L}(\nu_i) \sqcup \sigma_{1,i+1}(\mathcal{L}(\nu_1))$.*

As an immediate consequence,

Corollary 2. *For all $1 \leq i \leq m$, $\mathcal{L}(\nu_i) = \bigsqcup_{1 \leq k \leq i} \sigma_{1,k}(\mathcal{L}(\nu_1))$. In particular, for all $1 \leq i \leq m - 1$, $\mathcal{L}(\nu_{i+1}) = \mathcal{L}(\nu_i) \cup s_i(\mathcal{L}(\nu_i))$.*

We require the following additional lemma.

Lemma 2. *Let $1 \leq i \leq m - 1$. For all $\sigma \in \mathcal{L}(\nu_{i+1})$, there exists some $\sigma' \in \mathcal{L}(\nu_i)$ such that either*

- *σ and σ' have the same multiplicities of each label or*
- *σ and σ' have the same multiplicities for all j with $j \notin \{i, i+1\}$ and σ has exactly one more $i + 1$ and one fewer i boxes than σ'.*

Proof sketch: Clearly, for $\sigma \in \mathcal{L}(\nu_i) \subset \mathcal{L}(\nu_{i+1})$, we may take $\sigma' = \sigma$. Therefore, it suffices to take any $\sigma \in \mathcal{L}(\nu_{i+1}) \setminus \mathcal{L}(\nu_i)$. Let $\square_0$ denote the bottom left-most box of $\mathrm{dg}'(\nu_i)$ and, by identification, also of $\mathrm{dg}'(\nu_{i+1})$. From Lemma 1, we know that $\sigma(\square_0) = i + 1$. Reading the boxes of $\mathrm{dg}'(\nu_{i+1})$ (and dually of $\mathrm{dg}'(\nu_i)$) one row at a time left-to-right from bottom to top, let $\square_0, \square_1, \ldots, \square_r$ denote the longest sequence of boxes which are labeled either i or $i + 1$ by σ such that every $\square_{j+1}$ either attacks or is attacked by $\square_j$ for all $0 \leq j \leq r - 1$.

The set $\{\square_0, \square_1, \ldots, \square_r\}$ forms an *attacking chain*, i.e., that for each $1 \leq j \leq r - 1$ the only boxes of $\{\square_0, \square_1, \ldots, \square_r\}$ that either are attacked by or attack $\square_j$ are $\square_{j-1}, \square_{j+1}$, and furthermore, the boxes $\square_0$ and $\square_r$ are only either attacked by or attack $\square_1$ and $\square_{r-1}$, respectively. Define the labeling σ' of ν_i by $\sigma'(\square) := \sigma(\square)$ for all $\square \notin \{\square_0, \square_1, \ldots, \square_r\}$ and $\sigma'(\square_j) := s_i(\sigma(\square_j))$ for all $0 \leq j \leq r$. Then σ' satisfies the requirements of the lemma. $\square$

Example 4. *Here we give an example of the attacking chain argument used in the proof of Lemma 2. Consider the following non-attacking labeling of $\mu = (0,0,2,3,1,2,2,4)$:*

$$\sigma = \begin{array}{|c|c|c|c|c|c|c|c|}
 & & & & & & & 2 \\
 & & & 3 & & & & 1 \\
 & & 3 & 6 & & 4 & 2 & 7 \\
 & & 3 & 4 & 2 & 5 & 1 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}$$

The multiplicity vector of σ is $\mathrm{mult}(\sigma) = (2,3,3,2,1,1,1,1)$. The attacking chain in this example is the following sequence of boxes shown in order as a, b, c, d, e:

$$\begin{array}{|c|c|c|c|c|c|c|c|}
 & & & & & & & \; \\
 & & & e & & & & \; \\
 & & c & \; & & \; & d & \; \\
 & & a & \; & b & \; & \; & \; \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}$$

Note that the top right-most box has the label 2 but is not contained in the chain because the box e is neither attacked by nor attacks this 2-box. Flipping the labels $3 \leftrightarrow 2$ along the chain and shifting the left-most nonzero column to the left gives the labeling:

$$\sigma' = \begin{array}{|c|c|c|c|c|c|c|c|}
 & & & & & & & 2 \\
 & & & 2 & & & & 1 \\
 & 2 & & 6 & & 4 & 3 & 7 \\
 & 2 & & 4 & 3 & 5 & 1 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}$$

The multiplicity vector of σ' is $\mathrm{mult}(\sigma') = (2,4,2,2,1,1,1,1) = \mathrm{mult}(\sigma) + e_2 - e_3$.

Lemma 2 may be interpreted in the context of support sets via the mult map.

Proposition 3. *For all $1 \leq i \leq m-1$, as subsets of $\mathbb{Z}^n$,*

$$s_i(\mathrm{supp}(E_{\nu_i})) \subseteq \mathrm{supp}(E_{\nu_i}) \cup (\mathrm{supp}(E_{\nu_i}) + e_{i+1} - e_i).$$

3 M-convexity

3.1 A Geometric Lemma

In this section, we introduce a technical geometric lemma about preserving M-convexity required for the proof of Theorem 1. To start, M-convexity has a standard definition in terms of an exchange axiom coming from discrete convex analysis. We will rely on the following equivalent definition given in Theorem 1.9 of [30]:

Definition 6. *Let P be a polytope. Then P is a* ***generalized permutahedron*** *[33] if for all pairs of adjacent vertices $\mathbf{u}, \mathbf{v}$ of P, there exists $i, j \in [n]$ such that $\mathbf{u} - \mathbf{v} \in span(e_i - e_j)$. Let $S \subseteq \mathbb{Z}^n$. Then S is* ***M-convex*** *if $S = P \cap \mathbb{Z}^n$ for some generalized permutahedron P with vertices in $\mathbb{Z}^n$.*

Notice that the M-convexity property in invariant under $\mathbb{Z}^n$ lattice translations and simple root direction reflections s_i. Using Proposition 1, we obtain the following:

Corollary 3. *For all $\mu \in \mathbb{Z}^n$, $\operatorname{supp}(E_{\pi(\mu)}) = e_1 + s_1 \cdots s_{n-1}(\operatorname{supp}(E_\mu))$ and thus $\operatorname{Newton}(E_{\pi(\mu)}) = e_1 + s_1 \cdots s_{n-1}(\operatorname{Newton}(E_\mu))$. In particular, if $\operatorname{supp}(E_\mu)$ is M-convex, then so is $\operatorname{supp}(E_{\pi(\mu)})$.*

The next lemma is essential to the proof of the main theorem.

Lemma 3. *Let $L \subseteq \mathbb{Z}^n$ be an M-convex set and let $i, j \in \{1, \ldots, n\}$ be distinct. If $\sigma_{i,j} L \subseteq L \cup (L + e_i - e_j)$, then $L \cup \sigma_{i,j} L$ is M-convex.*

3.2 Proof of main theorem

With all of this in place, we may prove the M-convexity of supports for non-symmetric Macdonald polynomials.

Proof of Theorem 1. Let E_μ be the non-symmetric Macdonald polynomial for the composition $\mu = (\mu_1, \ldots, \mu_n)$. If $\mu = (0, \ldots, 0)$, $\operatorname{Newton}(E_{(0,\ldots,0)}) = \{0\}$ is trivially M-convex. Suppose for the sake of contradiction that there exists a composition μ^* such that E_{μ^*} is not M-convex. Let k be minimal such that there exists a composition μ such that $|\mu| = k$, and E_μ does not have M-convex support. Furthermore, choose some μ such that $\ell = \min(\{i : \mu_i > 0\})$ is minimal among all such μ with $|\mu| = k$ and does not have M-convex support.

Suppose that $\ell = 1$, so $\mu_1 > 0$. Since $\mu_1 > 0$, $(\mu_2, \ldots, \mu_n, \mu_1 - 1)$ is a valid composition. Furthermore, $|(\mu_2, \ldots, \mu_n, \mu_1 - 1)| = k - 1$, so $E_{(\mu_2, \ldots, \mu_{n-1}, \mu_1 - 1)}$ has M-convex support. Then, by Proposition 1, E_μ must also have M-convex support, a contradiction.

Thus, $\ell > 1$. Define $\mu'_i = \mu_i$ if $i > \ell$, 0 if $i \neq \ell - 1$, and μ_ℓ if $i = \ell - 1$. By the minimality of μ, $E_{\mu'}$ has M-convex support. Now, by Proposition 3, we may apply Lemma 3 to $\operatorname{supp}(E_{\mu'})$ to see that E_μ must also have M-convex support, yielding a contradiction. Therefore, for all compositions μ, E_μ must have M-convex support. Lastly, by the definition of M-convexity, this means that for all compositions μ, the support of E_μ is the set of integer points in its convex hull. Therefore, E_μ is SNP. $\square$

3.3 Conjecture of Monical, Tokcan, and Yong

In this section, we show that Theorem 1 resolves [29, Conjecture 3.8] of Monical, Tokcan, and Yong in the affirmative.

Definition 7. *Define the Bruhat order $\leq$ on $\mathbb{Z}^n$ as the transitive closure of the following relations. Given $\lambda \in \mathbb{Z}^n$ and $1 \leq i < j \leq n$ with $\lambda_i < \lambda_j$, we set $\lambda > \sigma_{ij}(\lambda)$, and if in addition $\lambda_j - \lambda_i > 1$, then $\sigma_{ij}(\lambda) > \lambda + e_i - e_j$.*

Monical, Tokcan, and Yong posed the following conjecture:

Conjecture 1. *Let $\mu \in \mathbb{Z}^n_{\geq 0}$. If $\beta \in \mathrm{conv}(\lambda|\lambda \leq \mu) \cap \mathbb{Z}^n_{\geq 0}$, then $\beta \leq \mu$.*

The Bruhat order relates to non-symmetric Macdonald polynomials via the following triangularity result of Sahi:

Proposition 4. *[34] For all $\mu \in \mathbb{Z}^n$, the non-symmetric Macdonald polynomial E_μ has a triangular expansion of the form $E_\mu = \sum_{\lambda \leq \mu} a_\lambda(q,t)x^\lambda$, with all $a_\lambda(q,t) \neq 0$ non-zero q,t rational functions. Therefore, $\{\lambda \mid \lambda \leq \mu\} = \mathrm{supp}(E_\mu)$, and so the Newton polytope of E_μ is the convex hull of the Bruhat lower order ideal of μ: $\mathrm{Newton}(E_\mu) = \mathrm{conv}(\lambda \mid \lambda \leq \mu)$.*

Theorem 1 implies that Conjecture 1 is true.

Corollary 4. *Conjecture 1 is true.*

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References

[1] K. Adiprasito, J. Huh, and E. Katz. "Hodge theory for combinatorial geometries". *Ann. of Math. (2)* **188**.2 (2018), pp. 381–452. DOI.

[2] S. An, K. Tung, and Y. Zhang. "Postnikov–Stanley polynomials are Lorentzian". 2024. DOI. arXiv:2412.02051.

[3] N. Anari, S. Oveis Gharan, and C. Vinzant. "Log-concave polynomials, entropy, and a deterministic approximation algorithm for counting bases of matroids". *59th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2018*. IEEE Computer Soc., Los Alamitos, CA, 2018, pp. 35–46. DOI.

[4] M. Besson, S. Jeralds, and J. Kiers. "Affine Demazure Weight Polytopes and Twisted Bruhat Orders". 2024. DOI. arXiv:2404.03142.

[5] M. Besson, S. Jeralds, and J. Kiers. "Weight polytopes and saturation of Demazure characters". *Math. Ann.* **388**.4 (2024), pp. 4449–4486. DOI.

[6] J. Blasiak, M. Haiman, J. Morse, A. Pun, and G. H. Seelinger. "Flagged LLT polynomials, nonsymmetric plethysm, and nonsymmetric Macdonald polynomials". 2025. DOI. arXiv:2506.09015.

[7] A. Borodin and I. Corwin. "Macdonald processes". *Probability Theory and Related Fields* **158**.1-2 (2014), pp. 225–400. DOI.

[8] P. Brändén and J. Huh. "Lorentzian polynomials". *Ann. of Math. (2)* **192**.3 (2020), pp. 821–891. DOI.

[9] F. Castillo, Y. Cid-Ruiz, F. Mohammadi, and J. Montaño. "Double Schubert polynomials do have saturated Newton polytopes". Vol. 11. 2023, Paper No. e100, 9. DOI.

[10] F. Castillo, Y. Cid-Ruiz, F. Mohammadi, and J. Montaño. "K-polynomials of multiplicity-free varieties". 2025. DOI. arXiv:2212.13091.

[11] I. Cherednik. "Double affine Hecke algebras and Macdonald's conjectures". *Ann. Math. (2)* **141**.1 (1995), pp. 191–216. DOI.

[12] Y. Cid-Ruiz, Y. Li, and J. Matherne. "Log-concavity of polynomials arising from equivariant cohomology". 2024. DOI. arXiv:2411.17572.

[13] L. Escobar and A. Yong. "Newton polytopes and symmetric Grothendieck polynomials". *C. R. Math. Acad. Sci. Paris* **355**.8 (2017), pp. 831–834. DOI.

[14] J. Fei. "Combinatorics of F-polynomials". *Int. Math. Res. Not. IMRN* 9 (2023), pp. 7578–7615. DOI.

[15] A. Fink, K. Mészáros, and A. St. Dizier. "Schubert polynomials as integer point transforms of generalized permutahedra". *Adv. Math.* **332** (2018), pp. 465–475. DOI.

[16] E. S. Hafner, K. Mészáros, L. Setiabrata, and A. St. Dizier. "M-convexity of vexillary Grothendieck polynomials via bubbling". *SIAM J. Discrete Math.* **38**.3 (2024), pp. 2194–2225. DOI.

[17] E. S. Hafner, K. Mészáros, and A. Vidinas. "Log-concavity of the Alexander polynomial". *Int. Math. Res. Not. IMRN* 13 (2024), pp. 10273–10284. DOI.

[18] J. Haglund, M. Haiman, and N. Loehr. "A combinatorial formula for Macdonald polynomials". *J. Amer. Math. Soc.* **18**.3 (2005), pp. 735–761. DOI.

[19] J. Haglund, M. Haiman, and N. Loehr. "A combinatorial formula for nonsymmetric Macdonald polynomials". *Amer. J. Math.* **130**.2 (2008), pp. 359–383. DOI.

[20] M. Haiman. “Hilbert schemes, polygraphs and the Macdonald positivity conjecture”. *Journal of the American Mathematical Society* **14**.4 (2001), pp. 941–1006. DOI.

[21] D. T. Hiep and K.-H. Nguyen-Dang. “Supersymmetric Schur polynomials have saturated Newton polytopes”. 2025. DOI. arXiv:2507.22528.

[22] J. Huh, J. P. Matherne, K. Mészáros, and A. St. Dizier. “Logarithmic concavity of Schur and related polynomials”. *Trans. Amer. Math. Soc.* **375**.6 (2022), pp. 4411–4427. DOI.

[23] S. Kumar. *Kac-Moody groups, their flag varieties and representation theory*. Vol. 204. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 2002, pp. xvi+606. DOI.

[24] I. G. Macdonald. “A new class of symmetric functions”. *Sémin. Lothar. Comb.* **20** (1988), b20a, 41.

[25] S. Mason. “An explicit construction of type A Demazure atoms”. *J. Algebraic Combin.* **29**.3 (2009), pp. 295–313. DOI.

[26] J. P. Matherne, A. H. Morales, and J. Selover. “The Newton polytope and Lorentzian property of chromatic symmetric functions”. *Selecta Math. (N.S.)* **30**.3 (2024), Paper No. 42, 35. DOI.

[27] A. Mattoo and M. Sherman-Bennett. “Saturation of Newton polytopes of type A and D cluster variables”. *Comb. Theory* **2**.2 (2022), Paper No. 6, 37. DOI.

[28] K. Mészáros and A. St. Dizier. “From generalized permutahedra to Grothendieck polynomials via flow polytopes”. *Algebr. Comb.* **3**.5 (2020), pp. 1197–1229. DOI.

[29] C. Monical, N. Tokcan, and A. Yong. “Newton polytopes in algebraic combinatorics”. *Selecta Math. (N.S.)* **25**.5 (2019), Paper No. 66, 37. DOI.

[30] K. Murota. *Discrete convex analysis*. Vol. 10. SIAM Monogr. Discrete Math. Appl. Philadelphia, PA: SIAM Society for Industrial and Applied Mathematics, 2003. DOI.

[31] D.-K. Nguyen, N. T. N. Giao, D. T. Hiep, and D. L. H. Thuy. “Newton polytope of good symmetric polynomials”. *C. R. Math. Acad. Sci. Paris* **361** (2023), pp. 767–775. DOI.

[32] G. Panova and C. Zhao. “The Newton polytope of the Kronecker product”. 2025.

[33] A. Postnikov. “Permutohedra, associahedra, and beyond”. *Int. Math. Res. Not. IMRN* 6 (2009), pp. 1026–1106. DOI.

[34] S. Sahi. “Some properties of Koornwinder polynomials”. Contemp. Math. **254** (2000), pp. 395–411. DOI.

[35] B. Wang, C. X. T. Zhang, and Z.-X. Zhang. “Newton polytopes of dual k-Schur polynomials”. *Adv. in Appl. Math.* **162** (2025), Paper No. 102773, 21. DOI.