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Dimensions of toggleability spaces

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Abstract. We establish a conjecture of Defant, Hopkins, Poznanović, and Propp concerning the dimensions of toggleability spaces for products of chains, shifted staircases, type-A root posets, and type-B root posets. Generalizing this result, we show that for a larger family of posets defined by restricted diagrams, the dimensions of toggleability spaces are equal to the rank of the poset plus one. As part of our approach, we build on the technique of rook statistics introduced by Chan, Haddadan, Hopkins, and Moci.

Keywords: rowmotion, homomesy, order ideal, antichain, toggleability statistics

1 Introduction

Rowmotion, one of the most studied actions in dynamical algebraic combinatorics, is an action on the distributive lattice of order ideals of a finite poset which sends an order ideal I of a poset $\mathcal{P}$ to the order ideal generated by the minimal elements of $\mathcal{P} \setminus I$. A large focus of research on rowmotion is concerned with identifying statistics that exhibit the property of homomesy under this action. Various authors have approached the study of homomesic statistics under rowmotion using toggleability statistics [3, 4, 7, 9, 10]. For a poset $\mathcal{P}$, $p \in \mathcal{P}$, and an order ideal I of $\mathcal{P}$, the toggleability statistic $\mathcal{T}_p(I)$ captures whether an element p is maximal in I, minimal in $\mathcal{P} \setminus I$, or neither. The order ideal indicator function $\mathbb{1}_p(I)$ indicates whether an element p is contained in I, while the antichain indicator function $\mathcal{T}_p^-(I)$ does the same for the antichain associated with I.

In [3], the authors study the subspaces of $\mathrm{Span}_{\mathbb{R}}(\mathbb{1}_p \mid p \in \mathcal{P})$ and $\mathrm{Span}_{\mathbb{R}}(\mathcal{T}_p^- \mid p \in \mathcal{P})$ consisting of elements which are linear combinations of toggleability statistics $\mathcal{T}_p$ and a constant, denoted $I_T(\mathcal{P})$ and $A_T(\mathcal{P})$, respectively. Every function in the spaces $I_T(\mathcal{P})$ and $A_T(\mathcal{P})$ is homomesic due to properties of toggleability statistics [9]. Restricting attention to products of two chains $[m] \times [n]$, shifted staircases $([n] \times [n]) \backslash \mathfrak{S}_2$, the type A root posets $\Phi^+(A_n)$, and the type B posets $\Phi^+(B_n)$, the authors of [3] conjectured values for both $\dim I_T(\mathcal{P})$ and $\dim A_T(\mathcal{P})$. Here, we provide a proof of their conjecture. Specifically, letting $\mathrm{rk}(\mathcal{P})$ denote the rank of a poset $\mathcal{P}$, we prove the following.

Theorem 1.1.

(*a*) *If* $\mathcal{P} = [m] \times [n]$ *for* $m, n \geq 2$, *then* $\dim I_T(\mathcal{P}) = \dim A_T(\mathcal{P}) = \mathrm{rk}(\mathcal{P}) + 1 = n + m - 1$.

(*b*) *If* $\mathcal{P} = ([n] \times [n]) \backslash \mathfrak{S}_2$ *for* $n \geq 2$, *then* $\dim I_T(\mathcal{P}) = \dim A_T(\mathcal{P}) = \mathrm{rk}(\mathcal{P}) + 1 = 2n - 1$.

(*c*) *If* $\mathcal{P} = \Phi^+(A_n)$ *for* $n \geq 2$, *then* $\dim I_T(\mathcal{P}) = \dim A_T(\mathcal{P}) = \mathrm{rk}(\mathcal{P}) + 1 = n$.

(*d*) *If* $\mathcal{P} = \Phi^+(B_n)$ *for* $n \geq 2$, *then* $\dim I_T(\mathcal{P}) = \dim A_T(\mathcal{P}) = \mathrm{rk}(\mathcal{P}) + 1 = 2n - 1$.

Moreover, we provide bases for each of the toggleability spaces $I_T(\mathcal{P})$ considered and, in proving Theorem 1.1, establish a number of results concerning $\dim I_T(\mathcal{P})$ and $\dim A_T(\mathcal{P})$ for more general posets $\mathcal{P}$. Through such results, we are led to two noteworthy extensions of Theorem 1.1.

Theorem 4.9. *For an integer partition* λ, *letting*

$$\mathcal{P}(\lambda) = \bigcup_{i=1}^{\ell} \{(i,j) \mid 1 \leq j \leq \lambda_i\}$$

with elements ordered lexicographically, N *denote the number of cells, and* C *the number of corner cells in the border strip of the Ferrers board of* λ, *we have that*

$$\dim I_T(\mathcal{P}) = \dim A_T(\mathcal{P}) = N - C.$$

Theorem 4.2. *Let* $\mathcal{P}$ *be the distributive lattice of order ideals of a finite, width-two poset. Then*

$$\dim A_T(\mathcal{P}) = \dim I_T(\mathcal{P}) = \mathrm{rk}(\mathcal{P}) + 1.$$

Our approach to Theorem 4.2 builds upon the technique of rook statistics introduced by Chan, Haddadan, Hopkins, and Moci in [2].

The remainder of the paper is organized as follows. In Section 2, we cover the requisite background from the theory of posets. Following this, the proof of Theorem 1.1 is split between Sections 3 and 4. In Section 3 we focus on the space $I_T(\mathcal{P})$, while in Section 4 we turn our focus to $A_T(\mathcal{P})$.

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2 Background

In this section, we cover the required background from the theory of posets. For more details, see [8, Chapter 3].

Recall that a **finite poset** $(\mathcal{P}, \preceq_{\mathcal{P}})$ consists of a finite set $\mathcal{P}$ along with a binary relation $\preceq$ between the elements of $\mathcal{P}$ which is reflexive, antisymmetric, and transitive. When no confusion will arise, we denote $(\mathcal{P}, \preceq_{\mathcal{P}})$ simply by $\mathcal{P}$, and $\preceq_{\mathcal{P}}$ by $\preceq$. For a poset $\mathcal{P}$ and $x, y \in \mathcal{P}$, if $x \preceq y$ and $x \neq y$, then we write $x \prec y$. In the case where $x \prec y$ and there exists no $z \in \mathcal{P}$ satisfying $x \prec z \prec y$, we refer to $x \prec y$ as a **cover relation** and write $x \lessdot y$. We refer to an element $x \in \mathcal{P}$ as **minimal** (resp., **maximal**) if there exists no $y \in \mathcal{P}$ satisfying $y \prec x$ (resp., $x \prec y$). All of the posets of interest in this paper are defined using “diagrams" as in [3].

A **diagram** is an array of finitely many cells in $\mathbb{N} \times \mathbb{N}$. Given a diagram D, one can associate a poset $\mathcal{P}(D)$ whose elements are the cells of D and where for $p_1, p_2 \in \mathcal{P}$, we have $p_1 \preceq p_2$ provided that the box associated with p_2 lies weakly southeast of the box associated with p_1. For a poset $\mathcal{P}$, if there exists a diagram D for which $\mathcal{P}$ is isomorphic to $\mathcal{P}(D)$, then we say that $\mathcal{P}$ is **defined** by the diagram D. When indexing cells in diagrams, we use matrix coordinates. We refer to a diagram as

- **connected** if there exists no pairs of cells which cannot be connected by a path of pairwise edge adjacent cells;
- **row convex** (resp., **column convex**) if between any two cells in the same row (resp., column), all cells between are contained as well; and
- **simply connected** if there exist no holes in the diagram, i.e., there exist no finite subsets S of $\mathbb{N} \times \mathbb{N}$ corresponding to empty positions in the diagram for which if $(i, j) \in S$, then each of $(i-1, j), (i+1, j), (i, j-1)$, and $(i, j+1)$ either belongs to S or corresponds to a cell in the diagram.

An example of a poset $\mathcal{P}$ defined by a diagram is provided in Example 2.1. See Example 2.1 for an illustration of these properties of diagrams.

Example 2.1. *Let $\mathcal{P} = \{a, b, c, d, e, f\}$ be the poset defined by the cover relations $a \lessdot c$; $b \lessdot c \lessdot d \lessdot f$; and $c \lessdot e \lessdot f$. Then $\mathcal{P}$ is defined by the diagram D illustrated in Figure 1 (a). The cells of D have been labeled by the corresponding elements of $\mathcal{P}$ to help make the identification clear.*

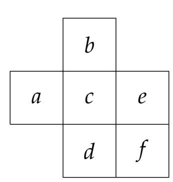


Figure 1: Diagrams defining a poset

For a simply connected diagram D, an **outward corner** of D associated to a cell $(i,j) \in D$ is a corner on the boundary of D of the form EN ($\lrcorner$) strictly to the northwest of (i,j) or a corner on the boundary of D of the form NE ($\ulcorner$) strictly to the southeast of (i,j). A diagram is said to have **no outward corners** if no cell in D has an outward corner.

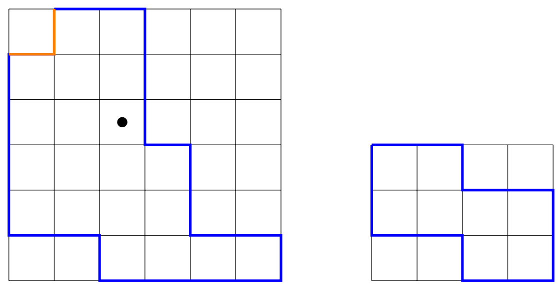
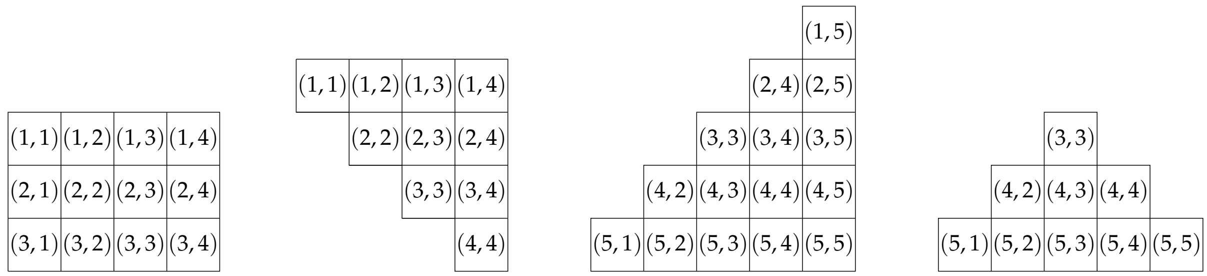


Figure 2: A diagram with an outward corner, colored in orange, associated to the cell (3,3) (left) and a diagram with no outward corners (right)

In this paper, we mainly focus on four families of posets. For $m,n \geq 2$, we denote by $[m] \times [n]$ the poset defined by the full $m \times n$ rectangular diagram. The remaining three families of posets of interest can all be viewed as certain induced subposets of $[n] \times [n]$ for $n \geq 2$:

- $([n] \times [n]) \backslash \mathfrak{S}_2$ for $n \geq 2$ denotes the poset defined by the diagram consisting of all cells in the full $n \times n$ square diagram lying on or above the diagonal of the square.
- $\Phi^+(A_n)$ for $n \geq 2$, denotes the posets defined by the diagram consisting of all cells in the full $n \times n$ square diagram lying on or below the anti-diagonal of the square.
- $\Phi^+(B_n)$ for $n \geq 2$, denotes the poset defined by the diagram consisting of all cells in the full $n \times n$ square diagram lying on or below both the diagonal and anti-diagonal of the square.

For a general poset $\mathcal{P}$, an **order ideal** is a subset $S \subseteq \mathcal{P}$ for which $x \in S$ and $y \prec x$ implies $y \in S$. Letting $\mathcal{J}(\mathcal{P})$ denote the set of order ideals of $\mathcal{P}$, with respect to posets $\mathcal{P}$ belonging to one of the four families defined above, our interest lies in two associated vector spaces which arise in the study of the dynamics of $\mathcal{J}(\mathcal{P})$ under the action of rowmotion. Recall from Section 1 that, if $I \in \mathcal{J}(\mathcal{P})$, then the **rowmotion** map $\mathrm{Row}(I)$ yields the order ideal generated by the elements of $\mathrm{Min}(P \setminus I)$. A statistic $\mathrm{st} : \mathcal{J}(\mathcal{P}) \to \mathbb{Z}_{\geq 0}$ is called d**-mesic** with respect to rowmotion if the average of st over every orbit of $\mathcal{J}(\mathcal{P})$ under rowmotion is equal to d. A statistic is **homomesic** if it is d-mesic for some d. Note that any linear combination of homomesic statistics will also be homomesic. Consequently, the collection of homomesic statistics forms a vector space.



(a) (b) (c) (d)

Figure 3: Diagrams defining (a) $[3] \times [4]$, (b) $([4] \times [4]) \backslash \mathfrak{S}_2$, (c) $\Phi^+(A_5)$, and (d) $\Phi^+(B_3)$

Many statistics of interest in the study of homomesy are defined as linear combinations of the following functions on order ideals. For a poset $\mathcal{P}$, define the **order ideal indicator function** $\mathbb{1}_p : \mathcal{J}(\mathcal{P}) \to \mathbb{R}$ and the **antichain indicator function** $\mathcal{T}_p^- : \mathcal{J}(\mathcal{P}) \to \mathbb{R}$ for $p \in \mathcal{P}$ defined by

$$\mathbb{1}_p(I) = \begin{cases} 1, & if\ p \in I \\ 0, & otherwise, \end{cases} \qquad \text{and} \qquad \mathcal{T}_p^-(I) = \begin{cases} 1, & \text{if } p \in \mathrm{Max}(I) \\ 0, & otherwise, \end{cases}$$

as well as the **toggleability statistic**, $\mathcal{T}_p : \mathcal{J}(\mathcal{P}) \to \mathbb{R}$ defined by

$$\mathcal{T}_p(I) = \begin{cases} 1, & if\ p \in \mathrm{Min}(\mathcal{P} \backslash I) \\ -1, & if\ p \in \mathrm{Max}(I) \\ 0, & otherwise. \end{cases}$$

Toggleability statistics are of special interest in the study of homomesy under rowmotion due to the following result by Striker, which implies that any linear combination of toggleability statistics and a constant is a homomesic statistic under rowmotion.

Lemma 2.2. *[9] For any $p \in P$, the statistic $\mathcal{T}_p$ is* 0*-mesic under rowmotion.*

Finally, for the aforementioned vector spaces of interest, given a finite poset $\mathcal{P}$, let $I_H(\mathcal{P})$ and $A_H(\mathcal{P})$ denote the subspaces of homomesic statistics within $\mathrm{Span}_{\mathbb{R}}(\mathbb{1}_p \mid p \in \mathcal{P})$ and $\mathrm{Span}_{\mathbb{R}}(\mathcal{T}_p^- \mid p \in \mathcal{P})$, respectively. The **toggleability spaces** of $\mathcal{P}$, denoted $I_T(\mathcal{P})$ and $A_T(\mathcal{P})$, are the collections of elements in $I_H(\mathcal{P})$ and $A_H(\mathcal{P})$, respectively, which are linear combinations of toggleability statistics $\mathcal{T}_p$ and a constant. We refer to $I_T(\mathcal{P})$ and $A_T(\mathcal{P})$ as the **order ideal** and **antichain toggleability spaces** of $\mathcal{P}$, respectively. Toggleability spaces were first considered in [3]. They are studied again in [7], where the authors completely describe the order ideal and antichain toggleability spaces for the family of fence posets. Here, as noted in Section 1, our main goal is to establish conjectured dimensions for the toggleability spaces associated with the four families of posets introduced above, detailed in Table 6.1 of [3].

3 Order Ideals

In this section, we focus on the conjectures of Defant, Hopkins, Poznanović, and Propp [3] concerning $\dim I_T(\mathcal{P})$ for posets $\mathcal{P}$ belonging to the four families of posets introduced in Section 2. We collect these results into Theorem 3.1 below.

Theorem 3.1.

(a) *If* $\mathcal{P} = [m] \times [n]$ *for* $m, n \geq 2$*, then* $\dim I_T(\mathcal{P}) = \mathrm{rk}(\mathcal{P}) + 1 = n + m - 1$.

(b) *If* $\mathcal{P} = ([n] \times [n]) \backslash \mathfrak{S}_2$ *for* $n \geq 2$*, then* $\dim I_T(\mathcal{P}) = \mathrm{rk}(\mathcal{P}) + 1 = 2n - 1$.

(c) *If* $\mathcal{P} = \Phi^+(A_n)$ *for* $n \geq 2$*, then* $\dim I_T(\mathcal{P}) = \mathrm{rk}(\mathcal{P}) + 1 = n$.

(d) *If* $\mathcal{P} = \Phi^+(B_n)$ *for* $n \geq 2$*, then* $\dim I_T(\mathcal{P}) = \mathrm{rk}(\mathcal{P}) + 1 = 2n - 1$.

Our proof of Theorem 3.1 was broken into two pieces: first we showed that $\dim I_T(\mathcal{P}) \leq \mathrm{rk}(\mathcal{P}) + 1$ and then $\dim I_T(\mathcal{P}) \geq \mathrm{rk}(\mathcal{P}) + 1$. For our proof that $\dim I_T(\mathcal{P}) \leq \mathrm{rk}(\mathcal{P}) + 1$ for the posets $\mathcal{P}$ of Theorem 3.1, we utilized Theorems 3.2 and 3.4 as well as Lemmas 3.5 and 3.6 below, which identify restrictions satisfied by elements of $I_T(\mathcal{P})$ for certain posets $\mathcal{P}$.

We start with Theorem 3.2, which concerns restrictions involving pairs of elements p_1 and p_4 in a poset $\mathcal{P}$ for which there exists $p_2, p_3 \in \mathcal{P}$ satisfying $p_1 \prec p_2 \prec p_4$ and $p_1 \prec p_3 \prec p_4$. Then, in Theorem 3.4, we show that Theorem 3.2 holds for all appropriate elements of posets defined by connected diagrams which are both row and column convex.

Theorem 3.2. *Let* $\mathcal{P}$ *be a poset with* $p_1, p_2, p_3, p_4 \in \mathcal{P}$ *satisfying* $p_1 \prec p_2 \prec p_4$ *and* $p_1 \prec p_3 \prec p_4$*. Setting* $S_i = Min(\mathcal{P} \backslash I(p_i))$ *for* $i \in [3]$*,* $S_{2,3} = Min(\mathcal{P} \backslash I(p_2, p_3))$*,* $S_2^1 = S_2 \backslash S_1$*, and* $S_3^1 = S_3 \backslash S_1$*, suppose that*

$$S_2^1, S_3^1, \{p_4\} \subseteq S_{2,3}, \quad S_1, S_{2,3} \setminus \{p_4\} \subseteq S_2 \cup S_3, \quad \textit{and} \quad S_{2,3} \cap S_1 = S_2 \cap S_3.$$

Then for

$$f = c + \sum_{p \in \mathcal{P}} c_p \mathcal{T}_p \in I_T(\mathcal{P}),$$

we have $c_{p_1} = c_{p_4}$.

Example 3.3. *Let $\mathcal{P}$ be the poset defined by the diagram D illustrated in Figure 4. For the elements $p_1, p_2, p_3, p_4 \in \mathcal{P}$ labeled in D, we have $p_1 \prec p_2 \prec p_4$ and $p_1 \prec p_3 \prec p_4$. Moreover, in the notation of Theorem 3.2,*

$$S_1 = \{s_1, s_3, p_3\}, \quad S_2 = \{s_1, p_3\}, \quad S_3 = \{s_1, s_3, s_6\}, \quad S_{2,3} = \{s_1, s_6, p_4\}, \quad S_2^1 = \emptyset, \quad S_3^1 = \{s_6\},$$

$$S_{2,3} \backslash (S_2^1 \cup S_3^1 \cup \{p_4\}) = \{s_1\}, \quad (S_2 \cap S_1) \backslash S_{2,3} = \{p_3\}, \quad \text{and} \quad (S_3 \cap S_1) \backslash S_{2,3} = \{s_3\}.$$

One can check that $p_1, p_2, p_3, p_4 \in \mathcal{P}$ satisfy the hypotheses of Theorem 3.2. Thus, applying Theorem 3.2, if $f = c + \sum_{p \in \mathcal{P}} c_p \mathcal{T}_p \in I_T(\mathcal{P})$, then $c_{p_1} = c_{p_4}$.

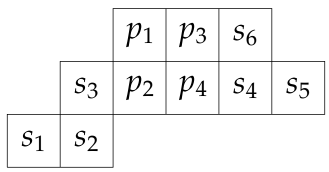


Figure 4: Example elements described in Theorem 3.2

Theorem 3.4. *Let $\mathcal{P}$ be a poset defined by a connected diagram which is both row and column convex. Suppose that $p_1, p_2, p_3, p_4 \in \mathcal{P}$ satisfy $p_1 \prec p_2 \prec p_4$ and $p_1 \prec p_3 \prec p_4$. Then for*

$$f = c + \sum_{p \in \mathcal{P}} c_p \mathcal{T}_p \in I_T(\mathcal{P}),$$

we have $c_{p_1} = c_{p_4}$.

Finally, Lemmas 3.5 and 3.6 concern further restrictions on elements of $I_T(\mathcal{P})$.

Lemma 3.5. *Let $\mathcal{P}$ be a poset, $M = \mathrm{Min}(\mathcal{P})$, and*

$$f = c + \sum_{p \in \mathcal{P}} c_p \mathcal{T}_p \in I_T(\mathcal{P}).$$

Then $c = -\sum_{p \in M} c_p$.

Lemma 3.6. *Let $\mathcal{P}$ be a poset defined by a connected diagram with $p, q_1, q_2 \in \mathcal{P}$ where*

$$\{r \in \mathcal{P} \mid r \preceq q_1\} \cap \{r \in \mathcal{P} \mid r \preceq q_2\} = \emptyset$$

and p is the unique element in $\mathcal{P}$ satisfying $q_1, q_2 \prec p$. Then for

$$f = c + \sum_{p \in P} c_p \mathcal{T}_p \in I_T(P),$$

we have $c_p = 0$.

Combining the results above allows us to prove that $\dim I_T(\mathcal{P}) \leq \mathrm{rk}(\mathcal{P}) + 1$ for the posets of Theorem 3.1. Moving to the proof of the opposite inequality, in Lemma 3.7 and Proposition 3.8 we identify certain elements of $I_T(\mathcal{P})$ for restricted posets $\mathcal{P}$.

Lemma 3.7. *Let $\mathcal{P}$ be a poset and $p \in \mathcal{P}$. If there exists unique $(r, q) \in \mathcal{P}^2$ for which $r \prec p \prec q$, then $\mathcal{T}_p \in I_T(\mathcal{P})$.*

Proposition 3.8. *Let $\mathcal{P}$ be a poset defined by a connected diagram which is both row and column convex, $M = \mathrm{Min}(\mathcal{P})$, and*

$$\mathcal{P}_\ell = \{(i, j) \in \mathcal{P} \mid i - j = \ell\}$$

for $\ell \in \mathbb{Z}$. Given $k \in \mathbb{Z}$, suppose that

(i) $\mathcal{P}_k, \mathcal{P}_{k\pm 1} \neq \emptyset$ and

(ii) $(i, j), (i + 1, j + 1) \in \mathcal{P}_k$ if and only if $(i, j + 1) \in \mathcal{P}_{k-1}$ and $(i + 1, j) \in \mathcal{P}_{k+1}$.

Then

$$f_k = \begin{cases} \displaystyle\sum_{\substack{(i,j)\in\mathcal{P} \\ i-j=k}} \mathcal{T}_{(i,j)}, & \mathcal{P}_k \cap M = \emptyset \\ \\ \displaystyle -1 + \sum_{\substack{(i,j)\in\mathcal{P} \\ i-j=k}} \mathcal{T}_{(i,j)}, & \mathcal{P}_k \cap M \neq \emptyset \end{cases}$$

belongs to $I_T(\mathcal{P})$.

Now, focusing on posets $\mathcal{P}$ in Theorem 3.1, we are able to identify collections of $\mathrm{rk}(\mathcal{P}) + 1$ linearly independent elements belonging to $I_T(\mathcal{P})$, finishing the proof of Theorem 3.1. Below, for the sake of space, we only include the result concerning $\mathcal{P} = \Phi^+(B_n)$.

Theorem 3.9. *If $\mathcal{P} = \Phi^+(B_n)$ for $n \geq 2$, then the following collection forms a basis of $I_T(\mathcal{P})$:*

$$\mathcal{B}_1 = \left\{ -1 + \sum_{\substack{(i,j)\in\mathcal{P} \\ i-j=2k}} \mathcal{T}_{(i,j)} : 0 < k \leq n - 1 \right\} \cup \left\{ \mathcal{T}_{(n,n)} - 1 \right\} \cup \left\{ \mathcal{T}_{(i,i)} : n < i \leq 2n - 1 \right\}.$$

Example 3.10. *For $\mathcal{P} = \Phi^+(B_3)$, we illustrate the basis $\mathcal{B}_1$ in Figure 5. Each diagram corresponds to an individual basis element with labels of cells p giving the coefficient of $\mathcal{T}_p$. The basis element corresponding to Figure 5 (c) requires the addition of the constant -1.*

Remark 3.11. *In the full version of this paper [1], Theorem 3.9 includes bases for the spaces $I_T(\mathcal{P})$ for all the posets of Theorem 3.1 given in terms of both toggleability statistics $\mathcal{T}_p$ as well as order ideal indicator functions $\mathbb{1}_p$.*

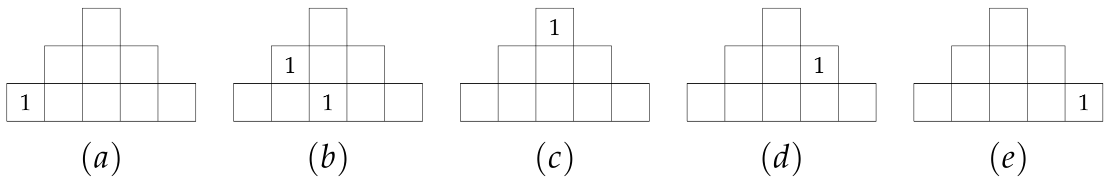


Figure 5: $\mathcal{B}_1$ for $\Phi^+(B_3)$

4 Antichains

In this section, we turn our attention to the antichain toggleability spaces $A_T(\mathcal{P})$ and discuss our approach to the antichain version of Theorems 3.1.

Theorem 4.1.

(a) *If* $\mathcal{P} = [m] \times [n]$ *for* $m, n \geq 2$, *then* $\dim A_T(\mathcal{P}) = \mathrm{rk}(\mathcal{P}) + 1 = n + m - 1$.

(b) *If* $\mathcal{P} = ([n] \times [n]) \backslash \mathfrak{S}_2$ *for* $n \geq 2$, *then* $\dim A_T(\mathcal{P}) = \mathrm{rk}(\mathcal{P}) + 1 = 2n - 1$.

(c) *If* $\mathcal{P} = \Phi^+(A_n)$ *for* $n \geq 2$, *then* $\dim A_T(\mathcal{P}) = \mathrm{rk}(\mathcal{P}) + 1 = n$.

(d) *If* $\mathcal{P} = \Phi^+(B_n)$ *for* $n \geq 2$, *then* $\dim A_T(\mathcal{P}) = \mathrm{rk}(\mathcal{P}) + 1 = 2n - 1$.

To start, we proved Theorem 4.2 below which generalizes Theorem 3.1 (a) and (b), and establishes that the equality $\dim A_T(\mathcal{P}) = \dim I_T(\mathcal{P})$ holds for posets defined by simply-connected diagrams with no outward corners. Such posets are exactly the distributive lattices of order ideals of finite, width-two posets.

Theorem 4.2. *Let $\mathcal{P}$ be a poset defined by a simply connected diagram D with no outward corners. Then* $\dim A_T(\mathcal{P}) = \dim I_T(\mathcal{P}) = \mathrm{rk}(\mathcal{P}) + 1$.

Our approach to Theorem 4.2 builds upon the technique of rook statistics introduced by Chan, Haddadan, Hopkins, and Moci in [2], which was later expanded by Hopkins [5, 6] and Defant, Hopkins, Poznanović, and Propp [3]. In the process of establishing Theorem 4.2, we find that Theorem 4.1 follows as a consequence of supporting results needed for the proof of Theorem 4.2.

First, we establish a relationship between $A_T(\mathcal{P})$ and $I_T(\mathcal{P})$ for certain posets $\mathcal{P}$. Ongoing, for $f, g : \mathcal{J}(\mathcal{P}) \to \mathbb{R}$, we write $f \equiv g$ to express that $f - g \in \mathrm{Span}(\mathcal{T}_p \mid p \in \mathcal{P})$.

Lemma 4.3. *Let $\mathcal{P}$ be a poset for which $\mathbb{1}_p \equiv \sum_{q \geq p} c_{q,p} \mathcal{T}_q^-$ with $c_{p,p} \neq 0$. Then* $\dim A_T(\mathcal{P}) = \dim I_T(\mathcal{P})$.

Next, we define certain combinations of the operators $\mathcal{T}^-_{(k,\ell)}$ and $\mathcal{T}^+_{(k,\ell)}$ for posets $\mathcal{P}$ defined by simply connected diagrams with no outward corners and specified elements $(i,j) \in \mathcal{P}$.

Definition 4.4. *For D a simply connected diagram with no outward corners and $(i,j) \in D$, define the rook statistic*

$$R^D_{(i,j)} = \sum_{i' \leq i,\ j' \leq j} \mathcal{T}^+_{(i',j')} + \sum_{i' \geq i,\ j' \geq j} \mathcal{T}^-_{(i',j')} - \sum_{\substack{(i'+1,j'),(i',j'+1) \in D \\ i'<i,\ j'<j}} \mathcal{T}^-_{(i',j')} - \sum_{\substack{(i'-1,j'),(i',j'-1) \in D \\ i'>i,\ j'>j}} \mathcal{T}^+_{(i',j')}$$

where $\mathcal{T}^{(\pm)}_{(\ell,k)} = 0$ if $(\ell,k) \notin D$. In addition, define $\tilde{R}^D_{(i,j)}$ to be the element of span$(\mathcal{T}^-_D)$ that is $\equiv -$ equivalent to $R^D_{(i,j)}$. See Figure 6 for an illustration of such a rook statistic.

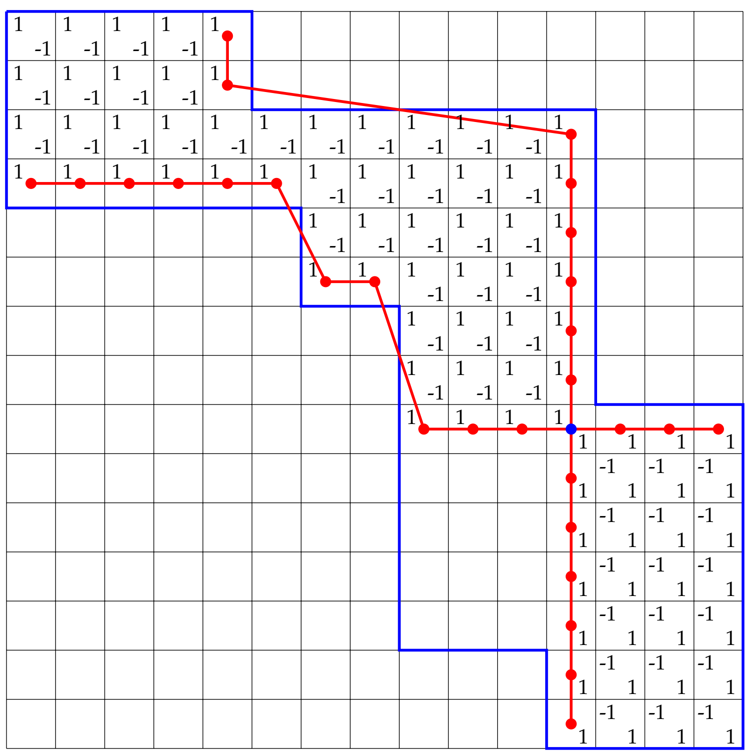


Figure 6: A rook for a shape with no outward corners. The associated reduced rook has a coefficient of 1 for the antichain indicator function for every cell with a red circle in it, and the cell with a blue circle has a coefficient of 2 for the associated antichain indicator function.

Now, Lemma 4.5 and Theorem 4.7 below establish properties of the functions $R^D_{(i,j)}$ and $\tilde{R}^D_{(i,j)}$. A consequence of Lemma 4.5 is essential to prove Theorem 4.1,

Lemma 4.5. *For a simply-connected diagram D with no outward corners and $(i,j) \in D$, we have $R^D_{(i,j)} = 1$.*

Corollary 4.6. *Let $\mathcal{P}$ be a poset defined by a simply-connected diagram D where for every cell (i,j) in the diagram D, there are no outward corners to the South East of (i,j). Then* $\dim A_T(\mathcal{P}) = \dim I_T(\mathcal{P})$

Theorem 4.7. *For $\mathcal{P}$ a poset defined by a simply connected diagram D with no outward corners, let B be the set of boxes corresponding to the maximal chain of $\mathcal{P}$ obtained by always going south if possible and east otherwise. Then the set $\{\tilde{R}^D_{(i,j)} : (i,j) \in B\}$ is linearly independent.*

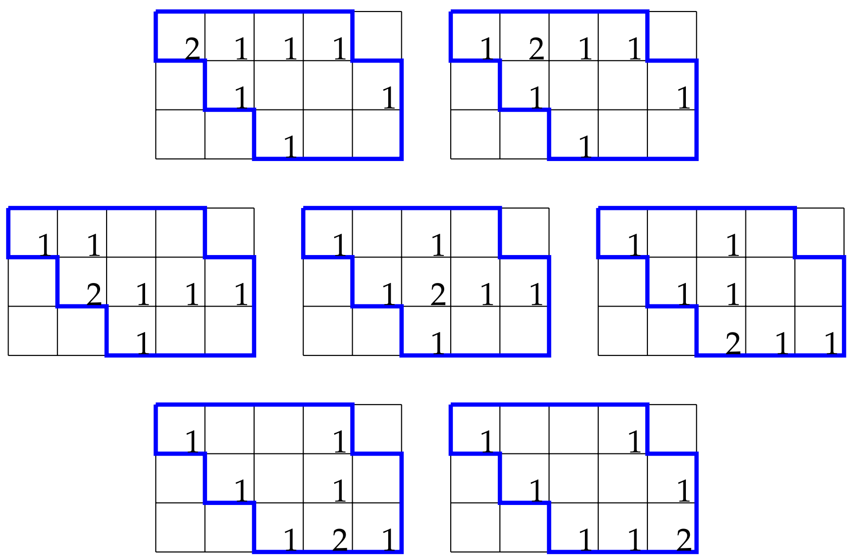


Figure 7: An example of the basis of reduced rooks in Theorem 4.7

Finally, to finish, we discuss one additional result we obtain as a consequence of the work above. Recall that for an **integer partition** $\lambda = (\lambda_1 \geq \ldots \geq \lambda_\ell) \in \mathbb{Z}^\ell_{>0}$, the **Ferrers diagram** $F(\lambda)$ of λ is the diagram consisting of the cells $\{(i,j) \mid 1 \leq i \leq \ell,\ 1 \leq j \leq \lambda_i\}$. The **border strip** of $F(\lambda)$, denoted $B(\lambda)$, is the collection of all cells $(i,j) \in F(\lambda)$ for which $(i+1,j+1) \notin F(\lambda)$. We refer to $(i,j) \in B(\lambda)$ as a **corner cell** if $(i+1,j),(i,j+1) \in B(\lambda)$. To the integer partition λ, we associate the poset $\mathcal{P}(\lambda)$ defined by $F(\lambda)$, as in Example 4.8.

Example 4.8. *The diagram $F(\lambda)$ defining the poset $\mathcal{P} = \mathcal{P}(\lambda)$ for $\lambda = (5,2,1,1)$ is illustrated in Figure 8, where the cells are labeled by the associated elements of $\mathcal{P}$. Note that $B(\lambda)$ corresponds to the collection of all cells in $F(\lambda)$ except for the one labeled $(1,1)$. Among these cells, the ones corresponding to $(1,2)$ and $(2,1)$ are the corner cells.*

With the notation above, the last main result of this section is as follows.

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$
$(2,1)$	$(2,2)$			
$(3,1)$				
$(4,1)$				

Figure 8: Diagram defining $\mathcal{P}(\lambda)$ for $\lambda = (5,2,1,1)$

Theorem 4.9. *Let λ be an integer partition, $\mathcal{P} = \mathcal{P}(\lambda)$, N denote the number of cells in $B(\lambda)$, and C denote the number of corner cells in $B(\lambda)$. Then $\dim I_T(\mathcal{P}) = \dim A_T(\mathcal{P}) = N - C$.*

Example 4.10. *For the poset $\mathcal{P} = \mathcal{P}(\lambda)$ of Example 4.8, using the notation of Theorem 4.9, we have $N = 8$ and $C = 2$. Thus, applying Theorem 4.9, we have $\dim I_T(\mathcal{P}) = 8 - 2 = 6$.*

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