

The automorphism groups of some noncommutative algebras

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A joint work with Secil, Palmieri and Zhang

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- I. Backgrounds.

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- III. Main results.
- The automorphism group of filtered algebras
 - The automorphism group of (-1) -quantum Weyl algebras
 - The automorphism group of W_n and $W_n[t]$
 - The automorphism group of the product of quantum Weyl algebras
 - The automorphism group of skew polynomial algebras

I. Backgrounds

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- 1992, Alev and Chamarie: Described the automorphism group of skew polynomial ring $k_q[x_1, \dots, x_n]$, where q is not a root of unity and $n \geq 2$.
- 1996, Alev and Dumas: The group of automorphisms of the first quantum Weyl algebra is an algebraic torus.

- 2001, Bavula and Jordan, Isomorphism problems and groups of automorphism for generalized Weyl algebras.

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- 2004, Shestakov and Umirbaev, The tame and the wild automorphisms of polynomial rings in three variables.
- 2008, Andruskiewitsch and Dumas, On the automorphisms of $U_q^+(g)$ (the positive part of the quantum enveloping algebra of a Lie algebra g).

- 2012, Suarez-Alvarez and Vivas, Automorphisms and isomorphisms of quantum generalized Weyl algebras.

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- 2012, Yakimov, The Andruskiewitsch-Dumas conjecture,
- 2012, Yakimov, The Launois-Lenagan conjecture.
- 2013, Goodearl and Yakimov, Unipotent and Nakayama automorphisms of quantum nilpotent algebras.

II. Definitions

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- Let B be an R -algebra and F be a commutative R -subalgebra of B such that $B_F := B \otimes_R F$ is finitely generated free over F . Then left multiplication defines a natural embedding of R -algebras $lm : B \rightarrow \text{End}_F(B_F) \cong M_n(F)$ where n is the rank $rk(B/F)$. **Regular trace**: $tr_{reg} : B \xrightarrow{lm} M_n(F) \xrightarrow{tr_{int}} F$.

Discriminant

- Let $tr : B \rightarrow F$ be a trace map and w be a fixed integer. Let $Z := \{z_i\}_{i=1}^w$ be a subset of B . The **discriminant** of Z is defined to be

$$d_w(Z : tr) = \det(tr(z_i z_j))_{w \times w} \in F.$$

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- The **w -discriminant ideal** (or w -discriminant R -module) $D_w(B, tr)$ is the R -submodule of F generated by the set of elements $d_w(Z : tr)$ for all $Z = \{z_i\}_{i=1}^w \subset B$.

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- Suppose B is an R -algebra which is finitely generated free over R . If Z is an R -basis of B , the **discriminant** of B is defined to be

$$d(B/R) = R^\times \cdot d_w(Z : tr),$$

where $R^\times$ is the set of units in R .

Locally Dominating

Suppose that $Y = \bigoplus_{i=1}^n kx_i$ generates A ($F_n A = (k \oplus Y)^n$) as an algebra.

A nonzero element $f := f(x_1, x_2, \dots, x_n) \in A$ is called **locally dominating** if, for every $g \in \text{Aut}(A)$, one has

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- if, further, $\deg y_{i_0} > 1$ for some i_0 , then $\deg f(y_1, \dots, y_n) > \deg f$.

Dominating

Assume that $\text{gr } A$ is a connected graded domain. A nonzero element $f \in A$ is called **dominating** if, for every filtered PI algebra T with $\text{gr } T$ a connected graded domain, and for every subset of elements $\{y_1, \dots, y_n\} \subset T$ that is linearly independent in the quotient k -module $T/F_0 T$, there is a lift of f , say $f(x_1, \dots, x_n)$, in the free algebra $k\langle x_1, \dots, x_n \rangle$, such that the following hold:

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- if, further, $\deg y_{i_0} > 1$ for some i_0 , then $\deg f(y_1, \dots, y_n) > \deg f$.
- We say the discriminant of A over R is **dominating** (**respectively, locally dominating**) if the discriminant ideal $D_w(A, \text{tr})$ is a principal ideal of R generated by a dominating (respectively, locally dominating) element.

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 - (IV) The morphisms in this category are isomorphisms of algebras.
- An algebra automorphism g of A is said to be **affine** if $\deg g(x_i) = 1$ for all i , or equivalently, $g(Y) \subset Y \oplus k$. If every $g \in \text{Aut}(A)$ is affine, we call $\text{Aut}(A)$ *affine*.

Triangular automorphism

- **Triangular automorphism:** For any $g \in \text{Aut}(A)$, $c \in k^\times$ and $r \in R$, the map

$$\sigma : t \rightarrow ct + r, \quad x \rightarrow g(x), \quad \text{for all } x \in A$$

is called a triangular automorphism of $A[t]$.

III. Main results

Automorphisms in Af

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$$\partial(x) = 0 \quad \text{for all } x \in A, \quad \partial(t) = r \quad \text{for some } r \in R.$$

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- $Aut(A)$ is an algebraic group that fits into an exact sequence

$$1 \rightarrow (k^\times)^r \rightarrow Aut(A) \rightarrow S \rightarrow 1$$

for some finite group S .

Automorphisms of filtered algebras

- **Theorem 2.** Assume k is a field of characteristic 0. Let A be a filtered algebra, finite over its center, such that the associated graded ring $\text{gr } A$ is a connected graded domain. Suppose that the v -discriminant $d_v(A/C(A))$ is dominating for some $v \geq 1$. Then the following hold.

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- **Theorem 2.** Assume k is a field of characteristic 0. Let A be a filtered algebra, finite over its center, such that the associated graded ring $\text{gr } A$ is a connected graded domain. Suppose that the v -discriminant $d_v(A/C(A))$ is dominating for some $v \geq 1$. Then the following hold.
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- **Theorem 4.** Assume that $\text{char } k \neq 2$. Let n be an even integer ≥ 4 and G be any group acting on W_n . Then the fixed subring W_n^G under the G -action is filtered Artin-Schelter Gorenstein.

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- $\text{Aut}(W_2[t_1^{\pm 1}, \dots, t_m^{\pm 1}]) = (S_2 \rtimes (k[t_1, \dots, t_m])^\times) \times ((k^\times)^m \rtimes \{\pm 1\}).$

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- If $\mathbb{Q} \subseteq k$, then every locally nilpotent derivation of W_n is zero.

Quantum Weyl algebra $V_n(\mathcal{A})$

- Let $\mathcal{A} := \{a_{ij} \mid 1 \leq i < j \leq n\}$ be a set of scalars in k . Define the (-1) -quantum Weyl algebra $V_n(\mathcal{A})$ to be generated by $\{x_1, x_2, \dots, x_n\}$ subject to the relations

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- If $a_{ij} = 0$ for all $i < j$, then this algebra is denoted by $k_{-1}[x_1, \dots, x_n]$.

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- If $a_{ij} = 1$ for all $i < j$, we get the algebra W_n .

Properties of Quantum Weyl algebra $V_n(\mathcal{A})$

- $V_n(\mathcal{A})$ is an iterated Ore extension
 $k[x_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n]$ where $\sigma_j : x_i \mapsto -x_i$ and
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- $V_n(\mathcal{A})$ is a filtered algebra with associated graded ring $\text{gr } V \cong k_{-1}[x_1, \dots, x_n]$.
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- The center of $k_{-1}[x_1, \dots, x_n]$ is

$$\begin{cases} k[x_1^2, \dots, x_n^2] & \text{if } n \text{ is even,} \\ k[x_1^2, \dots, x_n^2, \prod_i x_i] & \text{if } n \text{ is odd.} \end{cases}$$

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- If n is even, the center of V is $R := k[x_1^2, \dots, x_n^2]$, and V is finitely generated free over R of rank 2^n .

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- If n is even, then $V_n(\mathcal{A})$ is in Af .

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- If $\mathbb{Q} \subseteq k$, then every locally nilpotent derivation of $k_{-1}[x_1, \dots, x_n]$ is zero.

Automorphisms of product of quantum Weyl algebras

- **Theorem 8.** Let k be a field. Let $B = A_{q_1} \otimes \cdots \otimes A_{q_m}$ and assume that $q_i \neq 1$ for all $i = 1, \dots, m$. Then the following hold.

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- Every locally nilpotent derivation is zero.

Automorphisms of skew polynomial algebra

- An automorphism g of $k_{p_{ij}}[x_1, \dots, x_n]$ is called **elementary** if there is an s and an element f generated by $x_1, \dots, \widehat{x_s}, \dots, x_n$ such that

$$g(x_i) = \begin{cases} x_i & i \neq s \\ x_s + f & i = s. \end{cases}$$

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- An automorphism of $k_{p_{ij}}[x_1, \dots, x_n]$ is called **tame** if it is generated by affine and elementary automorphisms.
- **Theorem 10.** Let $A = k_{p_{ij}}[x_1, \dots, x_n]$ be a (not necessarily PI) skew polynomial algebra over the commutative domain k , and suppose that x_i is not central in A for all i . Let s_0 be some integer between 1 and n . Suppose that T_s is finite for all $s \neq s_0$. Then every automorphism of A is tame.

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Thank you!