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Luck and magic for Pitman–Stanley polytopes

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Abstract. Pitman–Stanley polytopes are ubiquitous in geometric combinatorics. They appear naturally in statistical contexts and are tightly connected with the theory of parking functions in combinatorics. The present article addresses the lattice point enumeration of these polytopes, through the lens of Ehrhart theory. We give a combinatorial interpretation of the h^*-polynomial of all Pitman–Stanley polytopes, and prove that they are real-rooted when the defining parameters of the polytope are positive. The crucial tool employed for this purpose is proving that Ehrhart polynomials of these Pitman–Stanley polytopes are magic positive. We achieve this by interpreting the coefficients of the Ehrhart polynomial in the magic basis via the combinatorics of "lucky" cars in a parking function.

Keywords: Lucky cars, Ehrhart positivity, Magic positivity, Pitman–Stanley polytopes

1 Introduction

In an influential paper, Pitman and Stanley [14] introduced a class of lattice polytopes, now commonly called *Pitman–Stanley polytopes*. The precise definition of a Pitman–Stanley polytope is as follows. Consider a vector $\mathbf{y} \in \mathbb{Z}^n_{\geq 0}$, and define

$$\Pi_n(\mathbf{y}) = \left\{ \mathbf{x} \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \text{ for each } k = 1, \dots, n \right\}.$$

(We customarily use the non-bold names x_i and y_i to denote the i-th coordinate of the boldface vectors $\mathbf{x}$ and $\mathbf{y}$ respectively.) These polytopes are especially relevant due to their appearance in diverse contexts within probability, statistics, algebraic combinatorics, and discrete geometry. Their face structure, volume computation and lattice point enumerations were addressed in the original work by Pitman and Stanley, but many aspects of the theory are still attracting mathematicians to this topic (see [11, 10, 5]).

In the present article we shall be concerned with the enumeration of lattice points in (dilations of) Pitman–Stanley polytopes. A celebrated theorem of Ehrhart [6] establishes

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that the number of lattice points in the m-th dilation of a lattice polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is interpolated by a polynomial of degree $d = \dim \mathcal{P}$. In other words, there exists a polynomial $\mathrm{ehr}_{\mathcal{P}} \in \mathbb{R}[t]$ such that $\mathrm{ehr}_{\mathcal{P}}(t) = \#(t\mathcal{P} \cap \mathbb{Z}^n)$, for every positive integer t. In current mathematics there is considerable interest in studying the coefficients of the Ehrhart polynomial of a polytope. If we write

$$\mathrm{ehr}_{\mathcal{P}}(t) = a_d\, t^d + a_{d-1}\, t^{d-1} + \cdots + a_1\, t + a_0, \tag{1.1}$$

then the following facts are known:

- The leading coefficient a_d equals the normalized volume of $\mathcal{P}$.
- The second leading coefficient a_{d-1} equals half the normalized volume of the boundary.
- The constant term a_0 equals the Euler characteristic of the Euclidean space, i.e., 1.

A consequence of the interpretation for a_d is that the Ehrhart polynomial of $\mathcal{P}$ provides a discrete analog for the notion of volumes.

Due to the geometric interpretations for the three coefficients above, one may be inclined to believe that something analogous happens for the remaining coefficients. In reality, the coefficients $a_1, a_2, \ldots, a_{d-2}$ are often very complicated and can in fact be negative. When all the coefficients $a_1, \ldots, a_{d-2}$ are non-negative, we say that $\mathcal{P}$ is Ehrhart positive. We recommend [12, 8] for surveys addressing Ehrhart positivity.

One of the results of Pitman and Stanley is a closed formula that shows that for any vector $\mathbf{y}$ the polytope $\Pi_n(\mathbf{y})$ is Ehrhart positive. Concretely, they gave a formula (see (2.2)) for $\mathrm{ehr}_{\Pi_n(\mathbf{y})}(t)$ as a sum of Catalan many terms, each of which is itself a polynomial with non-negative coefficients in t. It is nonetheless difficult to extract nice formulas for each individual coefficient of this polynomial, and no combinatorial interpretation is known for them (after multiplying by $(n-1)!$, which therefore turns each coefficient into a non-negative integer).

A considerable strengthening of Ehrhart positivity is the positivity of $\mathrm{ehr}_{\mathcal{P}}(t)$ in an alternative basis. If instead of using the monomial basis $1, t, t^2, \ldots, t^d$, we use the basis $\{t^i(1+t)^{d-i}\}_{i=0}^d$, we can write $\mathrm{ehr}_{\mathcal{P}}(t) = \sum_{i=0}^d c_i\, t^i (1+t)^{d-i}$ for some real numbers $c_0, \ldots, c_d$. The nonnegativity of all the c_i implies the nonnegativity of the a_i in equation (1.1). Following Ferroni and Higashitani [8], we say that $\mathrm{ehr}_{\mathcal{P}}(t)$ is *magic positive* if $c_i \geq 0$ for $i = 0, \ldots, d$. Our first main contribution is proving that $\Pi_n(\mathbf{y})$ have Ehrhart polynomials that are magic positive for $\mathbf{y} \in \mathbb{Z}^n_{>0}$. This result was announced in [8].

The specific strategy we employ to prove this result yields in fact something far more interesting than the mere nonnegativity of the coefficients c_i mentioned above. As it turns out, there is a nice *combinatorial interpretation* for these coefficients. In order to state our interpretation, we need to rely on one of the many frameworks in which Pitman–Stanley polytopes arise: that of $\mathbf{y}$-parking functions (see Definition 2.2 below for the details).

Theorem 1.1 (Thm. 3.7) *Let $\Pi_n(\mathbf{y})$ be a Pitman–Stanley polytope for $\mathbf{y} \in \mathbb{Z}^n_{>0}$. Let*

$$\mathrm{ehr}_{\Pi_n(\mathbf{y})}(t) = \sum_{i=0}^{n} c_i\, t^i (1+t)^{n-i}.$$

Then, for each $0 \leq i \leq n$, the integer number $n!c_i$ enumerates the $\mathbf{y}$-parking functions having exactly i lucky cars where the first spot is not available. In particular, each c_i is a nonnegative number, and therefore magic positivity holds.

For an explanation of the terms "lucky cars" and "first available spot" we refer the reader to Section 3. Note that there is a different notion of "lucky cars" recently introduced in [7]. The case $\mathbf{y} = \mathbf{1}$ of the result above retrieves a result by Gessel and Seo [9, Thm. 10.1] (see also [15]).

We now turn our attention to a different encoding of Ehrhart polynomials. For any lattice polytope $\mathcal{P}$ of dimension d, one can find a polynomial $h^*_{\mathcal{P}}(x)$ of degree at most d and satisfying: $\sum_{k\geq 0} \mathrm{ehr}_{\mathcal{P}}(k)\, z^k = h^*_{\mathcal{P}}(z)/(1-z)^{d+1}$. The polynomial appearing on the numerator of the right-hand-side is often called the *Ehrhart h^*-polynomial*. It is a famous result by Stanley that the coefficients of $h^*_{\mathcal{P}}(z)$ are always nonnegative integer numbers. Similar to how Ehrhart positivity is regarded as one of the most interesting classes of inequalities for the coefficients of $\mathrm{ehr}_{\mathcal{P}}(t)$, there is a hierarchy of attractive features that $h^*_{\mathcal{P}}(z)$ often displays: unimodality, log-concavity without internal zeros, or real-rootedness. Each of these three properties is stronger than the preceding ones (see [13, 3]). We suggest [4] and [8] for a detailed discussion of these classes of inequalities in the specific context of Ehrhart h^*-polynomials.

One of the many reasons why magic positivity is regarded as a strong feature for an Ehrhart polynomial is that, via a remarkable result of Petter Brändén [2], it implies that the h^*-polynomial of $\mathcal{P}$ is real-rooted (e.g., see [8, Theorem 4.19]). In other words, as a consequence of Theorem 1.1, we obtain the following result.

Theorem 1.2 *Let $\Pi_n(\mathbf{y})$ be a Pitman–Stanley polytope for $\mathbf{y} \in \mathbb{Z}^n_{>0}$. The polynomial $h^*_{\Pi_n(\mathbf{y})}(z)$ is real-rooted. In particular, its coefficients form a log-concave sequence without internal zeros and are unimodal.*

For an arbitrary polytope $\mathcal{P}$, the h^*-polynomial of $\mathcal{P}$ evaluated at $z = 1$ retrieves the (normalized) volume. The volumes of the Pitman–Stanley polytopes have a well-known combinatorial interpretation. For $\mathbf{y} \in \mathbb{Z}^n_{\geq 0}$ we have that $\mathrm{vol}\,\Pi_n(\mathbf{y}) = \#\,\mathrm{PF}_n(\mathbf{y})$, i.e., the number of $\mathbf{y}$-parking functions and the volume of $\Pi_n(\mathbf{y})$ coincide. It is reasonable to ask for a "natural" statistic on $\mathbf{y}$-parking function whose distribution matches the h^*-polynomial of $\Pi_n(\mathbf{y})$. The following new result provides one such natural statistic.

Theorem 1.3 *For any vector $\mathbf{y} \in \mathbb{Z}^n_{\geq 0}$, the h^*-polynomial of the Pitman–Stanley polytope*

$\Pi_n(\mathbf{y})$ *is given by*

$$h^*_{\Pi_n(\mathbf{y})}(z) = \sum_{\sigma\in \mathrm{PF}_n(\mathbf{y})} z^{\mathrm{asc}(\sigma)}.$$

where asc *is a statistic on* $\mathrm{PF}_n(\mathbf{y})$ *called modified ascents and defined as the number of ascents of* $\sigma \in \mathrm{PF}_n(\mathbf{y})$ *with an additional ascent if* $\sigma_1 > 1$.

In a recent preprint, Athanasiadis [1, Prop. 3.2] independently found the special case of this result for $\mathbf{y} = (s_1, 0^{s_1-1}, s_2, 0^{s_2-1}, \ldots, s_k, 0^{s_k-1})$ for positive integers $1 \leq s_1 < s_2 < \cdots < s_k = n$.

2 Background and notation

We denote by $\left(\!\binom{n}{k}\!\right) := \binom{n+k-1}{k}$ the **multiset binomial coefficient**, counting the number of multisets of size k that can be formed from a set of n distinct elements.

2.1 y-Extended permutations

Definition 2.1 ($\mathbf{y}$-Extended permutations) Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ be a vector of nonnegative integers. We associate to $\mathbf{s}$ the multiset $M_\mathbf{s} := \{1^{s_1}, 2^{s_2}, \ldots, n^{s_n}\}$; that is, the multiset containing exactly s_i copies of i, for each $i = 1, \ldots, n$. We denote by $\mathfrak{S}_\mathbf{s}$ the set of all permutations of the multiset $M_\mathbf{s}$; that is, all words of length $\sum_{i=1}^n s_i$ formed using the elements of $M_\mathbf{s}$ respecting multiplicities.

Let $\mathbf{y} = (y_1, \ldots, y_n)$ be a vector of non-negative integers. Define a collection of sets $Y_1, \ldots, Y_n$ as follows: $Y_i = \left\{\left(\sum_{j=1}^{i-1} y_j\right) + 1, \left(\sum_{j=1}^{i-1} y_j\right) + 2, \cdots, \sum_{j=1}^{i} y_j\right\}$, let $\pi \in \mathfrak{S}_\mathbf{s}$ be a multiset permutation, and let $r = \sum_{i=1}^n s_i$. We define a $\mathbf{y}$-*extended permutation* associated to π as any word $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_r)$ where $\sigma_j \in Y_{\pi_j}$ for each $j \in [r]$. We define $\mathfrak{S}_{\mathbf{s},\mathbf{y}}$ as the set of all $\mathbf{y}$-*extended permutations* associated with the multiset permutations in $\mathfrak{S}_\mathbf{s}$

2.2 Parking functions and y-parking functions

The next generalization of the classical parking functions was defined in [14, Sec. 5].

Definition 2.2 ($\mathbf{y}$-parking functions) Given $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{N}^n$, an $\mathbf{y}$-parking function is a sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ of positive integers such that their increasing rearrangement $b_1 \leq \cdots \leq b_n$ satisfies $b_i \leq y_1 + \cdots + y_i$. We denote this set by $\mathrm{PF}_n(\mathbf{y})$. The classical parking functions are the case $\mathbf{y} = \mathbf{1}$.

The $\mathbf{y}$-parking functions can be obtained from classical parking functions as follows in a process that we call *extension*. Given a parking function $\pi = (\pi_1, \ldots, \pi_n)$ in $\mathrm{PF}_n(\mathbf{1})$, if we replace each i in π by an integer in $\{y_1 + \cdots + y_{i-1} + 1, \ldots, y_1 + \cdots + y_i\}$, we obtain a $\mathbf{y}$-parking function. Conversely, every $\mathbf{y}$-parking function can be uniquely obtained this way. This gives the following identity.

Proposition 2.3 ([14, Thm. 11]) *For non-negative integers $\mathbf{y} = (y_1, \ldots, y_n)$ we have that*

$$\#\operatorname{PF}_n(\mathbf{y}) = \sum_{\pi \in \operatorname{PF}_n(\mathbf{1})} y_{\pi_1} \cdots y_{\pi_n}.$$

Example 2.4 For $\mathbf{y} = (2, 1)$, the $\mathbf{y}$-parking functions are

$$\{(1,1),(1,2),(2,1),(2,2)\} \cup \{(1,3),(2,3)\} \cup \{(3,1),(3,2)\},$$

which are obtained by extending the classical parking functions $(1,1)$, $(1,2)$, $(2,1)$. Moreover, $\#\operatorname{PF}_n(\mathbf{y}) = y_1^2 + y_1y_2 + y_2y_1$.

Let $\mathcal{I}_n$ be the set of weak compositions of n that are $\geq \mathbf{1}$ in dominance order.

$$\mathcal{I}_n := \left\{ (s_1, \ldots, s_n) \mid \sum_i s_i = n, \sum_{i=1}^{i} s_i \geq i, i = 1, \ldots, n-1 \right\}.$$

Each $\mathbf{s} \in \mathcal{I}_n$ encodes a non-decreasing sequence $b_1 \leq \cdots \leq b_n$ such that s_i counts the occurrences of the value i. The dominance condition guarantees that $b_i \leq i$ for all i, which is precisely the restriction that defines classical parking functions. Since each $\mathbf{y}$-parking function is a $\mathbf{y}$-extended permutation for a classical parking function, the following disjoint decomposition is obtained:

$$\operatorname{PF}_n(\mathbf{y}) = \bigsqcup_{\mathbf{s} \in \mathcal{I}_n} \mathfrak{S}_{\mathbf{s},\mathbf{y}} \tag{2.1}$$

On the other hand, Pitman and Stanley used $\mathcal{I}_n$ to give the following formulas for the Ehrhart polynomial and volume of $\Pi_n(\mathbf{y})$.

Corollary 2.5 (Pitman–Stanley [14, Eq. (33)]) *Let $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{Z}^n_{\geq 0}$, then*

$$\operatorname{ehr}_{\Pi_n(\mathbf{y})}(t) = \sum_{\mathbf{s} \in \mathcal{I}_n} \left(\!\!\binom{t \cdot y_1 + 1}{s_1}\!\!\right) \left(\!\!\binom{t \cdot y_2}{s_2}\!\!\right) \cdots \left(\!\!\binom{t \cdot y_n}{s_n}\!\!\right). \tag{2.2}$$

Example 2.6 For $n = 2$, we have $\operatorname{ehr}_{\Pi_2(y_1,y_2)}(t) = \left(\!\binom{ty_1+1}{1}\!\right)\left(\!\binom{ty_2}{1}\!\right) + \left(\!\binom{ty_1+1}{2}\!\right) = \frac{1}{2}(y_1^2 + 2y_1y_2)t^2 + (\frac{3}{2}y_1 + y_2)t + 1$. For $(y_1, y_2) = (2, 1)$, this gives $\operatorname{ehr}_{\Pi_2(2,1)}(t) = 4t^2 + 4t + 1$.

For special choices of $\mathbf{y}$, e.g., $\mathbf{y} = (a, b, b, \ldots, b)$, it is possible to write $\operatorname{ehr}_{\Pi_n(\mathbf{y}}(t)$ as a product [14, Thm. 13]. For general values of $\mathbf{y}$ one can write a determinantal formula for $\operatorname{ehr}_{\Pi_n(\mathbf{y}}(t)$ (see [14]).

For $\mathbf{y} \in \mathbb{Z}^n_{\geq 0}$ we have that $\operatorname{vol}\Pi_n(\mathbf{y}) = \#\operatorname{PF}_n(\mathbf{y})$. For certain values of $\mathbf{y}$, there are nice formulas for $\#\operatorname{PF}_n(\mathbf{y})$ (see[16]). For example, $|\operatorname{PF}_n(\mathbf{1})| = (n+1)^{n-1}$, the number of classical parking functions.

3 Lucky cars and magic positivity

Consider for each $\mathbf{y} \in \mathbb{Z}^n_{\geq 0}$ the polynomial $f_{\mathrm{PF}_n(\mathbf{y})}(t)$ defined as follows:

$$f_{\mathrm{PF}_n(\mathbf{y})}(t) := \sum_{\mathbf{s}\in\mathcal{I}_n} \binom{n}{s_1,\ldots,s_n} \prod_{l=1}^{s_1} ((y_1 - l)t + l) \cdot \prod_{i=2}^{n} \prod_{l=1}^{s_i} ((y_i - (l-1))t + (l-1)). \quad (3.1)$$

A computation shows that this is a transformation of the polynomial $\mathrm{ehr}_{\Pi_n(\mathbf{y})}(t)$. Written precisely:

$$\frac{1}{n!}(1+t)^n f_{\mathrm{PF}_n(\mathbf{y})}\left(\frac{t}{1+t}\right) = \mathrm{ehr}_{\Pi_n(\mathbf{y})}(t).$$

Moreover, the coefficients of the polynomial $f_{\mathrm{PF}_n(\mathbf{y})}(t)$ are exactly the coefficients of $n! \cdot \mathrm{ehr}_{\Pi_n(\mathbf{y})}(t)$ in the basis $\{t^i(t+1)^{n-i}\}_{i=0}^n$. We are therefore interested in proving that $f_{\mathrm{PF}_n(\mathbf{y})}(t)$ has nonnegative coefficients whenever $\mathbf{y} \in \mathbb{Z}^n_{>0}$.

Let $\mathbf{s}, \mathbf{y} \in \mathbb{Z}^n_{\geq 0}$ be vectors of non-negative integers such that $n = \sum_{i=1}^n s_i$. Consider the following polynomial:

$$f_{\mathbf{s},\mathbf{y}}(t) := \binom{n}{s_1,\ldots,s_n} \prod_{k=1}^{s_1} ((y_1 - k)t + k) \cdot \prod_{i=2}^{n} \prod_{k=1}^{s_i} ((y_i - (k-1))t + (k-1)). \quad (3.2)$$

By combining equations (3.1) and (3.2) we can write:

$$f_{\mathrm{PF}_n(\mathbf{y})}(t) = \sum_{\mathbf{s}\in\mathcal{I}_n} f_{\mathbf{s},\mathbf{y}}(t). \quad (3.3)$$

Definition 3.1 (Algebraic contribution) Let $\mathbf{s}$ and $\mathbf{y}$ be vectors of non-negative integers, let $\pi \in \mathfrak{S}_{\mathbf{s}}$ be a multiset permutation and let $L \subseteq [n]$. Consider the following notations:

$$l_{\pi,j}(\mathbf{y}) := \begin{cases} y_{\pi_j} - (\mu_{\pi,j} - 1) & \text{if } \pi_j > 1 \\ y_1 - \mu_{\pi,j} & \text{if } \pi_j = 1 \end{cases} \quad \text{and} \quad o_{\pi,j} := \begin{cases} \mu_{\pi,j} - 1 & \text{if } \pi_j > 1 \\ \mu_{\pi,j} & \text{if } \pi_j = 1 \end{cases}.$$

where $\mu_{\pi,j}$ for all $j \in [n]$ is the partial multiplicity of elements of $\pi \in \mathfrak{S}_{\mathbf{s}}$, formally given $j \in [n]$, define the partial multiplicity $\mu_{\pi,j} := \#\{k \leq j : \pi_k = \pi_j\}$. We define the *algebraic contribution of order α and β*, respectively, by

$$\mathrm{alg}^{(\alpha)}_{\mathbf{y}}(\pi, L) = \prod_{j\in L} y_{\pi_j} \cdot \prod_{j\in([n]\setminus L)} o_{\pi,j} \quad \text{and} \quad \mathrm{alg}^{(\beta)}_{\mathbf{y}}(\pi, L) = \prod_{j\in L} l_{\pi,j}(\mathbf{y}) \cdot \prod_{j\in([n]\setminus L)} o_{\pi,j}$$

These quantities satisfy an inclusion-exclusion relationship between those of order α and β for a fixed multiset permutation π.

Lemma 3.2 *Let $\mathbf{s}$ and $\mathbf{y}$ be vectors of non-negative integers, let $\pi \in \mathfrak{S}_{\mathbf{s}}$ be a multiset permutation and let $L \subseteq [n]$, then the algebraic contribution of order α and β satisfies the following:*

$$\mathrm{alg}^{(\alpha)}_{\mathbf{y}}(\pi, L) = \sum_{S\subseteq L} \mathrm{alg}^{(\beta)}_{\mathbf{y}}(\pi, S) \quad \textit{and} \quad \mathrm{alg}^{(\beta)}_{\mathbf{y}}(\pi, L) = \sum_{S\subseteq L} (-1)^{|L|-|S|} \mathrm{alg}^{(\alpha)}_{\mathbf{y}}(\pi, S)$$

The algebraic contributions are conveniently defined as a refinement of the polynomial $f_{\mathbf{s},\mathbf{y}}$ by multiset permutations $\pi \in \mathfrak{S}_{\mathbf{s}}$ and subsets $L \subseteq [n]$

Proposition 3.3 *Let $\mathbf{s} = (s_1, \dots, s_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ be vectors of non-negative integers and $n = \sum_{i=1}^{n} s_i$. The polynomial $f_{\mathbf{s},\mathbf{y}}(t)$ can be refined in terms of subsets $L \subseteq [n]$ and multiset permutations $\pi \in \mathfrak{S}_{\mathbf{s}}$*

$$f_{\mathbf{s},\mathbf{y}}(t) = \sum_{L \subseteq [n]} \sum_{\pi \in \mathfrak{S}_{\mathbf{s}}} \mathrm{alg}_{\mathbf{y}}^{(\beta)}(\pi, L) t^{|L|}$$

Remark 3.4 By combining Proposition 3.3 with the formulas in equations (2.1) and (3.3), we can write a formula for $f_{\mathrm{PF}_n(\mathbf{y})}(t)$ in terms of algebraic contributions indexed by subsets $L \subseteq [n]$ and classical parking functions $\pi \in \mathrm{PF}_n$:

$$f_{\mathrm{PF}_n(\mathbf{y})}(t) = \sum_{\mathbf{s} \in \mathcal{I}_n} f_{\mathbf{s},\mathbf{y}}(t) = \sum_{\pi \in \mathrm{PF}_n} \sum_{L \subseteq [r]} \mathrm{alg}_{\mathbf{y}}^{(\beta)}(\pi, L) t^{|L|} \tag{3.4}$$

We consider a generalization of the classical "parking protocol", imposing now that the first parking spot is never available. Let $w \in [m]^n$ be a word of length n over an alphabet of m letters. We interpret $w = (w_1, \dots, w_n)$ as the parking preferences of n cars with m available spots, each j-th car tries to park in its preferred spot w_j, but if the preferred spot is 1, or if it is already occupied, the car moves to the next available spot. If it does not find a free spot, it is considered to have failed.

We denote by $\mathrm{out}_m(w)$ the output vector, where $\mathrm{out}_m(w)_j$ represents the spot where the j-th car was parked, or $\bullet$ if it does not park in any spot.

- The lucky set: $\mathrm{Lucky}_m(w) = \{j \in [n] \mid \mathrm{out}_m(w)_j = w_j\}$.
- The number of lucky cars: $\mathrm{lucky}_m(w) = \#\mathrm{Lucky}_m(w)$.

Example 3.5 shows the output of this parking protocol.

Example 3.5 Consider the word $w = (6, 3, 1, 6, 2, 7, 6)$ with $m = 8$ and $n = 7$, then the output under the parking protocol where the first spot is not really available is: $\mathrm{out}_m(w) = (6, 3, 2, 7, 4, 8, \bullet)$. See Figure 1 to illustrate this process.

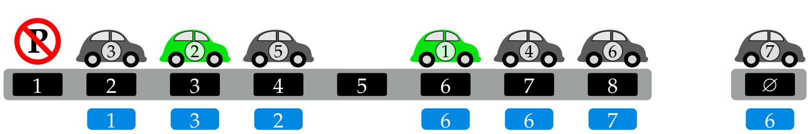

Figure 1: Example of parking protocol where the first spot is not available, the number inside the car represents its index, the spot number inside the black box, and its initial precedence inside the blue box.

For brevity, we will refer to the *parking protocol where the first spot is not available* simply as *parking protocol* throughout the remainder of this extended abstract.

Proposition 3.6 *Let $\mathbf{y} = (y_1, \dots, y_n)$ be vector of positive integers. Let $\sigma \in \mathrm{PF}_n(\mathbf{y})$ be a $\mathbf{y}$-parking function, if σ_j is the preference spot for j-th car, let $m := \sum_{i=1}^{k} y_i$ be the number of spots, then either all n cars park successfully, or the only car that cannot park is the last one in this parking protocol.*

We now re-state Theorem 1.1 in a more precise language, then we discuss an example, and finally we outline the sequence of lemmas and ideas used for its proof.

Theorem 3.7 *Let $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{Z}^n_{>0}$. Then the coefficients of the expansion $\mathrm{ehr}_{\Pi_n(\mathbf{y})}(t) = \sum_{j=1}^{n} c_j \cdot t^j (t+1)^{n-j}$ admit the following combinatorial interpretation: $d_j := n! \cdot c_j$ is the number of $\mathbf{y}$-parking functions having exactly j lucky cars under this parking protocol.*

Example 3.8 For $\mathbf{y} = (2, 1)$ and the $\mathbf{y}$-parking functions of Ex. 2.4, the parking protocols in Figure 2 give that $d_0 = 2$, $d_1 = 4$ and $d_2 = 2$. Indeed, consistently with Ex. 2.6,

$$2 \cdot \mathrm{ehr}_{\Pi_2(2,1)}(t) = 2t^0(1+t)^2 + 4t^1(1+t)^1 + 2t^2(1+t)^0.$$

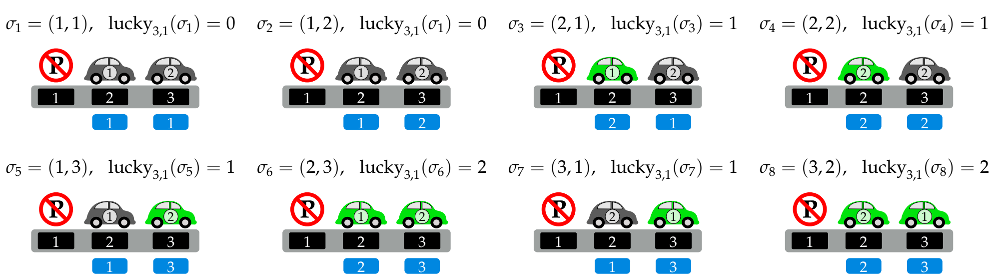

Figure 2: The parking protocol is illustrated for each $\mathbf{y}$-parking function $\sigma_i \in \mathrm{PF}_2(2,1)$. Here the green cars are lucky cars.

To prove Theorem 3.7, it suffices to prove the following result, which establishes a crucial relationship between the lucky statistic and algebraic contributions:

Theorem 3.9 *Let $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{Z}^n_{\geq 1}$, and let $L \subseteq [n]$ be a subset of car indices. Then the following equalities hold:*

$$\mathrm{lucky}^{(\beta)}_{\mathrm{PF}_n(\mathbf{y})}(L) = \sum_{\pi \in \mathrm{PF}_n} \mathrm{alg}^{(\beta)}_{\mathbf{y}}(\pi, L) \qquad \textit{and} \qquad \mathrm{lucky}^{(\alpha)}_{\mathrm{PF}_n(\mathbf{y})}(L) = \sum_{\pi \in \mathrm{PF}_n} \mathrm{alg}^{(\alpha)}_{\mathbf{y}}(\pi, L).$$

Where $\mathrm{lucky}^{(\beta)}_{\mathrm{PF}_n(\mathbf{y})}(L)$ denotes the number of $\mathbf{y}$-parking functions whose lucky set is equal to L, and $\mathrm{lucky}^{(\alpha)}_{\mathrm{PF}_n(\mathbf{y})}(L)$ denotes the number of $\mathbf{y}$-parking functions whose lucky set is contained in L in this parking protocol (see Table 1).

L	$\text{lucky}^{(\beta)}_{\text{PF}_n(\mathbf{y})}(L)$	$\vert_{y_1=2,y_2=1}$	$\text{lucky}^{(\alpha)}_{\text{PF}_n(\mathbf{y})}(L)$	$\vert_{y_1=2,y_2=1}$
$\varnothing$	2	2	2	2
$\{1\}$	$2y_1 + y_2 - 2$	3	$2y_1 + y_2$	5
$\{2\}$	$y_1 + y_2 - 2$	1	$y_1 + y_2$	3
$\{1,2\}$	$y_1^2 + 2y_1y_2 - 3y_1 - 2y_2 + 2$	2	$y_1^2 + 2y_1y_2$	8

Table 1: Number of (y_1, y_2)-parking functions with lucky set equal to (contained in) L.

Given that α and β are linked via inclusion-exclusion, it suffices to prove one of the two formulas. We focus on the second, because it involves nonnegative quantities and thus bypasses the necessity of handling cancellations in the summation.

We can interpret the expression of $\text{alg}^{(\alpha)}$ by a sequence of choices: for each $j \in [n]$, we choose y_{π_j} or $o_{\pi,j}$ different options, depending on whether $j \in L$. This motivates the following.

Algorithm 1 Algorithmic counting according to the lucky set

Input: $\mathbf{s}, \mathbf{y} \in \mathbb{Z}^n_{\geq 0}$, $\pi \in \mathfrak{S}_{\mathbf{s}}$, $L \subseteq [n]$, where $n = \sum_i s_i$ and $m = \sum_i y_i$.

Initialization: set $O_1^{(0)} = \{1\}$ and $O_i^{(0)} = \varnothing$ for $i \geq 2$.

Loop: for $j = 1, \ldots, n$:

1. If $j \in L$, choose $\sigma_j \in Y_{\pi_j}$ (a possible free preference spot).
2. If $j \notin L$, choose $\sigma_j \in O_{\pi_j}^{(j-1)}$ (an already occupied spot for previous cars with index $k < j$ such that $\pi_k = \pi_j$).
3. Apply the parking protocol to the prefix $\sigma^{(j)} = (\sigma_1, \ldots, \sigma_j)$ and define

$$O_{\pi_j}^{(j)} := O_{\pi_j}^{(j-1)} \cup (\{\text{out}_m(\sigma^{(j)})_j\} \setminus \{\bullet\}), \qquad O_i^{(j)} := O_i^{(j-1)} \text{ for } i \neq \pi_j.$$

Output: the resulting word $\sigma \in [m]^r$. The construction ensures that $\text{Lucky}_m(\sigma) \subseteq L$.

Remark 3.10 Assume that all choices at each step $j < n$ produce a prefix $\sigma^{(j)}$, where σ_k represents the preference of car $k \leq j$. Assume also that all cars successfully park (for all $k < j$ we have that $\text{out}_m(\sigma^{(j)})_k \neq \bullet$). Then, if $j \notin L$ we have $\#O_{\pi_j}^{(j-1)} = o_{\pi,j}$ and thus the number of distinct outputs of the algorithm for a given $\pi \in \mathfrak{S}_{\mathbf{s}}$ is precisely $\text{alg}^{(\alpha)}_{\mathbf{y}}(\pi, L)$.

Algorithm 1 defines a partition of the $\mathbf{y}$-parking function $\text{PF}_n(\mathbf{y})$ such that the lucky set is contained in L.

Proposition 3.11 *Let $\sigma \in \text{PF}_n(\mathbf{y})$ such that $\text{Lucky}_m(\sigma) \subseteq L$, and let $\mathbf{y} \in \mathbb{Z}^n_{\geq 0}$. Then there exists a unique multiset permutation π such that the algorithm 1, when applied to π , produces σ . Moreover two different multiset permutations produce different* $\mathbf{y}$ *- parking functions.*

The input for Algorithm 1 is a multiset permutation $\pi \in \mathfrak{S}_{\mathbf{s}}$, where $\mathbf{s}$ is a weak composition of n specifying the multiplicities of π.

We next define the *composition tracking process* during the execution of the algorithm. Let $\mathbf{s}^{(0)} := \mathbf{s}$. Whenever $\sigma_j \in Y_i$ with $i \neq \pi_j$, set $\mathbf{s}^{(j)}_{\pi_j} = \mathbf{s}^{(j-1)}_{\pi_j} - 1, \qquad \mathbf{s}^{(j)}_i = \mathbf{s}^{(j-1)}_i + 1$, and keep $\mathbf{s}^{(j)}_k = \mathbf{s}^{(j-1)}_k$ for $k \notin \{i, \pi_j\}$. As a result, at each step j, the prefix $\sigma^{(j)} := (\sigma_1, \ldots, \sigma_j)$ is the prefix of some $\mathbf{y}$-extended permutation in $\mathfrak{S}_{\mathbf{s}^{(j)},\mathbf{y}}$. Thus, at the end of the algorithm, the final word σ is in $\mathfrak{S}_{\mathbf{s}^{(n)},\mathbf{y}}$.

Remark 3.12 Assume that at some step j, during the execution of Algorithm 1, it happens that $\sigma_j \in Y_i$ with $i \neq \pi_j$. Then it must be the case that the j-th car is not lucky.

According to the definition of the parking protocol, if a car k does not park in its preferred spot σ_k, then $\mathrm{out}_m(\sigma^{(j-1)})_k > \sigma_k$, because cars only move to the right when their preferred spot is occupied.

Therefore, we have that $\pi_j < i$. This implies that the update $\mathbf{s}^{(j)}$ transfers units from a $s^{(j-1)}_{\pi_j}$ with smaller index π_j to a $s^{(j-1)}_i$ with larger index i.

Lemma 3.13 *Let $\mathbf{y} \in \mathbb{Z}^n_{>0}$. Suppose that during the execution of the Algorithm 1, the composition vector $\mathbf{s}^{(j)}$ is updated at step j. Then if $\mathbf{s}^{(j-1)} \in \mathcal{I}_n$ we have that $\mathbf{s}^{(j)}$ also belong to $\mathcal{I}_n$.*

Remark 3.14 The condition $\mathbf{y} \in \mathbb{Z}^n_{>0}$ is crucial for the dominance condition of $\mathcal{I}_n$ to be satisfied when updating the vector $\mathbf{s}^{(j)}$. Magic positivity may fail if $\mathbf{y}$ has zero entries.

Example 3.15 Consider $n = 5$, $\mathbf{y}_1 = (1,0,0,0,6)$, $L = \varnothing$, and the parking function $\pi = (1,1,1,1,1) \in \mathfrak{S}_{(5,0,0,0,0)}$. If we apply the algorithm and always select the maximum value of $O^{(j)}_i$ for each step $j = 1, \ldots, n$ we will obtain the word $\sigma = (1,2,3,4,5)$, This word belongs to $\mathfrak{S}_{\mathbf{s}_1,\mathbf{y}_1}$, where $\mathbf{s}_1 = (1,0,0,0,4) \notin \mathcal{I}_5$.

Conversely, if $\mathbf{y}_2 = (1,1,1,1,1)$, then the output of the algorithm is the same σ, but in this case $\sigma \in \mathfrak{S}_{\mathbf{s}_2,\mathbf{y}_2}$, where $\mathbf{s}_2 = (1,1,1,1,1) \in \mathcal{I}_5$.

We are now ready for a proof of Theorem 3.9 using the preceding developments.

Sketch of proof of Theorem 3.9. Given $\sigma \in \mathrm{PF}_n(\mathbf{y})$ with $\mathrm{Lucky}_m(\sigma) \subseteq L$, Proposition 3.11 and Remark 3.12 guarantee the existence of $\pi \in \mathrm{PF}_n$ generating σ via the algorithm. Each decision made corresponds to a factor in the multiplicative expression of $\mathrm{alg}^{(\alpha)}_{\mathbf{y}}(\pi, L)$, so every such σ is counted on the right-hand side.

Conversely, for each $\pi \in \mathrm{PF}_n$, Lemma 3.13 ensures that all intermediate compositions remain in $\mathcal{I}_n$, and the algorithm produces only $\mathbf{y}$-parking functions with $\mathrm{Lucky}_m(\sigma) \subseteq L$. Proposition 3.6 and Remark 3.10 then confirm that $\mathrm{alg}^{(\alpha)}_{\mathbf{y}}(\pi, L)$ enumerates precisely these outputs. □

4 Ascents of y-extended permutations and h^*-polynomials

We now present the following enumerative identity, which expresses the generating function of $\mathbf{y}$-extended permutations by ascent statistics

Definition 4.1 Given a $\mathbf{y}$-extended permutation $\sigma \in \mathfrak{S}_{\mathbf{s},\mathbf{y}}$, the set of *modified ascents*, denoted by $\mathrm{Asc}(\sigma) := \{j \in [0, r-1] : \sigma_j < \sigma_{j+1} \text{ and } \sigma_0 := 1\}$, where $r = \sum_{i=1}^{n} s_i$ and the number of modified ascents as $\mathrm{asc}(\sigma) := \#\mathrm{Asc}(\sigma)$.

Theorem 4.2 *Let* $\mathbf{s} = (s_1, \ldots, s_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$ *be vectors of non-negative integers with* $y_1 \geq 1$*, and let* $r := \sum_{i=1}^{n} s_i$ *be the total size of the multiset associated to* $\mathbf{s}$*. Then:*

$$\sum_{m=0}^{\infty} \left(\!\!\binom{m \cdot y_1 + 1}{s_1}\!\!\right) \prod_{i=2}^{n} \left(\!\!\binom{m \cdot y_i}{s_i}\!\!\right) z^m = \frac{\sum_{\sigma \in \mathfrak{S}_{s,y}} z^{\mathrm{asc}(\sigma)}}{(1-z)^{r+1}}. \tag{4.1}$$

Proof sketch. We construct a bijection between multiset-valued functions and $\mathbf{y}$-extended permutations using ascent structures. For each $\sigma \in \mathfrak{S}_{\mathbf{s},\mathbf{y}}$, we define a family of functions $f = (A_1, \ldots, A_n)$, where $A_1 \subseteq [m] \times [y_1] \cup \{(0,1)\}$ and $A_i \subseteq [m] \times [y_i]$, each A_i is a multiset of size s_i. Each such f determines a unique evaluation $f(\sigma) \in [0, m]^r$ obtained by reading the tuples in order prescribed by σ. The key notion of σ-compatibility requires that $f(\sigma)$ be weakly increasing, with strict inequalities at each ascent of σ.

Using this compatibility condition, we show that every f corresponds to a unique $\mathbf{y}$-extended permutation $\sigma \in \mathfrak{S}_{\mathbf{s},\mathbf{y}}$. This implies that each summand in eq. (4.1) equals $\sum_{\sigma \in \mathfrak{S}_{\mathbf{s},\mathbf{y}}} F_m^{\mathbf{s},\mathbf{y}}(\sigma)$ where $F_m^{\mathbf{s},\mathbf{y}}(\sigma)$ counts σ-compatible functions. A direct analysis of the weakly increasing sequences $f(\sigma)$ shows that $F_m^{\mathbf{s},\mathbf{y}}(\sigma) = \left(\!\binom{m-\mathrm{asc}(\sigma)+1}{r}\!\right)$, reducing the enumeration to the classical generating function for multiset binomial coefficients. $\square$

We are now ready to outline a proof for Theorem 1.3.

Proof sketch of Theorem 1.3. By Corollary 2.5 we have a formula for $\mathrm{ehr}_{\Pi_n(\mathbf{y})}(t)$ as a sum over compositions $\mathbf{s} \in \mathcal{I}_n$ with binomial multiset coefficients. This coincides with the generators of the $\mathbf{y}$-extended permutation series in Theorem 4.2.

Since the set of $\mathbf{y}$-parking functions decomposes as a disjoint union of $\mathbf{y}$-extended permutations (2.1), and since Ehrhart's generating series determines the h^*-polynomial, it follows that: $h^*_{\Pi_n(\mathbf{y})}(z) = \sum_{\sigma \in \mathrm{PF}_n(\mathbf{y})} z^{\mathrm{asc}(\sigma)}$. $\square$

Example 4.3 For $\mathbf{y} = (2,1)$ and the $\mathbf{y}$-parking functions of Example 2.4, we have that

$$\begin{aligned} h^*_{\Pi_2(2,1)}(z) &= z^{\mathrm{asc}(11)} + z^{\mathrm{asc}(12)} + z^{\mathrm{asc}(21)} + z^{\mathrm{asc}(22)} + z^{\mathrm{asc}(13)} + z^{\mathrm{asc}(23)} + z^{\mathrm{asc}(31)} + z^{\mathrm{asc}(32)} \\ &= z^0 + z^1 + z^1 + z^1 + z^1 + z^2 + z^1 + z^1 = 1 + 6z^1 + z^2. \end{aligned}$$

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