Sphere Packings, Lattices, and Kissing Configurations in $\mathbb{R}^n$

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Outline

What is a ball in $\mathbb{R}^n$?

- In $\mathbb{R}^n$, a ball with radius $r > 0$ that is centered at a point $x = (x_1, \ldots, x_n)$ is given by the set $\{y \in \mathbb{R}^n : ||x - y|| \leq r\}$

  where $||x - y|| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.

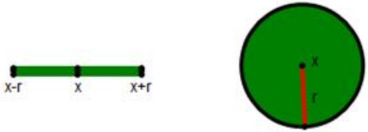


Figure: Balls in $\mathbb{R}$ and $\mathbb{R}^2$

- The boundary of a ball with radius r forms a *sphere* (with radius r) and consists of the points whose distance from the center is exactly r.

A natural geometric question. . .

The Sphere Packing Problem:

How densely can identical balls be packed into $\mathbb{R}^n$ so that their interiors don't overlap? (Such a configuration of balls is called a *sphere packing*.)

packing density = proportion of space covered by balls

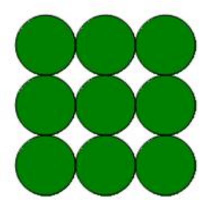


Figure: Sphere packing in 2 dimensions

A few reasons to care about Sphere Packings

- Stacking produce efficiently: how should a grocer wanting to maximize the amount of product available to customers choose to stack its oranges in the produce section?
- Sphere packings in $\mathbb{R}^n$ are the continuous analogue of error-correcting codes of length n
- Sphere packings are related to other areas of mathematics such as number theory and algebra

Example: communication over a noisy channel

- Let $\mathcal{S}$ denote a set of *signals* represented by points $x \in \mathbb{R}^n$, each satisfying $|x| \leq r$.
- A signal is an individual transmission at a given time. A message consists of a stream of signals.
- When one transmits $x \in \mathcal{S}$ over a noisy channel it is often the case that $y \neq x$ is received at the other end.
- However, it is likely that $|x - y| < \varepsilon$ where ε can be regarded as the noise level of the channel.

Communication over a noisy channel (continued)

How can we communicate without error?

- We can build our signal set S (i.e., our vocabulary) so that the distance between any two signals is at least 2ε.
- This will allow us to 'correct' any errors introduced during transmission.

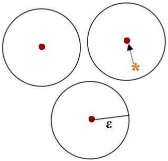


Figure: Good Signal Set

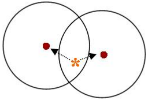


Figure: Signals Too Close

Communication over a noisy channel (continued)

- For reliable communication we want to ensure that each pair of signals is separated by distance of at least 2ε.
- However, for rapid communication we want to have a large vocabulary $\mathcal{S}$.
- This is sphere packing!!!
- Here we are trying to pack ε-balls into an r-ball as densely as possible.

Connections to number theory

- The sphere packing problem is closely related to number theoretic questions involving lattices in $\mathbb{R}^n$ (see below) and real positive-definite quadratic forms in n variables.

- A **lattice** Λ in $\mathbb{R}^n$ is the integral span of a vector space basis, i.e., for some vector space basis $v_1, \ldots, v_n$ for $\mathbb{R}^n$

$$\Lambda = \{k_1 v_1 + \cdots + k_n v_n : k_1, \ldots, k_n \in \mathbb{Z}\}.$$

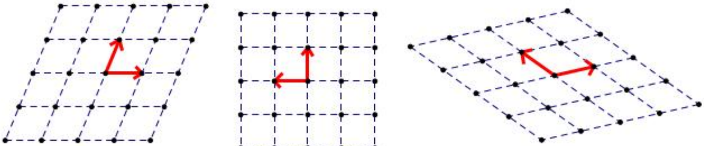
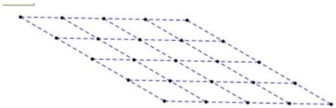


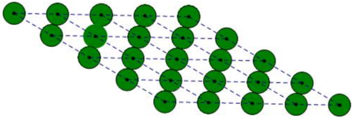
Figure: Lattice examples in $\mathbb{R}^2$

Using lattices to pack spheres in $\mathbb{R}^n$

From a lattice



to a sphere packing



The minimal distance between lattice points determines the maximum radius of the balls that can be centered onto each lattice point.

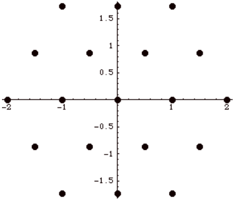


Figure: Hexagonal lattice generated by $\{(1,0),(\frac{1}{2},\frac{\sqrt{3}}{2})\}$

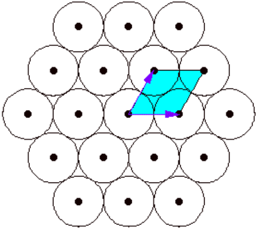


Figure: Hexagonal Lattice Sphere Packing and fundamental region

Using lattices to pack spheres in $\mathbb{R}^n$ (continued)

- For a lattice Λ in $\mathbb{R}^n$ define

$$\begin{aligned} ||\Lambda|| &= \text{length of shortest non-zero vectors in } \Lambda \\ \mathrm{disc}(\Lambda) &= \text{volume of a lattice fundamental region} \end{aligned}$$

- The density of a the sphere packing of a lattice Λ is given by the formula

$$\textbf{Density} = \frac{\textit{Volume of a ball with radius } \ \frac{1}{2}||\Lambda||}{\mathrm{disc}(\Lambda)}$$

- Note that $\mathrm{disc}(\Lambda)$ can be easily computed using any *generating matrix* M for Λ (i.e, a matrix whose rows consist of the coordinates of a lattice basis for Λ). More specifically, $\mathrm{disc}(\Lambda) = |\det(M)|$.

Known results for the Sphere Packing problem

- $n = 1$ (balls are closed intervals!)

  Optimal density $= 1$

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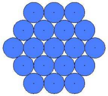

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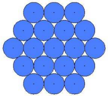
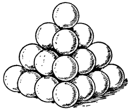


- $n = 3$
  - Kepler's Conjecture (1611)



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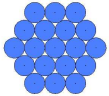

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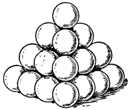

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  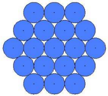

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  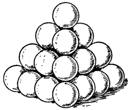

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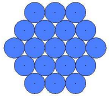

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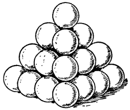

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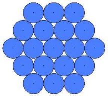

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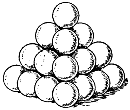

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- $n \geq 4$

  ???

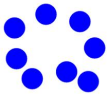
The Lattice Sphere Packing Problem

- While not all sphere packings are lattice packings, many of the densest sphere packings are.
- The lattice sphere packing problem is still very hard and the densest lattice sphere packings known only for dimensions $n \leq 8$ and $n = 24$.
- Note that the densest known lattices have lots of symmetries. In other words, the automorphism groups of the lattices are very large. This is especially true in dimensions 8 and 24 for the root lattice $\mathbb{E}_8$ and the Leech lattice Λ_{24}.

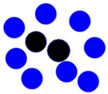
Question: Given that not general solution is likely to exist for the Sphere Packing problem or even the Lattice Sphere Packing problem, how can we attempt to solve these problems in high dimensions?

A simple lower bound for the optimal density in $\mathbb{R}^n$

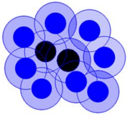
- Start with a sphere packing of unit balls.



- Add more unit balls into the packing until you run out of room.



- The resulting packing $\mathcal{P}$ has the property that if the sphere radii are doubled (while the sphere centers remain fixed) then one obtains a cover $\mathcal{P}'$ of $\mathbb{R}^n$. WHY? Thus the density of $\mathcal{P}$ is at least $\frac{1}{2^n}$



More lower bounds for the optimal sphere packing density in $\mathbb{R}^n$

- Minkowski (1905) $\frac{\zeta(n)}{2^{n-1}}$, where $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$.
- Rogers (1947) $\frac{n\zeta(n)}{2^{n-1}e(1-e^{-n})}$
- Ball (1992) $\frac{(n-1)\zeta(n)}{2^{n-1}}$
- Vance (2008) $\frac{3n}{2^{n-1}e(1-e^{-n})}$, for dimensions $n = 4m$

Note that $\zeta(n)$ is close to 1 for large values of n.

Why is the Sphere Packing Problem so difficult?

- Not all sphere packings have a density that is readily computable
- Many local maxima
- Each dimension behaves differently
- For the lattice version, while on can easily compute the volume of a fundamental lattice region (i.e., $\mathrm{disc}(\Lambda)$), calculating the length of the shortest non-zero vectors in a lattice extremely difficult and becomes incredibly complex as the dimension n increases.
- Hard to rule out implausible sphere configurations

The Kissing Number Problem:

How many unit balls (i.e., balls with radius 1) can be arranged tangent to a central unit ball so that their interiors do not overlap? (Note that such a configuration is called a *kissing configuration*)

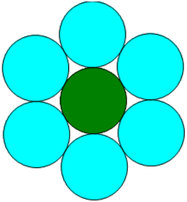


Figure: An optimal kissing configuration in $\mathbb{R}^2$

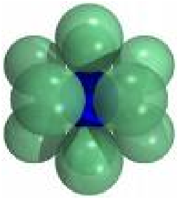
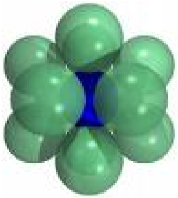


Figure: Kissing configuration in $\mathbb{R}^3$

- In $\mathbb{R}^3$, this problem is known as the *thirteen spheres problem* and can be restated as: how many billiard balls can be arranged around a central billiard ball so that the outer balls "kiss" the center ball?



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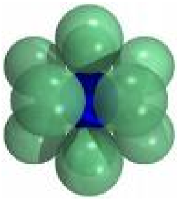

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- In 1694 Newton argued 12 balls and Gregory argued 13 balls
- In Newton proved correct in 1874 (nearly 200 years later!)

The kissing number for dimensions $1-4$, 8, and 24

Dimension	Kissing Number	year proven
1	2	
2	6	
3	12	1874
4	24	2003
8	240	1979
24	196560	1979

Question: Why do we know more about the geometry of $\mathbb{R}^8$ and $\mathbb{R}^{24}$ than say $\mathbb{R}^4$ and $\mathbb{R}^5$?

Possible research projects

- Continue working to improve on the lower bounds known for the optimal sphere packing density in high dimensions by looking at lattice sphere packings with interesting symmetries

- Further explore the connections between binary error-correcting codes, lattices, and kissing configurations.

- Consider more general packing problems (i.e., pack non-spherical convex bodies)

References

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- S. Vance, *A Mordell inequality for lattices over maximal orders*, to appear in Transactions of the AMS.
- S. Vance, *New lower bounds for the sphere packing density in high dimensions*, preprint.