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A modular (q,t)-Nekrasov–Okounkov formula and wreath Macdonald polynomials

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Abstract. We prove a modular analogue of the (q,t)-Nekrasov–Okounkov formula for all integers $r \geqslant 3$, originally conjectured by Walsh and Warnaar. Our proof is based on an analogue of the Carlsson–Nekrasov–Okounkov vertex operator for wreath Macdonald polynomials.

Keywords: Nekrasov–Okounkov formulas, wreath Macdonald polynomials, ext operator

1 Introduction

The celebrated Nekrasov–Okounkov formula gives a beautiful expansion for an arbitrary complex power of the Dedekind eta function. It was originally discovered by Nekrasov and Okounkov in their proof of Nekrasov's conjecture [12, Equation (6.12)], and also found independently around the same time by Westbury as a hook-length formula for the D'Arcais polynomials [18]. To state the formula, let λ denote an integer partition, which we identify with its Young diagram. For a cell $\square \in \lambda$ its hook length is denoted by $h_\lambda(\square)$. Full definitions may be found in Section 2. Then the Nekrasov–Okounkov formula states

$$\sum_{\lambda} T^{|\lambda|} \prod_{\square\in\lambda}\left(1-\frac{z}{h_\lambda^2(\square)}\right)=\prod_{k\geqslant 1}(1-T^k)^{z-1}. \tag{1.1}$$

Note that for $z=2$ the right-hand side is the Dedekind eta function when multiplied by a factor of $T^{1/24}$. For convergence one may either take $0<T<1$ and $z\in\mathbb{C}$ or $T\in\mathbb{C}$ with $|T|<1$ and $z\in\mathbb{R}$. Alternatively, it may be viewed as an identity in $\mathbb{Q}[z][[T]]$.

There are many generalizations, analogues and refinements of (1.1) in the literature. One important generalization is the (q,t)-Nekrasov–Okounkov formula, proved independently by Rains and Warnaar [14, Theorem 1.3] and Carlsson and Rodriguez Villegas

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[5, Theorem 1.0.2]. In order to write this down, let $a_\lambda(\square)$ and $l_\lambda(\square)$ denote the arm and leg length of a cell $\square \in \lambda$ respectively, so that $h_\lambda(\square) = a_\lambda(\square) + l_\lambda(\square) + 1$. Further define

$$(a_1, \dots, a_n; q_1, \dots, q_m)_\infty := \prod_{i=1}^{n} \prod_{j_1,\dots,j_m \geqslant 0} \left(1 - a_i q_1^{j_1} \cdots q_m^{j_m}\right),$$

the infinite multiple q-Pochhammer symbol. The (q,t)-Nekrasov–Okounkov formula is

$$\sum_\lambda T^{|\lambda|} \prod_{\square \in \lambda} \frac{(1 - uq^{a_\lambda(\square)+1} t^{l_\lambda(\square)})(1 - u^{-1} q^{a_\lambda(\square)} t^{l_\lambda(\square)+1})}{(1 - q^{a_\lambda(\square)+1} t^{l_\lambda(\square)})(1 - q^{a_\lambda(\square)} t^{l_\lambda(\square)+1})} = \frac{(uqT, u^{-1}tT; q, t, T)_\infty}{(T, tT; q, t, T)_\infty}. \quad (1.2)$$

Replacing $u = q^z$ and taking the limit $q \to 1$ recovers (1.1). In a different direction, specializing $u = (t/q)^{1/2}$ and then replacing $(q,t) \mapsto (q^{-2}, t^2)$ one obtains an identity conjectured by Hausel and Rodriguez Villegas [10, Conjecture 4.3.2] (our (q,t) is their (w,z)). This in turn is equivalent to the genus-one case of their much more general conjecture [10, Conjecture 4.2.1]. This remarkable conjecture gives an explicit expression for the mixed Hodge polynomial of the twisted character variety of a closed Riemann surface of genus g in terms of a plethystic logarithm involving a generalized hook-product.

It was the $g = 1$ case of the Hausel–Rodriguez Villegas conjecture which motivated the two proofs of the (q,t)-Nekrasov–Okounkov formula, both of which employ the theory of Macdonald polynomials. Rains and Warnaar derive (1.2) by evaluating a certain specialized sum of ordinary (that is, non-modified) Macdonald polynomials in two ways. On the other hand, Carlsson and Rodriguez Villegas derive the identity by way of the Carlsson–Nekrasov–Okounkov vertex operator, originally introduced in [3].

Another important generalization of (1.1) is Han's modular version wherein the product in the summand is replaced by a product over boxes with hook length divisible by r for a positive integer r [8, Theorem 1.4]. The mechanism behind this identity is the core-quotient construction for integer partitions, a family of bijections for each integer $r \geqslant 2$ decomposing a partition into its r-core and r-quotient, also known as the Littlewood decomposition; see Subsection 2.2. Han and Ji further used this construction to derive their "multiplication-addition theorem" [9] which allows for modular analogues of a vast swathe of identities involving hook lengths. While modular analogues of identities involving hook lengths are abundant, the same is not true for formulas involving arm and leg lengths such as (1.2). Nevertheless, Walsh and Warnaar conjectured a modular analogue of (1.2) based on similar identities in the case $q = 0$ or $t = 0$ [16, Conjecture 8.1]. Our first main result is a proof of their conjecture for all integers $r \geqslant 3$.

Theorem 1 ([1, Theorem 1.2])**.** *Let $r \geqslant 3$ be an integer. Then for any r-core α, we have*

$$\sum_{\substack{\lambda \\ \mathrm{core}(\lambda)=\alpha}} T^{|\mathrm{quot}(\lambda)|} \prod_{\substack{\square \in \lambda \\ h_\lambda(\square) \equiv 0 \bmod r}} \frac{(1 - uq^{a_\lambda(\square)+1} t^{l_\lambda(\square)})(1 - u^{-1} q^{a_\lambda(\square)} t^{l_\lambda(\square)+1})}{(1 - q^{a_\lambda(\square)+1} t^{l_\lambda(\square)})(1 - q^{a_\lambda(\square)} t^{l_\lambda(\square)+1})}$$

$$= \frac{1}{(T;T)_\infty^r} \prod_{i=1}^{r} \frac{(uq^i t^{r-i}T, u^{-1}q^{r-i}t^iT; q^r, t^r, T)_\infty}{(q^i t^{r-i}T, q^{r-i}t^iT; q^r, t^r, T)_\infty}. \quad (1.3)$$

Walsh and Warnaar conjectured this based on the $q = 0$ and $t = 0$ cases, combinatorial identities involving products over boxes with arm- and leg-length zero, respectively. They established those cases purely combinatorially using a variant of the core-quotient construction and an analogue of the "multiplication theorem" of Han and Ji [9]. However, it does not appear that there is such a nice combinatorial explanation for these modular variants at the (q,t)-level.

Our proof employs wreath Macdonald polynomials, a generalization of the Macdonald polynomials introduced by Haiman [7] which are intimately related with the core-quotient construction. The key tool is a wreath Macdonald polynomial analogue of the Carlsson–Nekrasov–Okounkov vertex operator, originally introduced in [3]. However, our construction is philosophically different to the one in that paper, being based on the wreath version of Tesler's identity, recently proved by Romero and the second author [15]. Beyond a proof of the Walsh–Warnaar conjecture, this allows for generalizations of the key results of [3], a new derivation of the quadratic norm evaluation for wreath Macdonald polynomials, and novel expressions for Pieri rules. Our only obstruction in the case $r = 2$ is that our work heavily relies on results from [17] involving the quantum toroidal algebra which are only proved for $r > 2$; the algebra for $r = 2$ is presented differently. Nevertheless, we believe our formulae should hold also for $r = 2$.

2 Partitions, cores, and quotients

2.1 Preliminaries

A partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ is a nonincreasing sequence of nonnegative integers with only finitely many nonzero entries:

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant 0.$$

The sum of the entries is well-defined; we denote it by $|\lambda| = \sum_{i \geqslant 1} \lambda_i$. The number of nonzero λ_i is called the length and is denoted by $\ell(\lambda)$. Later on, given a positive integer r, we will also work with r-tuples of partitions $\vec{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$. Entries in the tuple will be indexed using superscripts, while entries of an individual partition will be indexed by subscripts. We extend the notation $|\vec{\lambda}| = \sum_i |\lambda^{(i)}|$. Finally, we use $\geqslant$ to denote the dominance order on partitions of a fixed integer n: $\lambda \geqslant \mu$ if for all $k \geqslant 1$,

$$\lambda_1 + \cdots + \lambda_k \geqslant \mu_1 + \cdots + \mu_k.$$

For a partition λ, its Young diagram is the following set of lattice points:

$$\{(a,b) \in \mathbb{Z}^2 \mid 0 \leqslant a \leqslant \lambda_b - 1\}.$$

To each lattice point, we assign a box; we will draw the resulting arrangement of boxes following the French convention. Note that the bottom left corner has coordinate $(0,0)$. We will conflate a partition with its Young diagram and make statements like $(a,b) \in \lambda$.

3	2
5	4

2	1	0	
3	2	1	−1

Figure 1: The Young diagrams of the partitions $\lambda = (2,2)$ and $\mu = (4,3)$ with the mixed hook lengths $h_{\mu,\lambda}(\square)$ and $h_{\lambda,\mu}(\square)$ inscribed in each cell on the left and right, respectively, as defined in (2.1).

This visual representation clarifies some of the basic definitions concerning partitions. The *transpose* or *conjugate* partition ${}^t\lambda = ({}^t\lambda_1, {}^t\lambda_2, \ldots)$ is defined to be

$${}^t\lambda_i = \#\{1 \leqslant j \leqslant \ell(\lambda) \mid \lambda_j \geqslant i\}.$$

Its Young diagram is just the one for λ reflected across the diagonal line $y = x$. For a box $\square = (a,b) \in \mathbb{Z}^2_{\geqslant 0}$ (not necessarily in λ), we define its *arm-* and *leg-length* by

$$a_\lambda(\square) := \lambda_{b+1} - a - 1 \quad \text{and} \quad l_\lambda(\square) := {}^t\lambda_{a+1} - b - 1.$$

When $\square \in \lambda$, these quantities are easily visualized in the Young diagram: they count the number of boxes to the right of and above $\square$, respectively. For a pair of partitions λ, μ and a box $\square$ lying in one of them we define the *mixed hook-length* by

$$h_{\lambda,\mu}(\square) := a_\lambda(\square) + l_\mu(\square) + 1. \tag{2.1}$$

The ordinary hook-length of a box $\square \in \lambda$ is then $h_{\lambda,\lambda}(\square) = h_\lambda(\square)$. Finally, the *content* of a box $\square = (a,b)$ is $c_\square := b - a$. This is just the SW-to-NE diagonal that $\square$ lies on.

2.2 Cores and quotients

The core-quotient construction was originally introduced in the context of the modular representation theory of the symmetric group, but has long since taken on a life of its own as a fundamental combinatorial construction on integer partitions. We closely follow the notations of [17].

A *Maya diagram* is a function $m : \mathbb{Z} \longrightarrow \{\pm 1\}$ such that

$$m(n) = \begin{cases} -1 & n \gg 0, \\ 1 & n \ll 0. \end{cases}$$

To each such diagram we associate a visual representation using black and white beads. Namely, we consider a string of black and white beads indexed by $\mathbb{Z}$ where the bead at position n is black if $m(n) = 1$ and white if $m(n) = -1$, arranged in such a way that the index increases to the left. A notch is placed at $-\frac{1}{2}$ called the *central line*. The *charge* of a Maya diagram, $c(m)$, is defined by

$$c(m) = \#\{n \geqslant 0 \mid m(n) = 1\} - \#\{n < 0 \mid m(n) = -1\},$$

which is the number of black beads to the left of the central line minus the number of white beads to the right.

Integer partitions are in bijection with Maya diagrams of charge zero. Explicitly, the map is given by

$$\lambda \longmapsto m_\lambda(n) := \begin{cases} -1 & \text{if } n \in \{i - \lambda_i - 1 \mid i \geqslant 1\}, \\ 1 & \text{otherwise.} \end{cases}$$

To reconstruct the partition one counts the number of black beads to the left of each white bead, reading right-to-left, so that the ith bead from the right corresponds to λ_i. The empty partition corresponds to the *vacuum diagram*, which has only black beads to the right of the central line and only white beads to the left.

We are now ready to construct the core and the quotient of a partition, and so fix an integer $r \geqslant 2$. For each $0 \leqslant i \leqslant r-1$ we define the sub-Maya diagram $m_\lambda^{(i)}(n) := m_\lambda(i + nr)$ and denote its charge by c_i. Shifting the central line so that the resulting diagram has charge zero, we obtain a partition $\lambda^{(i)}$; see Figure 2. The r-quotient of λ is then

$$\operatorname{quot}(\lambda) := \big(\lambda^{(0)}, \dots, \lambda^{(r-1)}\big).$$

The r-core, denoted $\operatorname{core}(\lambda)$, is obtained by replacing each sub-diagram $m_\lambda^{(i)}$ by a vacuum diagram with the central line shifted so that it has charge c_i then reassembling the constituents into a single Maya diagram, which will have charge 0 since the sums of the charges c_i is zero.

An *r-ribbon* is a contiguous collection of r cells from the diagram of a partition λ such that: (i) the collection contains no 2×2 square of cells and (ii) the removal of these cells results in a valid Young diagram. For any partition, hooks of length r are in correspondence with both removable r-ribbons and pairs of black and white beads such that the white bead lies r positions to the right of the black bead. Swapping these two beads results in the removal of the r-ribbon. Thus another way to define $\operatorname{core}(\lambda)$ is as the unique partition obtained by removing r-ribbons until none remain. The quotient shows that this procedure is independent of the order in which ribbons are removed.

Theorem 2. *For every $r \geqslant 2$ the above defines a bijection*

$$\{\textit{partitions}\} \longrightarrow \{\textit{r-cores}\} \times \{\textit{partitions}\}^r$$

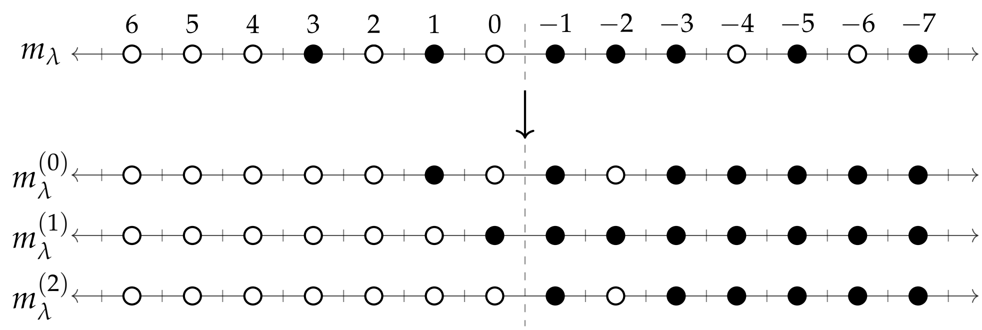

Figure 2: The Maya diagram corresponding to $\lambda = (6,5,2,1)$ (top) and the quotient diagrams of the same partition (bottom). We have that $\mathrm{core}(\lambda) = (1,1)$, and $(\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}) = ((2,1), \varnothing, 1)$, with charges $(0,1,-1)$.

$$\lambda \longmapsto \big(\mathrm{core}(\lambda), \mathrm{quot}(\lambda)\big)$$

such that $|\lambda| = |\mathrm{core}(\lambda)| + r|\mathrm{quot}(\lambda)|$. *Moreover,*

$$|\mathrm{quot}(\lambda)| = \#\{\square \in \lambda \mid h(\square) \equiv 0 \, mod(r)\}.$$

The final property of the quotient may be refined even further. Let $\mathcal{H}_\lambda$ denote the multiset of hook lengths in λ and $\mathcal{H}_\lambda^{(r)}$ the submultiset of those divisible by r. Then one may show that in fact

$$\mathcal{H}_\lambda^{(r)} = \bigcup_{i=0}^{r-1} r\mathcal{H}_{\lambda^{(i)}},$$

where rS for a multiset S means multiply all elements of S by r. This, combined with the $q = t$ case of (1.2), immediately implies the $q = t$ case of Theorem 1.

2.3 Nekrasov factors

The Nekrasov factor is a certain mixed hook polynomial occurring in mathematical physics, particularly in relation with the AGT correspondence. We work with a modular analogue defined by

$$N_{\lambda,\mu}(u) := \prod_{\substack{\square \in \lambda \\ h_{\mu,\lambda}(\square) \equiv 0 \bmod r}} \Big(1 - uq^{-a_\mu(\square)} t^{l_\lambda(\square)+1}\Big) \prod_{\substack{\square \in \mu \\ h_{\lambda,\mu}(\square) \equiv 0 \bmod r}} \Big(1 - uq^{a_\lambda(\square)+1} t^{-l_\mu(\square)}\Big), \tag{2.2}$$

where we recall that $h_{\lambda,\mu}(\square)$ denotes the mixed hook length (2.1). For a small example take $(r, \lambda, \mu) = (3, (2,2), (4,3))$ so that $N_{(2,2),(4,3)}(u) = (1-u)(1-ut/q^2)(1-uq^2/t)$. We remark that this factor may be expressed in terms of a plethystic exponential; see [1, Lemma 2.4].

3 Wreath Macdonald polynomials and the vertex operator

In this section we cover the basic properties of multi-symmetric functions and wreath Macdonald polynomials required to understand our results. The reader should consult [1, 15] for further details and clarifications.

3.1 Multi-symmetric functions and matrix plethysm

Let Λ be the ring of symmetric functions and further set $\Lambda_{q,t} := \Lambda \otimes \mathbb{C}(q,t)$. Of the usual distinguished elements of Λ (cf. [11]) we require only the power sums p_n, the complete homogeneous symmetric functions h_n, and the Schur functions s_λ. We heavily employ plethystic notation, viewing them as functions in an alphabet of variables X.

For our choice of r, we consider the rings $\Lambda^{\otimes r}$ and $\Lambda_{q,t}^{\otimes r}$. The tensorands will be indexed from 0 to $r-1$, which we view as elements of $\mathbb{Z}/r\mathbb{Z}$. To the tensorand indexed by $i \in \mathbb{Z}/r\mathbb{Z}$, we assign an alphabet $X^{(i)}$; we call i the *color* of the alphabet $X^{(i)}$. Given $f \in \Lambda$, we denote by $f[X^{(i)}] \in \Lambda^{\otimes r}$ to be the element that is f in the tensorand indexed by i and 1 elsewhere. For example, we have colored power sums $p_n[X^{(i)}]$, which, varying over n and i, are polynomial generators of $\Lambda^{\otimes r}$. A general element $f \in \Lambda^{\otimes r}$ can be a nontrivial function in every color of variable, so we denote it by $f[X^\bullet]$. To an r-tuple of partitions $\vec{\lambda} = \left(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)}\right)$, we can associate the *multi-Schur* function

$$s_{\vec{\lambda}}[X^\bullet] := s_{\lambda^{(0)}}[X^{(0)}] s_{\lambda^{(1)}}[X^{(1)}] \cdots s_{\lambda^{(r-1)}}[X^{(r-1)}].$$

Varying over all r-tuples of partitions, the multi-Schur functions give a basis of $\Lambda^{\otimes r}$.

Because Λ is a polynomial ring in the power sums $\{p_1, p_2, \ldots\}$, we can define ring homomorphisms out of Λ by specifying the image of each p_n. The usual plethysm, which we call here scalar plethysm, gives a convenient shorthand for certain homomorphisms of interest. Given a series $E = E(u_1, u_2, \ldots)$ in some parameters $u_1, u_2, \ldots$, we set

$$p_n[EX] = E(u_1^n, u_2^n, \ldots) p_n[X]$$

and extend algebraically to define $f[EX]$ for any $f \in \Lambda$. We can make sense of the case when E is a rational function by taking series expansions. For instance,

$$p_n\left[\frac{X}{1-q}\right] = \frac{p_n[X]}{1-q^n}.$$

For $\Lambda^{\otimes r}$, there is the richer structure of matrix plethysm. Namely, we now take an $r \times r$ matrix (whose rows and columns are indexed by $\mathbb{Z}/r\mathbb{Z}$, and which need not be invertible) $M = \left(E_{i,j}(u_1, u_2, \ldots)\right)$ where each $E_{i,j} = E_{i,j}(u_1, u_2, \ldots)$ is a series. The image of $p_n[X^{(i)}]$ is given by

$$p_n[MX^{(i)}] := \sum_{j \in \mathbb{Z}/r\mathbb{Z}} p_n[E_{j,i} X^{(j)}],$$

from which we define $f[MX^\bullet]$ for a general $f \in \Lambda^{\otimes r}$. Two particular instances of this will be important for us: define σ and ι by

$$p_n[\sigma X^{(i)}] := p_n[X^{(i+1)}], \quad \text{and} \quad p_n[\iota X^{(i)}] := p_n[X^{(-i)}],$$

respectively. Setting $(-)^T$ as the transpose matrix, note that $\sigma^T = \sigma^{-1}$ and $\iota^T = \iota$.

Given a series $E = E(u_1, u_2, \ldots)$, we set

$$p_n[E] := E(u_1^n, u_2^n, \ldots).$$

This is extended algebraically to define $f[E]$ for $f \in \Lambda$. To evaluate elements of $\Lambda^{\otimes r}$, our inputs are subdivided along color:

$$E^\bullet = \sum_{i \in \mathbb{Z}/r\mathbb{Z}} E^{(i)}.$$

For $f \in \Lambda^{\otimes r}$, we define $f[E^\bullet]$ by mapping $p_n[X^{(i)}] \mapsto p_n[E^{(i)}]$. Thus, one can view $E^\bullet$ as a vector plethysm.

There are two very important plethystic operators used in what follows. One is the plethystic exponential

$$\Omega[X] = \exp\left(\sum_{k>0} \frac{p_k[X]}{k}\right) = \sum_{n \geqslant 0} h_n[X]. \tag{3.1}$$

Although this is an infinite sum of elements in Λ, each summand of fixed degree is finite. Alternatively, it may be viewed as an element of the completion of Λ by degree. In $\Lambda^{\otimes r}$, we have $\Omega[X^{(i)}]$, and for any matrix plethysm M, we can make sense of

$$\Omega[MX^{(i)}] = \Omega\left[\sum_{j \in \mathbb{Z}/r\mathbb{Z}} E_{j,i} X^{(j)}\right] = \Omega\left[E_{0,i} X^{(0)}\right] \Omega\left[E_{1,i} X^{(1)}\right] \cdots \Omega\left[E_{r-1,i} X^{(i)}\right].$$

The second important operator is the translation operator $\mathcal{T}[X]$, which is the ring automorphism on Λ defined by

$$\mathcal{T}[X](p_n[X]) = p_n[X] + 1.$$

This is similarly extended to matrix plethysm as above.

3.2 Wreath Macdonald polynomials

Wreath Macdonald polynomials were introduced by Haiman [7], where he conjectured their existence and an analogue of the Macdonald positivity conjecture. These were later proved by Bezrukavnikov–Finkelberg [2], with an alternative proof of existence being given in [17]. For much more detail see the survey [13]. The following is equivalent to the definition given by Haiman.

Definition 3. Given a partition λ, the *wreath Macdonald polynomial* $H_\lambda[X^\bullet;q,t] \in \Lambda_{q,t}^{\otimes r}$ is the unique multi-symmetric function satisfying

- $H_\lambda\left[(1-q\sigma^{-1})X^\bullet;q,t\right] \in \operatorname{span}\left\{s_{\mathrm{quot}(\mu)} \mid \mu \geqslant \lambda \text{ and } \mathrm{core}(\mu)=\mathrm{core}(\lambda)\right\}$;
- $H_\lambda\left[(1-t^{-1}\sigma^{-1})X^\bullet;q,t\right] \in \operatorname{span}\left\{s_{\mathrm{quot}(\mu)} \mid \mu \leqslant \lambda \text{ and } \mathrm{core}(\mu)=\mathrm{core}(\lambda)\right\}$;
- $H_\lambda[1]=1$.

When we have no need to emphasize the (q,t)-dependence, we may simply denote it by $H_\lambda[X^\bullet]$ or even H_λ.

The degree of a multisymmetric function is the sum of the degrees of each component. Since the matrix plethysms in the above definition preserve degree, $\deg(H_\lambda) = |\mathrm{quot}(\lambda)|$. For $r=1$ the wreath Macdonald polynomials reduce to the modified Macdonald polynomials, and there is much interest in generalizing properties of these well-studied objects to the wreath case. This almost always involves delicate combinatorics of partitions, cores and quotients. Bases of $\Lambda^{\otimes r}$ are naturally indexed by r-tuples of partitions. If we fix an r-core α, then $\{H_\lambda \mid \mathrm{core}(\lambda)=\alpha\}$ does indeed form a basis of $\Lambda_{q,t}^{\otimes r}$. One can view the r-core as imposing an order on r-tuples of partitions by going backwards along the core-quotient decomposition and applying the usual dominance order.

Recall that Λ possesses the Hall inner product $\langle -,-\rangle$, for which the Schur functions form an orthonormal basis. We use the same notation for the tensor product pairing on $\Lambda^{\otimes r}$; now the multi-Schur functions form an orthonormal basis:

$$\langle s_{\vec{\lambda}}, s_{\vec{\mu}}\rangle = \delta_{\vec{\lambda},\vec{\mu}}.$$

The (modified) *wreath Macdonald pairing* $\langle -,-\rangle'_{q,t}$ on $\Lambda_{q,t}^{\otimes r}$ is defined to be

$$\langle f,g\rangle'_{q,t} := \left\langle f\left[\iota X^\bullet\right], g\left[(1-q\sigma^{-1})(t\sigma-1)X^\bullet\right]\right\rangle.$$

Moreover, this pairing is symmetric (see the adjoint relation [15, Lemma 3.10]). For $r \geqslant 3$ the H_λ are not orthonormal with respect to this scalar product. The dual basis to $\{H_\lambda\}$ is given by the "dagger" polynomials

$$H_\lambda^\dagger = H_\lambda^\dagger[X^\bullet;q,t] := H_\lambda[-\iota X^\bullet;q^{-1},t^{-1}].$$

Proposition 4 ([15, Proposition 2.12]). *For λ and μ with* $\mathrm{core}(\lambda)=\mathrm{core}(\mu)$, $\langle H_\lambda^\dagger, H_\mu\rangle'_{q,t}$ *is nonzero if and only if $\lambda=\mu$.*

The inner product $\langle H_\lambda^\dagger, H_\lambda\rangle'_{q,t}$ was computed in [13]. In Subsection 3.3 we will see that it is a special case of our second main result, leading to an independent derivation of this quantity.

3.3 The vertex operator

We are now ready to introduce our main construction: the wreath analogue of the Carlsson–Nekrasov–Okounkov vertex operator.

Definition 5. We define the vertex operator

$$\mathsf{W}(u) := \Omega\left[\frac{(1-u^{-1})X^{(0)}}{(1-q\sigma^{-1})(t\sigma-1)}\right]\mathcal{T}\left[(1-uqt)X^{(0)}\right].$$

As already mentioned this is simply the wreath Macdonald polynomial analogue of the vertex operator of [3] (see also [4] for the Jack polynomial case). In fact their operator is most naturally interpreted in terms of the torus-equivariant K-theory of the Hilbert scheme of points on $\mathbb{C}^2$, denoted $\mathrm{Hilb}_n(\mathbb{C}^2)$. Our $\mathsf{W}(u)$ extends this story to the $(\mathbb{Z}/r\mathbb{Z})$-fixed loci $(\mathrm{Hilb}_n(\mathbb{C}^2))^{\mathbb{Z}/r\mathbb{Z}}$, in perfect analogy with Haiman's original motivation for introducing the wreath Macdonald polynomials, but we will not discuss this here; see [1] for details.

The key result needed in the proof of Theorem 1 is the following.

Theorem 6. *For any integer $r \geqslant 3$ the vertex operator $\mathsf{W}(u)$ reproduces the modular Nekrasov factor (2.2) in the following sense: for λ and μ with the same r-core, we have*

$$\left\langle H_\mu^\dagger, \mathsf{W}(u)H_\lambda\right\rangle'_{q,t} = u^{-|\mathrm{quot}(\mu)|}N_{\lambda,\mu}(u).$$

This is the wreath Macdonald polynomial analogue of [5, Theorem 3.0.1]; again the Jack polynomial case may be found in [3, Proposition 2]. Carlsson, Nekrasov and Okounkov prove their result by reducing the identity to the Cherednik–Macdonald–Mehta identity for Macdonald polynomials. We take a quite different approach, instead relying on the wreath Macdonald polynomial analogue of the Tesler identity [6] recently proved by Romero and the second author [15]. Using this interpretation we further give some new expressions for Pieri coefficients for wreath Macdonald polynomials [1, §4.3].

Notice that we have no problem setting $u = 1$ in the definition of $\mathsf{W}(u)$ and the Nekrasov factor (2.2).

Corollary 7 ([13, Theorem 3.32])**.** *We have*

$$\langle H_\lambda^\dagger, H_\lambda\rangle'_{q,t} = N_{\lambda,\lambda}(1) = \prod_{\substack{\square\in\lambda \\ h_\lambda(\square)\equiv 0 \bmod r}} \left(1-q^{-a_\lambda(\square)}t^{l_\lambda(\square)+1}\right)\left(1-q^{a_\lambda(\square)+1}t^{-l_\lambda(\square)}\right).$$

3.4 Outline of the proof of Theorem 1

Our proof relies on taking the graded trace of the operator $\mathsf{W}(u)$. While this is standard, the following lemma gives an expression for such a trace in slightly more generality. Here T^D denotes the degree operator: $T^D f = T^{\deg(f)}f$.

Lemma 8. *Let T be a variable such that $|T| \ll 1$; we assume that the magnitude of T is less than all other variables involved. For matrix plethysms A and B, we have*

$$\mathrm{Tr}_{\Lambda_{q,t}^{\otimes r}}\left(\Omega[BX^\bullet]\mathcal{T}[AX^\bullet]T^D\right) = \frac{1}{(T;T)_\infty^r}\Omega\left[\frac{T}{1-T}\sum_{i,j=0}^{r-1}(A^TB)_{i,j}\right]. \tag{3.2}$$

Now one may identify the sum-side of Theorem 1 as the graded trace of $\mathsf{W}(u)$ with respect to the basis $\{H_\lambda : \mathrm{core}(\lambda) = \alpha\}$ for some fixed r-core α. It follows from combining Theorem 6 and Corollary 7 that we have

$$\mathsf{W}(u)T^D H_\lambda = \sum_{\substack{\mu \\ \mathrm{core}(\mu)=\alpha}} T^{|\mathrm{quot}(\lambda)|}u^{-|\mathrm{quot}(\mu)|}\frac{N_{\lambda,\mu}(u)}{N_{\mu,\mu}(1)}H_\mu.$$

Thus at $\mu = \lambda$ we can rewrite the matrix elements as in (1.3) up to replacing t by $1/t$. Some elementary manipulations of plethystic exponentials using Lemma 8 produce the correct product-side, and complete the proof.

Note that the wreath Macdonald polynomials depending on the choice of r-core α is what explains the core-independence of the Walsh–Warnaar conjecture, something which is entirely unclear from a combinatorial perspective. However, Walsh and Warnaar further conjecture that an elliptic deformation of the sum-side of Theorem 1, in which the ratio of Nekrasov factors is replaced by an appropriate ratio of modified theta functions, is also independent of the choice of r-core. This would correspond to a modular analogue of the elliptic (q,t)-Nekrasov–Okounkov formula, derived by Rains and Warnaar as a consequence of the equivariant Dijkgraaf–Moore–Verlinde–Verlinde (DMVV) formula [14, Appendix A].

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