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Two Proofs of Rubey's Lattice Conjecture

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Abstract. In 2012, Rubey introduced a natural partial order on the set of reduced pipe dreams of a permutation in which cover relations correspond to generalized chute moves. He conjectured that this poset is a lattice. We discuss two proofs of Rubey's conjecture that we found independently and simultaneously in two groups. In addition, we show that this lattice is semidistributive and polygonal. A simple test for comparison in the Rubey lattice is given in terms of tableaux associated to reduced pipe dreams.

Keywords: Pipe dream, generalized chute move, lattice, semidistributive lattice

1 Introduction

Consider the n-th staircase Young diagram $\diagup_n$, which is the Young diagram of the partition $(n, n-1, n-2, \ldots, 1)$ drawn using English conventions. We index the rows of $\diagup_n$ as $1, \ldots, n$ from top to bottom. A *pipe dream* is a filling of the boxes of $\diagup_n$ in which each box along the southeast boundary is filled with an *elbow tile* and every other box is filled with either a *cross tile* or a *bump tile*. This produces an arrangement of n *pipes* that travel from the north side of $\diagup_n$ to the west side of $\diagup_n$ (see Figure 1). We label each pipe with an element of the set $[n] = \{1, \ldots, n\}$ according to the column where it exits on the west side of $\diagup_n$. Let $\mathfrak{S}_n$ denote the symmetric group of permutations of $[n]$. Given a pipe dream D, we can read the labels of the pipes along the west side of $\diagup_n$ from top to bottom to obtain a permutation $w_D \in \mathfrak{S}_n$ in one-line notation; we say D is a

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pipe dream for w_D. A pipe dream is *reduced* if no two pipes cross each other more than once. For $w \in \mathfrak{S}_n$, let $\mathrm{PD}(w)$ denote the set of reduced pipe dreams for w. Pipe dreams encode some of the algebraic, enumerative, geometric, and probabilistic properties related to Schubert polynomials [6, 12, 13, 16, 17]. Reduced pipe dreams first appeared in the work of Bergeron and Billey [3] following the work of several authors [4, 10, 9, 14, 15]. Bergeron and Billey defined natural raising and lowering operators called *chute moves* and *ladder moves*, which connect the set of all reduced pipe dreams for a fixed permutation.

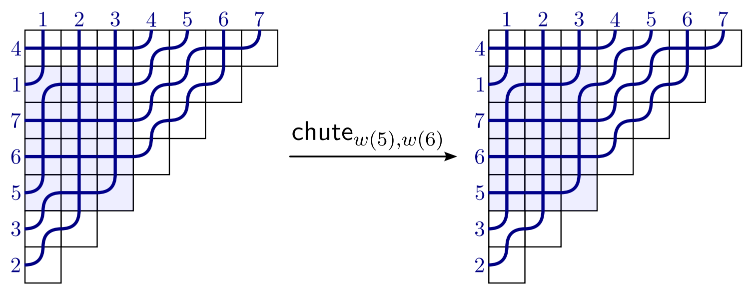

Figure 1: Two reduced pipe dreams in $\mathrm{PD}(w)$ where $w = 4176532$. The pipe dream on the right is obtained from the one on the left by applying a generalized chute move that takes place in the shaded rectangle. The chute move is written as $\mathsf{chute}_{w(5),w(6)}$ rather than $\mathsf{chute}_{5,3}$ in light of Proposition 3.6.

Consider a rectangle R of boxes in $◸_n$. We denote by $\mathsf{NW}(R)$, $\mathsf{NE}(R)$, $\mathsf{SW}(R)$, and $\mathsf{SE}(R)$ the boxes in the northwest, northeast, southwest, and southeast corners of R, respectively. Now consider a pipe dream $D \in \mathrm{PD}(w)$ such that $\mathsf{NW}(R)$ and $\mathsf{SW}(R)$ are filled with bump tiles, $\mathsf{SE}(R)$ is filled with a bump or elbow tile, and all other boxes in R are filled with cross tiles. We can apply a *generalized chute move* to D by changing the bump tile in $\mathsf{SW}(R)$ to a cross tile and changing the cross tile in $\mathsf{NE}(R)$ to a bump tile (see Figure 1); this results in a new pipe dream D', which is also in $\mathrm{PD}(w)$. If the cross tile in D in the box $\mathsf{NE}(R)$ involves pipes $w(i)$ and $w(j)$, then we write $D' = \mathsf{chute}_{w(i),w(j)}(D)$. The inverse of a generalized chute move is called a *generalized ladder move*.

In 2012, building off of work of Bergeron–Billey [3] and Serrano–Stump [19], Rubey [18] defined a partial order $\leq_{\mathrm{chute}}$ on $\mathrm{PD}(w)$ by declaring that $D \leq_{\mathrm{chute}} D'$ if one can obtain D' from D by applying a sequence of generalized chute moves (see Figure 2). Rubey proved that $\mathrm{PD}(w)$ has a minimum element and a maximum element [18, Theorem 3.8]; his argument was a mild modification of a previous argument due to Bergeron and Billey [3]. Rubey also conjectured that $\mathrm{PD}(w)$ is a lattice [18, Conjecture 2.8]. In the special case when w begins with 1 and avoids the pattern 1243, Ceballos, Padrol, and Sarmiento [8, Theorem 5.9] proved Rubey's conjecture by showing that $\mathrm{PD}(w)$ is isomorphic to a certain ν-Tamari lattice (in fact, every ν-Tamari lattice arises in this way).

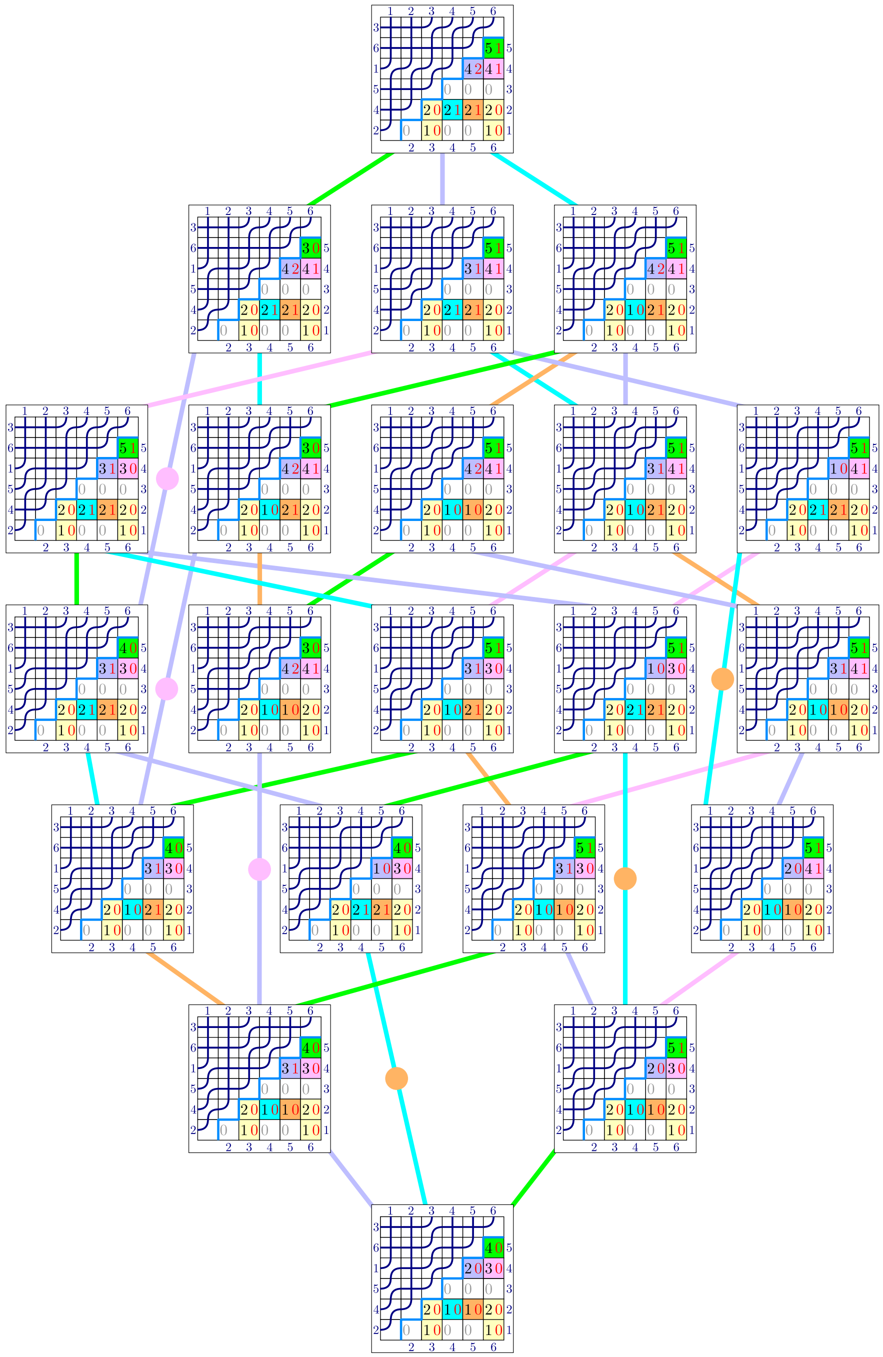

Figure 2: The poset PD(361542), with each element represented by a pipe dream D, the inversions tableau $\Theta(D)$ (with black numbers), and the Lehmer tableau $\Phi(D)$ (with red numbers). Each edge is colored according to which box corresponds to the pair of pipes involved in the generalized chute move. Discs on edges are colored according to additional boxes that change in the Lehmer tableau.

This extended abstract summarizes the works of two groups (one based in Cambridge, Massachusetts and the other based in Seattle, Washington) that settled Rubey's conjecture independently around the same time, resulting in the articles [2, 5]. While the approaches share some common features, they are different enough that we believe it is worth describing both.

Theorem 1.1 ([2, Theorem 1.1] and [5, Theorem 1.1])**.** *For every permutation $w \in \mathfrak{S}_n$, the poset* $\mathrm{PD}(w)$ *is a lattice.*

In light of Theorem 1.1, we will refer to $\mathrm{PD}(w)$ as the *Rubey lattice* of w. It turns out that Rubey lattices have additional structure beyond the lattice property.

Theorem 1.2 ([2, Theorem 1.2])**.** *For every permutation $w \in \mathfrak{S}_n$, the lattice* $\mathrm{PD}(w)$ *is semidistributive and polygonal. Moreover, every polygon in* $\mathrm{PD}(w)$ *is a diamond or a pentagon.*

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2 Background

Let $\mathfrak{S}_n$ be the symmetric group of permutations on $[n] = \{1, 2, \ldots, n\}$. We represent $w \in \mathfrak{S}_n$ by its one-line notation $w = w(1)w(2)\ldots w(n)$. The set of *inversions* of w is

$$\mathrm{Inv}(w) = \{(i,j) \in [n]^2 \mid i < j,\ w(i) > w(j)\}.$$

The *diagram* of w is obtained from the $n \times n$ array with a dot in position $(i, w(i))$ for $1 \leq i \leq n$ by removing all cells in the array that are in the hook shape weakly to the right of or below a dot, including the dots. The remaining cells form the (Rothe) diagram $\mathbf{D}(w)$. Thus,

$$\begin{aligned}\mathbf{D}(w) &= \{(i,j) \in [n]^2 \mid j < w(i) \text{ and } i < w^{-1}(j)\}\\ &= \{(i, w(j)) \in [n]^2 \mid i < j \text{ and } w(i) > w(j)\},\end{aligned}$$

as the cells of $\mathbf{D}(w)$ are in bijection with the inversions of w.

Let $\triangle_{n-1}$ denote the $(n-1)$-th staircase Young diagram drawn by reflecting French conventions across a vertical axis. We index the rows of $\triangle_{n-1}$ as $1, \ldots, n-1$ from bottom to top, and we index the columns as $2, \ldots, n$ from left to right. For $1 \leq i < j \leq n$, we index the box in row i and column j by the pair (i, j). By identifying $\triangle_{n-1}$ with its set of boxes, we can write

$$\triangle_{n-1} = \{(i,j) \in \mathbb{Z}^2 : 1 \leq i < j \leq n\}.$$

Let $w \in \mathfrak{S}_n$. The *inversions diagram* of w is the set $\mathrm{ID}(w)$ of boxes in $\triangle_{n-1}$ indexed by the inversions of w; we often represent $\mathrm{ID}(w)$ by coloring its boxes in $\triangle_{n-1}$. For examples of inversions diagrams, see the shaded boxes for each poset element of Figure 2. For $\mathcal{D} \subseteq \triangle_{n-1}$, we define a *tableau* T of shape $\mathcal{D}$ to be a filling of $\mathcal{D}$ with integers. We denote the number in box (i, j) by $T(i, j)$.

Let Q be a finite poset with partial order $\leq$. For $\alpha, \beta \in Q$ with $\alpha \leq \beta$, the *interval* between α and β is the set $[\alpha, \beta] = \{\gamma \in Q : \alpha \leq \gamma \leq \beta\}$. If $\alpha < \beta$ and $[\alpha, \beta] = \{\alpha, \beta\}$, then we say β *covers* α and write $\alpha \lessdot \beta$. A *maximum* element of Q is an element $\hat{1} \in Q$ such that $\alpha \leq \hat{1}$ for all $\alpha \in Q$. Similarly, a *minimum* element of Q is an element $\hat{0} \in Q$ such that $\hat{0} \leq \alpha$ for all $\alpha \in Q$.

A *polygon* in a poset is an interval $[\alpha, \beta]$ of cardinality at least 4 that is the union of two chains whose intersection is $\{\alpha, \beta\}$. A *diamond* is a polygon of cardinality 4. A *pentagon* is a polygon of cardinality 5.

A *lattice* is a poset Q such that any two elements $\alpha, \beta \in Q$ have a greatest lower bound $\alpha \wedge \beta$ and a least upper bound $\alpha \vee \beta$. We call $\alpha \wedge \beta$ and $\alpha \vee \beta$ the *meet* and *join* (respectively) of α and β. A lattice Q is *polygonal* if for all $\gamma, \gamma' \in Q$ such that γ and γ' cover $\gamma \wedge \gamma'$ or are covered by $\gamma \vee \gamma'$, the interval $[\gamma \wedge \gamma', \gamma \vee \gamma']$ is a polygon.

An *anti-isomorphism* from a poset Q to a poset Q' is a bijection $\varphi : Q \to Q'$ such that for all $\alpha, \beta \in Q$, we have that $\alpha \leq \beta$ in Q if and only if $\varphi(\beta) \leq \varphi(\alpha)$ in Q'.

Let Q be a finite lattice. We say Q is *meet-semidistributive* if for all $\alpha, \beta \in Q$ such that $\alpha \leq \beta$, the set $\{\gamma \in Q : \gamma \wedge \beta = \alpha\}$ has a maximum element. We say Q is *join-semidistributive* if for all $\alpha, \beta \in Q$ such that $\alpha \leq \beta$, the set $\{\gamma \in Q : \gamma \vee \alpha = \beta\}$ has a minimum element. We say Q is *semidistributive* if it is both meet-semidistributive and join-semidistributive.

3 The Cambridge approach

Our proof strategy is as follows. We define a *Lehmer tableau*, which is a certain type of bottom-right-aligned staircase shape with nonnegative integer entries. We also define a *Lehmer poset* by taking the set $\mathrm{LT}(w)$ of all Lehmer tableaux corresponding to elements of $\mathrm{PD}(w)$ and imposing the partial order $\leq$ on Lehmer tableaux given by componentwise comparison of entries. We show that there is a poset isomorphism $\Phi : \mathrm{PD}(w) \to \mathrm{LT}(w)$ and that $\mathrm{LT}(w)$ is a lattice, allowing us to conclude that $\mathrm{PD}(w)$ is a lattice.

3.1 Inversions tableaux and Lehmer tableaux

The following definitions were given in [1]; a similar construction can be found in Elizabeth Kelly's Ph.D. Thesis [11].

Definition 3.1. Let $w \in \mathfrak{S}_n$. An *inversions tableau* T for w is a filling of $\triangle_{n-1}$ with numbers in $\{0, 1, \ldots, n-1\}$ which satisfies:

1. $T(i,j) = 0$ if and only if $(i,j) \notin \mathrm{ID}(w)$.
2. For $1 \le i < j < k \le n$, either $T(i,j) \le T(i,k) \le T(j,k)$ or $T(i,j) \ge T(i,k) \ge T(j,k)$.
3. No two shaded boxes in a column have the same entry.
4. $T(i,i+1) \le i$ for all $1 \le i < n$.

Definition 3.2. Let T be an inversions tableau for a permutation w. The *Lehmer tableau* of T is the tableau $\Lambda(T)$ defined as follows. For each $(i,j) \in \mathrm{ID}(w)$, let $\Lambda(T)(i,j)$ be the number of integers $k \in [T(i,j)-1]$ that do not appear below box (i,j) in T.

Let $\mathrm{IT}(w)$ denote the set of inversions tableaux for w. Let $\mathrm{LT}(w) = \Lambda(\mathrm{IT}(w))$ denote the set of Lehmer tableaux for w. We view $\mathrm{LT}(w)$ as a poset under the componentwise comparison order. Figure 2 shows many inversions tableaux and their corresponding Lehmer tableaux.

3.2 The poset isomorphism

Consider a reduced pipe dream $D \in \mathrm{PD}(w)$. The pipes labeled $w(i)$ and $w(j)$ cross each other if and only if (i,j) is an inversion of w. Moreover, if (i,j) is an inversion of w, then pipes $w(i)$ and $w(j)$ cross exactly once because D is reduced. For each $(i,j) \in \mathrm{ID}(w)$, let $\mathsf{row}_D(i,j)$ be the index of the row in D where pipes $w(i)$ and $w(j)$ cross. Let $\Theta(D)\colon \triangle_{n-1} \to \mathbb{Z}_{\ge 0}$ be the tableau defined by

$$\Theta(D)(i,j) = \begin{cases} \mathsf{row}_D(i,j) & \text{if } (i,j) \in \mathrm{ID}(w); \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.3 ([1, proof of Theorem 3.13])**.** *For each $w \in \mathfrak{S}_n$, the map Θ is a bijection from $\mathrm{PD}(w)$ to $\mathrm{IT}(w)$.*

Composing Θ with Λ, we have a bijection $\Phi = \Lambda \circ \Theta\colon \mathrm{PD}(w) \to \mathrm{LT}(w)$.

Theorem 3.4 ([2, Theorem 6.14])**.** *For $w \in \mathfrak{S}_n$, $\Phi\colon \mathrm{PD}(w) \to \mathrm{LT}(w)$ is a poset isomorphism.*

Thus, two pipe dreams are related by chute moves if and only if their Lehmer tableaux are comparable elementwise. In order to prove Theorem 3.4, we need to define chute moves on inversions tableaux and Lehmer tableaux. Consider a box $(i,j) \in \mathrm{ID}(w)$, and let $T(i,j) = a$. Let b be the smallest integer that is greater than a and does not appear below (i,j) in T. We can *increment* box (i,j) by either replacing a by b if b does not appear in column j, or exchanging a and b if b appears in column j. Let $\uparrow_{(i,j)} T$ be the tableau obtained from T by incrementing box (i,j).

Lemma 3.5 ([1, Section 6]). *If T is an inversions tableau for a permutation w, then $\Lambda(\uparrow_{(i,j)} T)$ is obtained from $\Lambda(T)$ by adding 1 to the entry in box (i,j).*

It follows from Lemma 3.5 that increments commute with each other. Thus, it makes sense to speak about incrementing a multiset $\mathcal{M}$ of boxes in a column-injective tableau. Let $\uparrow_{\mathcal{M}} T$ denote the tableau obtained from T by incrementing $\mathcal{M}$. For each box (i,j), we obtain the entry $\Lambda(\uparrow_{\mathcal{M}} T)(i,j)$ by adding the multiplicity of (i,j) in $\mathcal{M}$ to the entry $\Lambda(T)(i,j)$. The following proposition shows how chute moves relate to incrementing.

Proposition 3.6 ([1, Proposition 6.3]). *Let $D_1, D_2 \in \mathrm{PD}(w)$ be reduced pipe dreams for a permutation $w \in \mathfrak{S}_n$, and let (x_0, y_0) be an inversion of w. Let $T_1 = \Theta(D_1)$ and $T_2 = \Theta(D_2)$. Let $p_0 = T_1(x_0, y_0)$, and let q_0 be the smallest integer that is greater than p_0 and that does not appear below (x_0, y_0) in T. Then*

$$D_2 = \mathsf{chute}_{w(x_0),w(y_0)}(D_1)$$

if and only if there exists a set $Y \subseteq \{y_0+1, \ldots, n\}$ such that the following conditions hold:

- *$T_2 = \uparrow_{\mathcal{B}} T_1$, where $\mathcal{B} = \{(x_0, y_0)\} \cup \{(x_0, y) : y \in Y\}$;*
- *$T_1(x_0, y) = T_2(y_0, y) = p_0$ and $T_2(x_0, y) = T_1(y_0, y) = q_0$ for every $y \in Y$;*
- *no box in column y_0 has entry q_0 in T_1.*

Moreover, if $D_2 = \mathsf{chute}_{w(x_0),w(y_0)}(D_1)$, then $w(Y) = \{w(y) | y \in Y\}$ is the set of indices of the pipes that cross vertically through the rectangle $\mathbf{R}(D_1, D_2)$ in D_1 (and also in D_2).

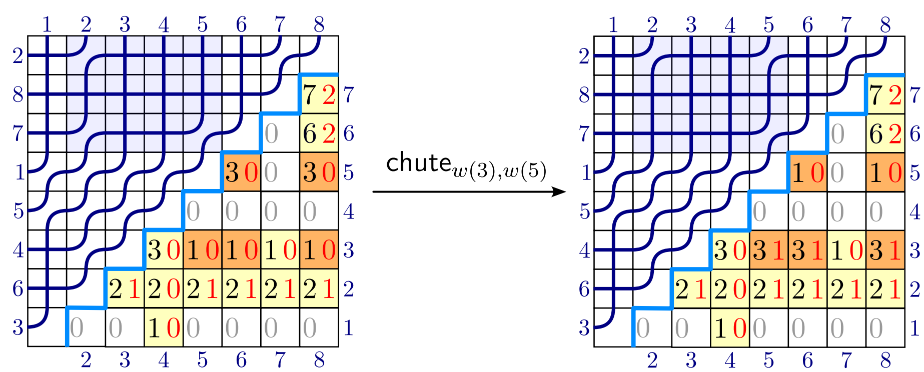

Figure 3: Applying a chute move to a reduced pipe dream, along with the corresponding chute move on inversions tableau and Lehmer tableau.

Corollary 3.7 ([1, Corollary 6.16]). *If $D, D' \in \mathrm{PD}(w)$ and $D \leq_{\text{chute}} D'$, then $\Phi(D) \leq \Phi(D')$.*

Corollary 3.7 follows from Lemma 3.5 and Proposition 3.6. It is the easier direction of the poset isomorphism; the harder direction appears in Section 6 of [2], where we construct a sequence of chute moves between any two comparable Lehmer tableaux.

3.3 The lattice property

Theorem 1.1 follows from Theorem 3.4 by studying $\mathrm{LT}(w)$. We use the following result of Björner, Edelman, and Ziegler.

Proposition 3.8 ([7, Lemma 2.1]). *Let Q be a finite poset with a minimum element and a maximum element. Suppose that for all distinct $\gamma_0, \gamma_1, \gamma_2 \in Q$ satisfying $\gamma_0 \lessdot \gamma_1$ and $\gamma_0 \lessdot \gamma_2$, the elements γ_1 and γ_2 have a least upper bound in Q. Then Q is a lattice.*

Thus, to prove the lattice property, it is enough to show that for $D_1, D_2 \in \mathrm{PD}(w)$ covering $D_0 \in \mathrm{PD}(w)$, the pipe dreams D_1, D_2 have a least upper bound.

We show that in several cases, the tableau L obtained from $\Phi(D_1)$ and $\Phi(D_2)$ by taking the maximum of each box entry-wise is also a Lehmer tableau. Thus, L is the least upper bound of $\Phi(D_1)$ and $\Phi(D_2)$ in the Lehmer poset. By Theorem 3.4 the desired property holds on the corresponding pipe dreams as well. For cases where the element-wise maximum is not a valid Lehmer tableau, we prove the desired property by using symmetries of pipe dreams to reduce to the previous case. Using a related approach, we also prove Theorem 1.2.

4 The Seattle approach

In this section, we outline an alternative proof that $\mathrm{PD}(w)$ is a lattice by giving a recursive algorithm to compute the meet of any two reduced pipe dreams $D_1 \wedge D_2$ for a fixed permutation. We use transposition symmetry to compute the join from the meet algorithm. The approach is to identify which tiles are movable by a sequence of generalized chute moves, and then move one tile at a time until there is agreement in a fixed order. We also define pipe dream tableaux to give a quick comparison test for any two pipe dreams in the Rubey lattice.

4.1 The move operation

Fix a positive integer n and a permutation $w \in \mathfrak{S}_n$. For $D \in \mathrm{PD}(w)$, if the tile at position (i, j) in D is a cross tile, we write $D(i, j) = +$. Similarly, if the tile at position (i, j) in D is a bump or elbow tile, we write $D(i, j) = \bullet$. For the remainder of this section, we will not distinguish between bump and elbow tiles.

Definition 4.1. Let $D \in \mathrm{PD}(w)$. If $D(i, j) = +$ and $\{h \in [1, i-1] \mid D(h, j) = \bullet\}$ is nonempty, we say that (i, j) is *movable* in D, and we define

$$h_{ij}(D) = \max\{h \in [1, i-1] \mid D(h, j) = \bullet\}.$$

Similarly, define

$$k_{ij}(D) = \min\{k \in [j+1, n] \mid D(i, k) = \bullet\}.$$

For any cross tile (i,j) in D, $k_{ij}(D)$ is well-defined since $D(i,n) = \bullet$ for all reduced pipe dreams D representing $w \in \mathfrak{S}_n$; but $h_{ij}(D)$ is only defined when (i,j) is movable in D.

We now introduce the move operation $\mathcal{M}_{ij}$, which is the key tool in the Seattle proof of the lattice property of $\mathrm{PD}(w)$. The pipe dream $\mathcal{M}_{ij}(D)$ will be defined for $D \in \mathrm{PD}(w)$ such that $D(i,j) = +$ and (i,j) is movable in D. The effect of $\mathcal{M}_{ij}$ will be to turn the tile (i,j) in D into a bump tile by a particular sequence of generalized ladder moves depending on D. This sequence is defined recursively in [5, Definition 3.6]. We omit the recursive definition of $\mathcal{M}_{ij}(D)$ here and give an explicit characterization of $\mathcal{M}_{ij}(D)$ instead.

Definition 4.2. Let $D \in \mathrm{PD}(w)$ such that (i,j) is a movable cross tile in D. Define $\mathrm{Path}_{ij}(D)$ to be the lattice path generated by the procedure below.

(i) Let $(p,q) = (i, k_{ij}(D))$ and initialize $\mathrm{Path}_{ij}(D)$ as $\{(p,q)\}$.

(ii) Let $p' = \max\{r < p : \exists t \in [j, k_{ij}(D)]$ such that $D(r,t) = \bullet\}$ and trace up from (p,q) to (p',q) — include $\{(p',q),(p'+1,q),\ldots,(p-1,q)\}$ into $\mathrm{Path}_{ij}(D)$.

(iii) Let $q' = \min\{t \geq j \mid D(p',t) = \bullet\}$ and trace left from (p',q) to (p',q') — include $\{(p',q'),(p',q'+1),\ldots,(p',q-1)\}$ into $\mathrm{Path}_{ij}(D)$.

(iv) If $q' = j$, return $\mathrm{Path}_{ij}(D)$. Otherwise, set $(p,q) = (p',q')$ and return to Step (ii).

The tiles $(i, k_{ij}(D))$ and $(h_{ij}(D), j)$ are the *endpoints* of $\mathrm{Path}_{ij}(D)$.

Definition 4.3. Let $D \in \mathrm{PD}(w)$ such that (i,j) is a movable cross tile in D. Let $h = h_{ij}(D)$ and $k = k_{ij}(D)$. Then $\mathcal{M}_{ij}(D)$ is equal to the output of the procedure below.

(i) For each $r \in [h,i]$, if there is a bump tile in row r of $\mathrm{Path}_{ij}(D)$, replace (r,j) with a bump tile.

(ii) For each $c \in [j,k]$, if there is a bump tile in column c of $\mathrm{Path}_{ij}(D)$, replace (i,c) with a bump tile.

(iii) Replace each bump tile in $\mathrm{Path}_{ij}(D)$ with a cross tile, except for the endpoints (h,j) and (i,k).

Example 4.4. In the submatrix of a pipe dream below, $\mathrm{Path}_{61}(D)$ consists of the tiles in cyan. The cross tile $(6,1)$ is movable in D, and $M_{61}(D)$ can be computed by Definition 4.3.

$$\begin{array}{cccccccc}
\cdot & \cdot & \cdot & + & \cdot & + & \cdot & + \\
+ & + & + & + & + & + & \cdot & + \\
+ & + & \cdot & + & \cdot & \cdot & + & + \\
+ & + & + & + & + & \cdot & + & \cdot \\
+ & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & \cdot
\end{array}
\xrightarrow{\mathcal{M}_{61}}
\begin{array}{cccccccc}
\cdot & + & + & + & \cdot & + & \cdot & + \\
+ & + & + & + & + & + & \cdot & + \\
\cdot & + & + & + & + & + & + & + \\
\cdot & + & + & + & + & + & + & + \\
+ & + & + & + & + & + & + & + \\
\cdot & \cdot & \cdot & + & \cdot & \cdot & + & \cdot
\end{array}$$

The following is an immediate corollary of [5, Proposition 3.10].

Proposition 4.5. *For all $D \in \mathrm{PD}(w)$ and all tiles (i, j) that are movable in D, $\mathcal{M}_{ij}(D)$ is a reduced pipe dream for w. Furthermore, $\mathcal{M}_{ij}(D) \leq_{\text{chute}} D$.*

4.2 Meet-computing algorithm

In Theorem 4.8, we give a recursive algorithm whose output can be shown to be the unique greatest lower bound of $\{D_1, D_2\}$ for any pipe dreams $D_1, D_2 \in \mathrm{PD}(w)$. Therefore, meets exist for all $D_1, D_2 \in \mathrm{PD}(w)$, so $\mathrm{PD}(w)$ is a meet-semilattice in the partial order induced by generalized chute moves. There is an order anti-isomorphism between $\mathrm{PD}(w)$ and $\mathrm{PD}(w^{-1})$ given by the transpose $D \mapsto D^t$, so $D_1 \vee D_2 = (D_1^t \wedge D_2^t)^t$, and hence $\mathrm{PD}(w)$ is also a join-semilattice for all w, so it is a lattice.

Definition 4.6. Let $\prec$ be the total order on $[n]^2$ defined by $(i, j) \prec (i', j')$ if $i > i'$ or $i = i'$ and $j < j'$. Let $D_1, D_2 \in \mathrm{PD}(w)$ such that $D_1 \neq D_2$, and define the *principal disagreement* between D_1 and D_2 to be the minimal tile (i, j) with respect to $\prec$ such that $D_1(i, j) \neq D_2(i, j)$.

Lemma 4.7 ([5, Lemma 4.5])**.** *Let $D_1, D_2 \in \mathrm{RPD}(w)$ such that $D_1 \neq D_2$. Let (i, j) be the principal disagreement between D_1 and D_2, and assume that $D_1(i, j) = +$ while $D_2(i, j) = \bullet$. Then (i, j) is movable in D_1.*

Theorem 4.8 ([5, Theorem 4.2])**.** *Let $D_1, D_2 \in \mathrm{PD}(w)$. Then $D_1 \wedge D_2$ exists and is given by the output of the following recursive algorithm:*

(i) If $D_1 = D_2$, return D_1.

(ii) If $D_1 \neq D_2$, let (i, j) be the principal disagreement between D_1 and D_2. If $D_1(i, j) = +$ and $D_2(i, j) = \bullet$, then return $\mathcal{M}_{ij}(D_1) \wedge D_2$. Otherwise, return $D_1 \wedge \mathcal{M}_{ij}(D_2)$.

To prove Theorem 4.8, we induct on the size of lower order ideals $\langle (D_1, D_2) \rangle$ in the product order on $\mathrm{PD}(w)^2$, where our induction hypothesis is that for all $D_1, D_2 \in \mathrm{PD}(w)$ with $\#\langle (D_1, D_2) \rangle < m$, $D_1 \wedge D_2$ exists and is computed by the above algorithm. For the details of this argument, see [5]. We note that this paper uses the convention that the Rubey lattice is the dual poset of $\mathrm{PD}(w)$, so Theorem 4.8 is originally stated as a join-computing algorithm.

4.3 Pipe dream tableaux

Observe from Section 2 that pipes $x < y$ cross in any/every pipe dream for w if and only if $(x, y) \in \mathrm{Inv}(w^{-1})$. Additionally, if pipe x crosses pipe y and $x < y$, then x enters the crossing vertically and y enters horizontally. Such crosses will be referred to as *vertical*

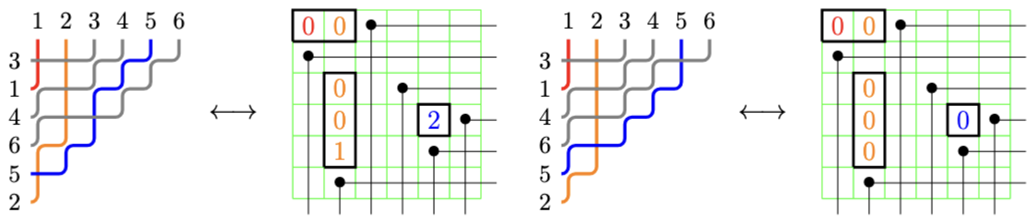

Figure 4: Pipe dreams $D, D_{\text{top}} \in \text{PD}(314652)$ with their corresponding pipe dream tableaux on the left and right, respectively.

or *horizontal crosses* of pipe x or y respectively. Furthermore, a bump tile where pipe x enters from the north and exits to the west is a *left bump* of pipe x. Since each pipe traverses from the top to the left boundary of a pipe dream, each pipe must have at least one left bump.

Definition 4.9. Let $D \in \text{PD}(w)$. The *pipe dream tableau* $\mathbf{T}(D)$ is a filling of $\mathbf{D}(w)$ where t is placed in cell $(w^{-1}(y), x)$ if there are t left bumps of pipe x in any row north of the crossing between pipes $x < y$. Denote the entry in cell $(w^{-1}(y), x)$ corresponding to the crossing between pipes $x < y$ of $\mathbf{T}(D)$ by $\mathbf{T}_D(x, y)$.

Definition 4.10. Suppose $D_1, D_2 \in \text{PD}(w)$. Define a partial order $\trianglelefteq$ on pipe dream tableaux by $\mathbf{T}(D_1) \trianglelefteq \mathbf{T}(D_2)$ if $\mathbf{T}_{D_1}(x, y) \leq \mathbf{T}_{D_2}(x, y)$ for each $(x, y) \in \text{Inv}(w^{-1})$.

Lemma 4.11 ([5, Lemma 5.6]). *A pipe dream $D \in PD(w)$ is uniquely determined by $\mathbf{T}(D)$.*

Theorem 4.12 ([5, Theorem 1.3]). *Let $w \in \mathfrak{S}_n$. If D_1 and D_2 are reduced pipe dreams for w,*

$$D_1 \leq_{\text{chute}} D_2 \iff \mathbf{T}(D_1) \trianglelefteq \mathbf{T}(D_2).$$

Pipe dream tableaux are closely related to the inversions and Lehmer tableaux from the Cambridge approach, as well as Kelly's flagged inversion fillings. The pipe dream and Lehmer tableaux for $w \in \mathfrak{S}_n$ with poset structures induced by entrywise comparison are both isomorphic to $\text{PD}(w)$. However, for $(x, y) \in \text{Inv}(w^{-1})$ and $D \in \text{PD}(w)$, $\mathbf{T}_D(x, y)$ is not necessarily equal to $\Phi(D)(x, y)$. Let $D \in \text{PD}(w)$ and let $T = \Theta(D)$ be the corresponding inversions tableau. For each $(x, y) \in \text{Inv}(w^{-1})$, let $\tilde{\Lambda}(T)(w^{-1}(y), x)$ be the number of integers $k \in [T(x, y) - 1]$ that do not appear in column y of T. Then, $\tilde{\Lambda}(T)$ is the pipe dream tableau $\mathbf{T}(D)$. See [5, Rem. 5.17] for argument details and Definition 3.2 for a comparison with the definition of $\Lambda : \text{IT}(w) \to \text{LT}(w)$.

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