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Flagged Hamel–Goulden formulas

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Abstract. We obtain Hamel–Goulden-type ribbon decomposition determinantal formulas for flagged supersymmetric Schur functions. As an application, we derive corresponding new determinantal formulas for dual refined canonical stable Grothendieck polynomials. These results generalize and produce a number of new determinantal formulas for these symmetric functions including Jacobi–Trudi and skew Giambelli-type hook determinants.

Keywords: Flagged Schur functions, dual stable Grothendieck polynomials, supersymmetric functions, determinantal formulas

1 Introduction

Determinantal formulas for Schur functions admit a remarkable unifying approach via ribbon decompositions due to Hamel and Goulden [5], which generalizes Jacobi–Trudi and their dual Nägelsbach–Kostka determinants, Giambelli and Lascoux–Pragacz formulas [13].

In this extended abstract, we present new generalizations of Hamel–Goulden formulas for *flagged supersymmetric Schur functions* with two doubly infinite sets of variables. As the main application, we obtain new determinantal formulas for *dual refined canonical stable Grothendieck polynomials*.

Dual stable Grothendieck polynomials can be viewed as a K-theoretic analogue of Schur functions due to Lam and Pylyavskyy [11]. One of their most general versions (with two sets of extra refined parameters) is known as the family of dual *refined canonical* stable Grothendieck polynomials which were introduced and studied by Hwang, Jang, Kim, Song, and Song in [6, 7]. These functions generalize refined versions by Galashin, Grinberg, and Liu [4], and canonical versions by the second author [17].

Determinantal formulas for dual stable Grothendieck polynomials have received considerable attention, especially Jacobi–Trudi-type formulas, see [12, 17, 8, 9, 2, 10, 6, 7, 15] for some related works. Generalizing such formulas for Hamel–Goulden identities becomes more technical, as even Giambelli-type formulas obtained by Lascoux and Naruse [12] were not obvious generalizations. In contrast, other various generalizations of Schur

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functions have straightforward analogues of ribbon decomposition formulas, see e.g. [14].

It turns out that to obtain Hamel–Goulden-type formulas for dual stable Grothendieck polynomials it is useful to consider this problem for more general families of flagged supersymmetric functions. The crucial technical part in deriving such formulas for flagged functions is to find appropriate tableaux formulas and to define *induced ribbon flags* used in the enumeration determinants. For the full text with proofs, see [1].

2 Preliminaries

2.1 Variables notation

We denote $[m] := \{1, \ldots, m\}$. We shall use the following notation for variables. For a set of variables $\mathbf{x} = (x_i)$ we denote $\mathbf{x}_m = (x_i)_{i \in [m]} = (x_1, \ldots, x_m)$ and $\mathbf{x}_{a,b} = (x_i)_{i \in [a,b]} = (x_a, \ldots, x_b)$ if $a \leq b$ (and $\mathbf{x}_{a,b} = \varnothing$ if $a > b$). We also denote $\overline{\mathbf{x}} = (-x_i)$ for the negation.

2.2 Partitions

A *partition* is a sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers $\lambda_1 \geq \cdots \geq \lambda_\ell$, where $\ell = \ell(\lambda)$ is its *length*. The set $[\lambda] := \{(i,j) : 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$ is the *Young diagram* (or shape) of λ. The *conjugate* partition λ' of λ is the partition with the transposed Young diagram. For partitions λ and μ with $[\mu] \subseteq [\lambda]$, the *skew partition* λ/μ has the diagram (or shape) $[\lambda/\mu] := [\lambda] \setminus [\mu]$. We refer to the elements of diagrams as boxes or cells, and use the English notation for drawing them (like matrices).

For a cell $\gamma = (i,j)$ of a diagram, its *content* is given by $c(\gamma) := j - i$ and we also denote $\mathrm{row}(\gamma) = i, \mathrm{col}(\gamma) = j$ for its row and column indices.

Notice that knowing the content of a single cell in the shape λ/μ, completely determines contents of all other cells. We use this property of shapes and define a *refined shape* (or *r-shape* for short) as a pair $(\lambda/\mu, c)$ for $c \in \mathbb{Z}$. The r-shape is a shape whose content of the bottom-left cell is c. Informally, r-shape is a shape with 'shifted' contents. We call the number c as the *root content* of the r-shape. Then the usual shape λ/μ is an r-shape $(\lambda/\mu, 1 - \lambda'_1)$.

2.3 Elementary supersymmetric functions

For sets of variables $\mathbf{x} = (x_i), \mathbf{y} = (y_i)$, we define the *elementary and complete homogeneous supersymmetric functions* $e_n(\mathbf{x}/\mathbf{y}), h_n(\mathbf{x}/\mathbf{y})$ as follows:

$$e_n(\mathbf{x}/\mathbf{y}) := \sum_{i=0}^{n} (-1)^{n-i} e_i(\mathbf{x}) h_{n-i}(\mathbf{y}), \quad h_n(\mathbf{x}/\mathbf{y}) := \sum_{i=0}^{n} (-1)^{n-i} h_i(\mathbf{x}) e_{n-i}(\mathbf{y})$$

via the usual elementary e_i and complete homogeneous symmetric functions h_i.

2.4 Dual stable Grothendieck polynomials

For sets of variables $\mathbf{x} = (x_1, x_2, \ldots)$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots)$ the *dual refined canonical stable Grothendieck polynomial* $g_{\lambda/\mu}(\mathbf{x}; \boldsymbol{\alpha}; \boldsymbol{\beta})$ is a symmetric function (in the $\mathbf{x}$ variables) which can be defined as follows:

$$g_{\lambda/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta}) := \det \left[e_{\lambda'_i - i - \mu'_j + j}(\mathbf{x}, \overline{\boldsymbol{\alpha}}_{j-1}, \boldsymbol{\beta}_{\lambda'_i - 1} \,/\, \overline{\boldsymbol{\alpha}}_{i-1}, \boldsymbol{\beta}_{\mu'_j}) \right]_{1 \le i,j \le n},$$

where $n \geq \lambda_1$. These functions were introduced and studied in [6, 7] (the version written here differs by $\boldsymbol{\alpha} \to \overline{\boldsymbol{\alpha}}$). Note that for $\boldsymbol{\alpha} = 0, \boldsymbol{\beta} = 0$ these functions specialize to the Schur functions $s_{\lambda/\mu}(\mathbf{x})$; in general, $g_{\lambda/\mu}(\mathbf{x}; \boldsymbol{\alpha}; \boldsymbol{\beta}) = s_{\lambda/\mu}(\mathbf{x}) + \text{lower degree terms}$. For $\boldsymbol{\alpha} = 0, \boldsymbol{\beta} = (1, 1, \ldots)$ they specialize to the dual stable Grothendieck polynomials $g_{\lambda/\mu}(\mathbf{x})$ introduced in [11]; for $\boldsymbol{\alpha} = 0$ they specialize to the refined version introduced in [4]; and for $\boldsymbol{\alpha} = (\alpha, \alpha, \ldots), \boldsymbol{\beta} = (\beta, \beta, \ldots)$ they specialize to canonical version introduced in [17]. Notably, the following duality

$$\omega(g_{\lambda/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})) = g_{\lambda'/\mu'}(\mathbf{x}; \boldsymbol{\beta}, \boldsymbol{\alpha})$$

holds for the action of the standard involutive ring automorphism ω of the ring of symmetric functions. The functions $g_{\lambda/\mu}(\mathbf{x}; \boldsymbol{\alpha}, \boldsymbol{\beta})$ also have combinatorial formula using marked reverse plane partitions [6, 7]; here we will use for them another new formula using super tableaux.

3 Flagged supersymmetric Schur polynomials

We use the known notions of column and row flags of symmetric functions, see e.g. [16, 14].

Definition 1 (Flags). *Given r-shape λ/μ and $n \geq \lambda_1$.*

(Column flags) The vectors $\mathbf{a} = (a_i), \mathbf{b} = (b_i) \in \mathbb{Z}^n$ satisfying the following conditions

$$a_i - a_{i+1} \leq \mu'_i - \mu'_{i+1} + 1, \quad b_i - b_{i+1} \leq \lambda'_i - \lambda'_{i+1} + 1, \quad i \geq 1, \text{ whenever } \mu'_i < \lambda'_{i+1},$$

are called column flags. (We shall refer to them as flags.)

(Row flags) The vectors $\mathbf{a} = (a_i), \mathbf{b} = (b_i) \in \mathbb{Z}^n$ satisfying the following conditions

$$a_i - a_{i+1} \leq 0, \quad b_i - b_{i+1} \leq 0, \quad i \geq 1, \text{ whenever } \mu_i < \lambda_{i+1},$$

are called row flags.

(Conjugate flags) For flags $\mathbf{a}, \mathbf{b}$ *define the conjugate flags* $\mathbf{a}' = (a_i'), \mathbf{b}' = (b_i')$ *given by*

$$a_i' = a_i + c(\gamma_i), \quad b_i' = b_i + c(\delta_i), \quad i \geq 1,$$

where γ_i, δ_i *denote the top and bottom cells of i-th column of the diagram. (We define conjugation this way to include refined shapes with shifted contents.) Note that if* $\mathbf{a}, \mathbf{b}$ *are column flags for the shape* λ/μ*, then* $\mathbf{a}', \mathbf{b}'$ *are row flags for the shape* λ'/μ' *(and vice versa).*

We define flagged supersymmetric Schur functions as follows.

Definition 2 (Flagged supersymmetric Schur functions)**.** *For a skew r-shape* λ/μ *and (doubly infinite) sets of variables* $\mathbf{y} = (y_i)_{i \in \mathbb{Z}}$ *and* $\mathbf{z} = (z_i)_{i \in \mathbb{Z}}$ *we define the skew flagged supersymmetric Schur functions as follows:*

$$\mathsf{S}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{y}/\mathbf{z}) := \det\left[e_{\lambda_i'-i-\mu_j'+j}(\mathbf{y}_{a_j,b_i} \,/\, \mathbf{z}_{a_j',b_i'})\right]_{1 \leq i,j \leq n},$$

where $n \geq \lambda_1$ *and* $\mathbf{a}, \mathbf{b}$ *are column flags.*

For $\mathbf{z} = 0$ this function specializes to flagged Schur function. Without flag restrictions (i.e. letting $a_i \to -\infty, b_i \to \infty$), this function specializes to supersymmetric Schur function.

We similarly define the *row flagged supersymmetric Schur functions* $\overline{\mathsf{S}}^{\mathbf{a}',\mathbf{b}'}_{\lambda/\mu}(\mathbf{y}/\mathbf{z})$ with row flags $\mathbf{a}', \mathbf{b}'$ as follows:

$$\overline{\mathsf{S}}^{\mathbf{a}',\mathbf{b}'}_{\lambda/\mu}(\mathbf{y}/\mathbf{z}) = \det\left[h_{\lambda_i-\mu_j-i+j}(\mathbf{y}_{a_j',b_i'}/\mathbf{z}_{a_j,b_i})\right]_{1 \leq i,j \leq \ell(\lambda)}.$$

Then the two functions are related via the following duality (which shows that it is enough to consider just one of them).

Proposition 1 (Duality for column and row flagged functions)**.** *The following duality holds:*

$$\mathsf{S}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{y}/\mathbf{z}) = \overline{\mathsf{S}}^{\mathbf{a}',\mathbf{b}'}_{\lambda'/\mu'}(\overline{\mathbf{z}}/\overline{\mathbf{y}}).$$

3.1 g-specialization

We are mainly interested in the following specialization of flagged supersymmetric Schur functions which reduces them to dual refined canonical stable Grothendieck polynomials.

Definition 3 (g-specialization)**.** *Let m be a given positive integer. Define the g-specialization as the following substitution of variables* $\mathbf{y} = (y_i), \mathbf{z} = (z_i)$

$$g : y_i \to \begin{cases} x_i, & \text{if } i \in [m], \\ \beta_{i-m}, & \text{if } i > m, \\ -\alpha_{-i+1}, & \text{if } i \leq 0, \end{cases} \qquad z_i \to \begin{cases} 0, & \text{if } i \in [m], \\ -\alpha_{i-m}, & \text{if } i > m, \\ \beta_{-i+1}, & \text{if } i \leq 0. \end{cases}$$

We then define the flagged dual Grothendieck enumerator

$$\mathrm{g}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{x}_m;\boldsymbol{\alpha},\boldsymbol{\beta}) := \mathrm{S}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{y}/\mathbf{z})\,|_g$$

as the function under g-specialization of flagged supersymmetric Schur function.

Proposition 2. *Let $m \in \mathbb{Z}_{\geq 0}$ and $\mathbf{a} = (a_i), \mathbf{b} = (b_i)$ be flags. Let us denote $\tau(k) = -k+1$ and $\eta(k) = k - m$. Then we have*

$$\mathrm{g}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{x}_m;\boldsymbol{\alpha},\boldsymbol{\beta}) = \det\left[e_{\lambda'_i-i-\mu'_j+j}(\mathbf{x}_{u_j,v_i},\overline{\boldsymbol{\alpha}}_{\tau(b_i),\tau(a_j)},\boldsymbol{\beta}_{\eta(a_j),\eta(b_i)}\;/\;\overline{\boldsymbol{\alpha}}_{\eta(a'_j),\eta(b'_i)},\boldsymbol{\beta}_{\tau(b'_i),\tau(a'_j)})\right]_{1\leq i,j\leq n},$$

where $n \geq \lambda_1$ and $u_j = max(1, a_j)$, $v_i = min(m, b_i)$ (we also set $\alpha_i = \beta_i = 0$ if $i \leq 0$). In particular, if $\mathbf{a} = (a_i), \mathbf{b} = (b_i) \in \mathbb{Z}^n$ are the following column flags:

$$a_i = -i+2, \quad b_i = \lambda'_i + m - 1, \quad i \in [n],$$

then flagged supersymmetric Schur functions and flagged dual Grothendieck enumerator specialize to dual refined canonical stable Grothendieck polynomials

$$\mathrm{S}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{y}/\mathbf{z})\,|_g = \mathrm{g}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{x}_m;\boldsymbol{\alpha},\boldsymbol{\beta}) = g_{\lambda/\mu}(\mathbf{x}_m;\boldsymbol{\alpha},\boldsymbol{\beta})$$

under g-specialization.

4 Tableaux formulas

In this section we describe combinatorial interpretation for the functions $\mathrm{S}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{y}/\mathbf{z})$.

Definition 4. *A flagged $\mathbb{Z}$-SSYT with flags $\mathbf{a} = (a_i), \mathbf{b} = (b_i)$ is a semistandard tableau (i.e. weakly increasing along rows from left to right and strictly increasing along columns from top to bottom) $T = (T_{i,j})$ filled with integers so that $a_j \leq T_{i,j} \leq b_j$.*

Example 1. *For $\lambda/\mu = (5,5,5,3,3)/(2,1)$, and flags $\mathbf{a} = (-2,-2,-1,0,0)$, $\mathbf{b} = (7,6,7,7,6)$, an example of $\mathbb{Z}$-SSYT is shown in Figure 1a.*

We now consider tableaux with extra primed entries $\mathbb{Z}' = \{\ldots, -2', -1', 0', 1', 2', \ldots\}$.

Definition 5. *For a given flagged $\mathbb{Z}$-SSYT $\tilde{T} = (\tilde{T}_{i,j})$, we produce flagged super tableau $T = (T_{i,j})$ of the same shape as follows:*

$$T_{ij} = \tilde{T}_{ij} \in \mathbb{Z} \quad or \quad T_{ij} = (\tilde{T}_{ij} + c(i,j))' \in \mathbb{Z}'.$$

$$\begin{array}{|c|c|c|c|c|}
\cline{3-5} \multicolumn{2}{c|}{} & 0 & 0 & 1 \\
\cline{2-5} \multicolumn{1}{c|}{} & -2 & 1 & 2 & 3 \\
\hline 1 & 1 & 3 & 4 & 4 \\
\hline 2 & 2 & 6 & \multicolumn{2}{c}{} \\
\cline{1-3} 5 & 6 & 7 & \multicolumn{2}{c}{} \\
\cline{1-3}
\end{array}$$

(a) Flagged $\mathbb{Z}$-SSYT.

$$\begin{array}{|c|c|c|c|c|}
\cline{3-5} \multicolumn{2}{c|}{} & 2' & 0 & 1 \\
\cline{2-5} \multicolumn{1}{c|}{} & -2' & 1 & 4' & 3 \\
\hline -1' & 1 & 3 & 4 & 4 \\
\hline 2 & 0' & 5' & \multicolumn{2}{c}{} \\
\cline{1-3} 5 & 6 & 5' & \multicolumn{2}{c}{} \\
\cline{1-3}
\end{array}
\qquad
\begin{array}{|c|c|c|c|c|}
\cline{3-5} \multicolumn{2}{c|}{} & 2' & 3' & 5' \\
\cline{2-5} \multicolumn{1}{c|}{} & -2' & 2' & 2 & 3 \\
\hline -1' & 0' & 3 & 4 & 6' \\
\hline -1' & 2 & 5' & \multicolumn{2}{c}{} \\
\cline{1-3} 1' & 3' & 7 & \multicolumn{2}{c}{} \\
\cline{1-3}
\end{array}$$

(b) Flagged super tableaux.

Figure 1: Examples of tableaux

Note that every choice of T induces unique super tableau. Let $ST_{\lambda/\mu}(\mathbf{a}, \mathbf{b})$ be the set of all flagged super tableaux produced from the set of $\mathbb{Z}$-SSYT of r-shape λ/μ with flags $\mathbf{a}, \mathbf{b}$. Notice that in T, the vectors $\mathbf{a}', \mathbf{b}'$ are row flags for primed elements. For $r \in \mathbb{Z}$, define the weight $wt(T_{ij})$ as

$$wt(T_{ij}) := \begin{cases} y_r & \text{if} \quad T_{ij} = r, \\ -z_r & \text{if} \quad T_{ij} = r', \end{cases}$$

and define the weight $wt(T)$ of the super tableau T as

$$wt(T) = \prod_{ij} wt(T_{ij}).$$

Example 2. *Some super tableaux produced from the $\mathbb{Z}$-SSYT in Example 1 are shown in Figure 1b. Let T_1, T_2 be the left and right tableaux respectively, then*

$$wt(T_1) = -y_0 y_1^3 y_2 y_3^2 y_4^2 y_5 y_6 z_{-2} z_{-1} z_0 z_2 z_4 z_5^2 \qquad wt(T_2) = y_2^2 y_3^2 y_4 y_7 z_{-2} z_{-1}^2 z_0 z_1 z_2^2 z_3^2 z_5^2 z_6$$

We obtain the following combinatorial formula for $\mathrm{S}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{y}/\mathbf{z})$.

Theorem 1 (Tableaux formula for flagged supersymmetric Schur functions)**.** *The following tableau formula holds:*

$$\mathbf{S}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{y}/\mathbf{z}) = \sum_{T \in ST_{\lambda/\mu}(\mathbf{a},\mathbf{b})} wt(T). \tag{4.1}$$

5 Ribbon decomposition formulas

5.1 Outer ribbon decompositions

In this subsection, we define the Hamel–Goulden # operation [5] on ribbons via cutting strips due to [3].

(a) An outer ribbon decomposition. (b) Cutting strip $\Theta(\lambda/\mu)$.

Figure 2

A *ribbon* is a connected skew partition whose diagram does not contain a 2×2 block of cells. For a ribbon θ, we call its bottom leftmost cell as the *head*, and its top rightmost cell as the *tail*.

For any skew r-shape λ/μ, its *outer ribbon decomposition* $\Theta = (\theta_1, ..., \theta_k)$ is a decomposition of λ/μ into ribbons such that the head of every ribbon θ_i lies on the left or bottom perimeter and the tail lies on the top or right perimeter of the diagram. See Figure 2a for an example. For a cell $\gamma \in \theta_i$, we say that it *goes up* if the cell adjacent above is also in θ_i, or if γ is the tail of θ_i and lies on top perimeter of λ/μ; similarly, we say it *goes right* if the cell adjacent to the right is in θ_i, or if γ is the tail of θ_i and lies on the right perimeter of λ/μ. Additionally, we define the direction for the top-right cell γ in λ/μ. Suppose that $\gamma \in \theta_i$. We say that it goes up, if the adjacent cell below γ is also in θ_i; similarly, we say it goes right, if the cell adjacent to the left is also in θ_i.

Let us denote the head and the tail of θ_i in the r-shape λ/μ by δ_i and γ_i, respectively. Each ribbon θ_i can be seen as a sub-diagram of $\Theta(\lambda/\mu)$, i.e. θ_i consists of cells, whose contents lie in the interval $[c(\delta_i), c(\gamma_i)]$. In general, let p, q be contents of some cells in the Young diagram s.t. $p \leq q$ and define $\Theta(p, q)$ as the sub-ribbon of $\Theta(\lambda/\mu)$ whose cell contents lie in $[p, q]$, and also $\Theta(p, q) = \varnothing$, if $p = q + 1$, and $\Theta(p, q)$ is undefined, if $p > q + 1$. Then we define

$$\theta_i \# \theta_j := \Theta(c(\delta_i), c(\gamma_j)).$$

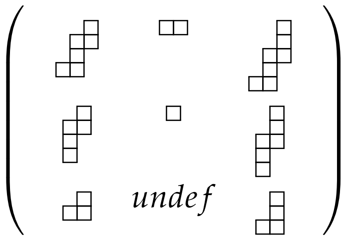
Notice that $\theta_i \# \theta_j$ is an r-shape $(\theta_i \# \theta_j, c(\delta_i))$, for all i, j.

In the formulas, we shall consider shapes only with connected diagrams λ/μ.

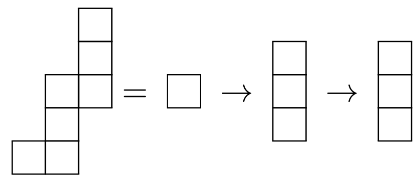
Let us now define the right-arrow operation $\rightarrow$ on Young diagrams D_1, D_2. The diagram $D_1 \rightarrow D_2$ is constructed as follows: stack the bottom left cell of D_2 to the right of the upper right cell of D_1.

Definition 6 (Canonical decomposition)**.** *For a ribbon θ, the expression $\theta = \rho_1 \rightarrow ... \rightarrow \rho_k$ is called the canonical decomposition, where each ρ_i is a vertical strip.*

Example 3. *Let $\lambda = (4, 4, 4, 2), \mu = (2, 1, 0, 0)$. An outer ribbon decomposition and the corresponding cutting strip are shown in Figure 2a, 2b. There are three ribbons in this decomposition: $\theta_1 = \Theta[-3, 2], \theta_2 = \Theta[-2, -2], \theta_3 = \Theta[0, 3]$. The ribbon matrix $[\theta_i \# \theta_j]$ is shown in Figure 3a.*



(a) Ribbon matrix $(\theta_i \# \theta_j)_{i,j}^3$



(b) Canonical decomposition of $\theta_1 \# \theta_3$

Figure 3

5.2 Hamel-Goulden formulas

In this section we state Hamel-Goulden-type formulas for the functions $\mathsf{S}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{y}/\mathbf{z})$.

The following is a key definition for obtaining ribbon determinantal formulas.

Definition 7 (Induced ribbon flags)**.** *For a skew r-shape λ/μ and its outer ribbon decomposition $(\theta_1, \ldots, \theta_k)$, let $\theta_i \# \theta_j = \rho_1^{ij} \to \ldots \to \rho_{k_{ij}}^{ij}$ be corresponding canonical decompositions. Denote by δ_r^{ij} and γ_r^{ij} the bottom and top cells of ρ_r^{ij}. Let $M_r^{ij} := \max(c(\delta_r^{ij}))$ and $m_r^{ij} := \min(c(\gamma_r^{ij}))$, where $\max(c), \min(c)$ denote the rightmost and the leftmost cells with the content c in the r-shape λ/μ. For flags $\mathbf{a} = (a_i), \mathbf{b} = (b_i)$, define the induced ribbon flags $\mathbf{a}^{ij} = (a_r^{ij}), \mathbf{b}^{ij} = (b_r^{ij})$ as follows:*

$$a_r^{ij} = a_{col(m_r^{ij})} - (\mu'_{col(m_r^{ij})} + 1) + row(m_r^{ij}), \qquad b_r^{ij} = b_{col(M_r^{ij})} - \lambda'_{col(M_r^{ij})} + row(M_r^{ij}),$$

with $a^{ij} = b^{ij} = \varnothing$ if $\theta_i \# \theta_j = \varnothing$ or $\theta_i \# \theta_j = undefined$. It can be shown that $\mathbf{a}^{ij}, \mathbf{b}^{ij}$ are indeed flags for shapes $\theta_i \# \theta_j$.

Our main formulas are now stated as follows.

Theorem 2 (Hamel–Goulden formulas for flagged supersymmetric Schur functions)**.** *Let $(\theta_1, \ldots, \theta_k)$ be an outer ribbon decomposition of λ/μ. Then the following determinantal formulas hold:*

$$\mathsf{S}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{y}/\mathbf{z}) = \det \left[\mathsf{S}^{\mathbf{a}^{ij},\mathbf{b}^{ij}}_{\theta_i \# \theta_j}(\mathbf{y}/\mathbf{z}) \right]_{1 \le i,j \le k}$$

where $\mathsf{S}^{\mathbf{a}^{ij},\mathbf{b}^{ij}}_{\varnothing}(\mathbf{y}/\mathbf{z}) = 1$ and $\mathsf{S}^{\mathbf{a}^{ij},\mathbf{b}^{ij}}_{undefined}(\mathbf{y}/\mathbf{z}) = 0$.

The proof of this result is combinatorial and generally relies on the Lindström-Gessel-Viennot lemma. We construct certain weighted lattice which we call *super lattice*, based on the ribbon decomposition and given flags. We then show that paths correspond to super tableaux with induced ribbon flags, and establish weight-preserving bijection between nonintersecting path systems and flagged super tableaux.

Taking the g-specialization we now obtain Hamel–Goulden-type formulas for dual refined canonical stable Grothendieck polynomials with determinant entries written via flagged dual Grothendieck enumerators.

Corollary 1 (Hamel–Goulden formulas for dual refined canonical stable Grothendieck polynomials)**.** *Let $(\theta_1, \ldots, \theta_k)$ be an outer ribbon decomposition of λ/μ, and $\mathbf{a} = (a_i), \mathbf{b} = (b_i)$ be column flags defined as follows: $a_i = -i+2, b_i = \lambda'_i + m - 1, i \geq 1$. Then the following determinantal formulas hold:*

$$g_{\lambda/\mu}(\mathbf{x}_m; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \det \left[\mathrm{g}^{\mathbf{a}^{ij}, \mathbf{b}^{ij}}_{\theta_i \# \theta_j}(\mathbf{x}_m; \boldsymbol{\alpha}, \boldsymbol{\beta}) \right]_{1 \leq i,j \leq k},$$

where the induced ribbon flags are given by

$$\mathbf{a}^{ij} = (-c(m^{ij}_r) - \mu'_{col(m^{ij}_r)} + 1)_{r \in [k^{ij}]}, \qquad \mathbf{b}^{ij} = (row(M^{ij}_r) + m - 1)_{r \in [k^{ij}]}.$$

Example 4. *Consider the shape and its outer ribbon decomposition from Example 3. Let the flags be $\mathbf{a} = (1, 0, -1, -2)$, $\mathbf{b} = (6, 6, 5, 5)$. Then $\mathbf{a}' = (-1, 0, 1, 1)$, $\mathbf{b}' = (3, 4, 5, 6)$ and the induced ribbon flags are the following:*

$\mathbf{a}^{11} = (2, 0, -1)$	$\mathbf{b}^{11} = (6, 6, 5)$	$\mathbf{a}'^{11} = (-1, 0, 1)$	$\mathbf{b}'^{11} = (3, 4, 6)$
$\mathbf{a}^{12} = (2, 1)$	$\mathbf{b}^{12} = (6, 6)$	$\mathbf{a}'^{12} = (-1, -1)$	$\mathbf{b}'^{12} = (3, 4)$
$\mathbf{a}^{13} = (2, 0, -2)$	$\mathbf{b}^{13} = (6, 6, 5)$	$\mathbf{a}'^{13} = (-1, 0, 1)$	$\mathbf{b}'^{13} = (3, 4, 6)$
$\mathbf{a}^{21} = (0, -1)$	$\mathbf{b}^{21} = (6, 5)$	$\mathbf{a}'^{21} = (0, 1)$	$\mathbf{b}'^{21} = (4, 6)$
$\mathbf{a}^{22} = (1)$	$\mathbf{b}^{22} = (6)$	$\mathbf{a}'^{22} = (-1)$	$\mathbf{b}'^{22} = (4)$
$\mathbf{a}^{23} = (0, -2)$	$\mathbf{b}^{23} = (6, 5)$	$\mathbf{a}'^{23} = (0, 1)$	$\mathbf{b}'^{23} = (4, 6)$
$\mathbf{a}^{31} = (0, -1)$	$\mathbf{b}^{31} = (5, 5)$	$\mathbf{a}'^{31} = (0, 1)$	$\mathbf{b}'^{31} = (5, 6)$
$\mathbf{a}^{32} = \varnothing$	$\mathbf{b}^{32} = \varnothing$	$\mathbf{a}'^{32} = \varnothing$	$\mathbf{b}'^{32} = \varnothing$
$\mathbf{a}^{33} = (0, -2)$	$\mathbf{b}^{33} = (5, 5)$	$\mathbf{a}'^{33} = (0, 1)$	$\mathbf{b}'^{33} = (5, 6)$

Then we have

$$\mathsf{S}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{y}/\mathbf{z}) = \det \left[\mathsf{S}^{\mathbf{a}^{ij}, \mathbf{b}^{ij}}_{\theta_i \# \theta_j}(\mathbf{y}/\mathbf{z}) \right]_{1 \leq i,j \leq 3},$$

For $m = 3$, the flags $\mathbf{a}, \mathbf{b}$ also apply for g-specialization and we have:

$$g_{\lambda/\mu}(\mathbf{x}_m; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \det \left[\mathrm{g}^{\mathbf{a}^{ij}, \mathbf{b}^{ij}}_{\theta_i \# \theta_j}(\mathbf{x}_m; \boldsymbol{\alpha}, \boldsymbol{\beta}) \right]_{1 \leq i,j \leq 3}.$$

Strict flags. We can also slightly change the definition of ribbon flags, and get another Hamel-Goulden-type formulas. Let us define *strict flags* $\tilde{\mathbf{b}} = (\tilde{b}_i), \tilde{\mathbf{a}} = (\tilde{a}_i)$ as flags where the flag $\tilde{\mathbf{a}}$ respects more strict inequality: $\tilde{a}_i - \tilde{a}_{i+1} \leq 1$. Then we define flags for the ribbons $\theta_i \# \theta_j$ as follows. Define flags $\tilde{\mathbf{b}}^{ij} = (\tilde{b}^{ij}_r)_r, \tilde{\mathbf{a}}^{ij} = (\tilde{a}^{ij}_r)_r$:

$$\tilde{a}^{ij}_r = \tilde{a}_{col(min(\gamma^{ij}_r))} \qquad \tilde{b}^{ij}_r = \tilde{b}_{col(max(\delta^{ij}_r))} - \lambda'_{col(max(\delta^{ij}_r))} + row(max(\delta^{ij}_r))$$

We also prove that the same Hamel–Goulden-type formulas hold for these strict flags.

6 Some special cases

In this section we show some new formulas for special cases of outer decompositions: Jacobi-Trudi and skew Giambelli type formulas.

6.1 Jacobi-Trudi formulas

Let $n \in \mathbb{Z}_{>0}$, $c \in \mathbb{Z}$ and $\mathbf{p}, \mathbf{q}$ be column flags for the r-shape (n, c) (single row with shifted content c). Let us denote

$$h^{\mathbf{p},\mathbf{q}}_{n,c}(\mathbf{y}/\mathbf{z}) := \mathsf{S}^{\mathbf{p},\mathbf{q}}_{(n)}(\mathbf{y}/\mathbf{z}),$$

where the polynomials on both sides are indexed by the r-shape (n, c). From Theorem 1 it follows that:

$$h^{\mathbf{p},\mathbf{q}}_{n,c}(\mathbf{y}/\mathbf{z}) = \sum^{n}_{\substack{i_1 \le \ldots \le i_n \\ p_k \le i_k \le q_k,\ k \in [n]}} \prod_{r=1}^{n} (y_{i_r} - z_{i_r + c + r - 1}).$$

Corollary 2 (Jacobi–Trudi-type formulas)**.** *Let* $\mathbf{a}, \mathbf{b}$ *be some flags for (the usual shape)* λ/μ*. The following formulas hold:*

$$\mathsf{S}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{y}/\mathbf{z}) = \det \left[h^{\mathbf{a}^{ij},\mathbf{b}^{ij}}_{\lambda_i - \mu_j - i + j,\, \mu_j + 1 - j}(\mathbf{y}/\mathbf{z}) \right]_{1 \le i,j \le n},$$

where $n \ge \ell(\lambda)$ *and* $\mathbf{a}^{ij}, \mathbf{b}^{ij}$ *are induced ribbon flags for the r-shape* $(\lambda_i - \mu_j - i + j, \mu_j + 1 - j)$*.*

Under g-specialization this gives different formula for dual refined canonical stable Grothendieck functions from the known Jacobi-Trudi-type formula obtained in [6, 7]; on the other hand, we can deduce the known formula from the row flagged supersymmetric Schur function.

6.2 Skew Giambelli hook formulas

We need *Frobenius notation* to write Giambelli-type formulas. For a partition λ, let (k, k) be the rightmost diagonal cell in λ, i.e. k is the size of the *Durfee square*. Let u_i, v_i be the number of cells on the right side and below the diagonal cell (i, i), for $i \in [k]$. We then write $\lambda = (u_1, ..., u_k | v_1, ..., v_k)$. Similarly, let $\mu = (c_1, ..., c_\ell | d_1, ..., d_\ell)$. We denote hook partitions $(u + 1, 1^v)$ as $(u|v)$. Let $\mathbf{a}, \mathbf{b}$ be flags for the shape λ/μ, and $\Theta = (\theta_1, \ldots, \theta_{k+\ell})$ an outer decomposition of λ/μ into hooks (as in Fig.4). In this decomposition we have $k + \ell$ ribbons of three types: vertical, horizontal, and hooks:

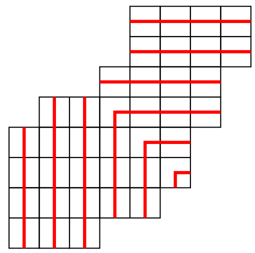


Figure 4: Ribbon decomposition into hooks

We specify the ribbons we get using # operation for ribbons in Θ:

$$\begin{cases} \Theta(-v_i, u_j) = (u_j|v_i) \quad \text{for} \quad i,j \in [k] \qquad \text{hook} \\ \Theta(-v_i, -d_j - 1) = 1^{v_i - d_j} \quad \text{for} \quad i \in [k]\,, j \in [\ell] \qquad \text{vertical} \\ \Theta(c_i + 1, u_j) = u_j - c_i \quad \text{for} \quad i \in [\ell]\,, j \in [k] \qquad \text{horizontal} \\ \Theta(c_i + 1, -d_j - 1) = undefined \quad \text{for} \quad i \in [\ell]\,, j \in [\ell] \end{cases}$$

Let us define the following ribbon flags:

- $\mathbf{b}^{ij} = (b_r^{ij})_r, \mathbf{a}^{ij} = (a_r^{ij})_r$ for hooks $(u_j|v_i)$, $i, j \in [k]$
- $\mathbf{p}^{ij} = (p_r^{ij})_r, \mathbf{q}'^{ij} = (q_r^{ij})_r$ for vertical ribbons $1^{v_i - d_j}$, $i \in [k]\,, j \in [\ell]$
- $\tilde{\mathbf{p}}^{ij} = (\tilde{p}_r^{ij})_r, \tilde{\mathbf{q}}^{ij} = (\tilde{q}_r^{ij})_r$ for horizontal ribbons $u_j - c_i$, $i \in [\ell]\,, j \in [k]$

Corollary 3 (Skew Giambelli formulas for flagged supersymmetric Schur functions)**.** *The following formulas hold:*

$$\mathrm{S}^{\mathbf{a},\mathbf{b}}_{\lambda/\mu}(\mathbf{y}/\mathbf{z}) = (-1)^{\ell} \det \begin{bmatrix} \left[\mathrm{S}^{\mathbf{a}^{ij},\mathbf{b}^{ij}}_{u_j|v_i}\right]_{i,j\in[k]} & \left[\mathrm{S}^{\mathbf{p}^{ij},\mathbf{q}^{ij}}_{1^{v_i-d_j}}\right]_{\substack{i\in[k]\\ j\in[\ell]}} \\ \left[\mathrm{S}^{\tilde{\mathbf{p}}^{ij},\tilde{\mathbf{q}}^{ij}}_{u_j-c_i}\right]_{\substack{i\in[\ell]\\ j\in[k]}} & \mathbf{0} \end{bmatrix}.$$

Under g-specialization this also gives new skew Giambelli formulas for dual refined canonical stable Grothendieck functions which were not known before.

We also note that in the special case for dual stable Grothendieck polynomials (for $\boldsymbol{\alpha} = 0$ and $\boldsymbol{\beta} = (1, 1, \ldots)$) of straight shape $\mu = \varnothing$, this formula gives the Giambelli-type formula obtained by Lascoux–Naruse [12].

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