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Pointed Hopf actions on commutative domains

joint w/ P. Ettinger

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$k = \overline{k}$, $\text{char } k = 0$

A study in NCIT:

(such actions have been examined since the early 1800s)

Classic IT:

G ^{faithful}
(sometimes) finite group

$R = k[x_1, \dots, x_n]$
poly'l ring

& study R^G , $R \rtimes G$

and various other constructions & properties arising from action

(previous talks of conference)
outline

NCIT
(for this talk)

H

^{unfaithful}

($\exists$ induced action of $H/I \cong A$ for $I \neq 0$ Hopf ideal of H)

A

finite dim'l Hopf algebra

'quantum poly'l ring'

& study (w/o unfaithful-ness)

$A^\#$, $A \# H$

etc, etc.

(actions not well understood)

(results assume that a Hopf exists)

Basic Question: When does such a Hopf action exist?

Lots of choices for A

Important from viewpoint of comm alg/ AG

(Commutative)

(noncom)

(PI)

(not PI)

(Artin-Schelter reg)

(domain)

(not domain)

(graded) (a quantization/deformation of R)
(any algebra, that looks like R)

Traditionally two choices for H

Semisimple

(as an $\mathbb{K}$ -alg)

pointed

(all simple H -comods are 1-dim)

$\Leftrightarrow H^*$ is basic

this story is much more complicated



Theorem [Teturov-10, 2014]

If H is a semisimple Hopf algebra that acts inner faithful on a commutative domain,

then H is cocommutative (so, $H \cong \mathbb{K}G$, G finite group).

Program led by Andruskiewitsch & Schneider to classify all finite dim'd pointed ^{Hopf} algebras H .

$$\{x \in H \mid \Delta(x) = g \otimes x + x \otimes g', g, g' \in G(H)\}$$

Conj [AS] 2000 Such H are generated by group-like & skew primitive elts.

$$\{g \in H \mid \Delta(g) = g \otimes g'\}$$

$\equiv G(H)$

Theorem [Angiono, 2013] Conjecture is true for H w/ $G(H)^{=G}$ abelian.

so we assume this.

$\leadsto$ All such H are 'lifts of' $\underbrace{B(V)}_P \# \mathbb{K}G$ ($\text{gr } H \cong B(V) \# \mathbb{K}G$)

Nichols algebra, not a Hopf alg, but a braided Hopf algebra in $\mathcal{G}\text{-YD}$

G -graded G -mods.

③

$B(V) \leftarrow$ hard to understand / present
 $\parallel$

$\mathbb{K}\langle x_1, \dots, x_n \rangle$

x_i are the skew primitive elts of H

$\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1$

$\rightarrow I$

hard to understand

Restrict again to H of finite Cartan type $\leftarrow$ all variations of small quantum groups $u_q(g)$

Here, $g_i x_j = q_{ij} x_j g_i$

with

$q_{ij} q_{ji} = q^{a_{ij}}$

for (a_{ij}) a Cartan matrix of a finite root system

Claim

finite root system $\rightarrow$ Cartan type

Example

$H = u_q(sl_2)$
 $m = \text{ord}(q^2)$

gen by

$k, e, f \in EG(H)$

e, f

$\Delta(e) = k \otimes e + e \otimes 1$

k, f

$\Delta(k) = k \otimes k + k f \otimes 1$

$ke = q^2 ef$

$kf = q^{-2} fk$

$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}$

$k^m = 1, e^m = f^m = 0$

$\text{so } (q_{ij}) = \begin{pmatrix} q^2 & q^{-2} \\ q^2 & q^{-2} \end{pmatrix}$

$q_{12} q_{21} = q_{11}^0$

$q_{21} q_{12} = q_{22}^0$

$a_{ij} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

finite Cartan. $u_q(sl_2)$ is of type $A_1 \times A_1$ ✓

finite (dim = m^3)

pointed by $\langle g, x \rangle$ skew-primitive
 $\uparrow$
 grouplike

Boils down to actions on fields.

Lemma A commutative domain, $\mathbb{Q}_A$ quotient field.

If $H \curvearrowright A$ inner faithfully, then $H \curvearrowright \mathbb{Q}_A$ inner faithfully.

[Pf/ $H \simeq A \simeq H \simeq A \tilde{J}^{-1}$ $\tilde{J}$ multip closed subset of $A^\#$
 A integral over $A^\#$ [Skogabin]
 $J = \text{nonzero elts of } A$ get $H \simeq A \tilde{J}^{-1} \simeq H \simeq A J^{-1} = Q_A$
 $A \otimes_{A^\#} A^\# \tilde{J}^{-1}$]

Definition A Hopf algebra H/K is Galois-theoretical if H acts
 free-faithfully & K -linearly on a field containing K .

Proposition

- ① Any finite group alg is GT
- ② H semisimple & GT $\Rightarrow H \simeq K[G]$
- ③ GT preserved under taking
 - * Hopf subalgebra
 - * $\otimes$
 - * Hopf dual
 - * triesto $\begin{cases} 2\text{-cocycle (triv. aut)} \\ \text{Drinfeld (mod } A) \end{cases}$

Theorem: H fin dim'l pointed GT Hopf alg w/ H -module field L . Then

- ① $L^\# = L^{G(H)}$ (skew primitive elts of H act by zero on $L^{G(H)}$)
- ② $L^\# \subseteq L$ is Galois w/ Galois group $G(H)$.

have generalizations for actions on commutative domains
 on Azumaya algebras.

	on commutative domain A	on Azumaya algebra A w/ $Z = Z(A)$
H finite dim'l action $\nearrow$	$A^\# = A^{H_0}$	$Z \cap A^\# = Z \cap A^{H_0} \quad (Z^\# = Z^{H_0})$
H finite dim'l & pointed ($H_0 = (K[G])$)	$A^\# = A^G$	$Z^\# = Z^G$

Theorem

① The following are examples of findim'l pointed GT Hopf algebras of finite Cartan type.

<u>GT Hopf algebra</u>	<u>finite Cartan type</u>
* $T(n)$ (Taft alg)	A_1
* $E(n)$ (dim 2^{n+1})	$A_1^{\times n}$
* $h(p, 1)$ (boson alg)	$A_1 \times A_1$
* H_{81} (81-dim'l)	A_2
* $u_q(sl_2) \rightarrow$ $\left\{ \begin{array}{l} k \cdot z = q^{-2}z \\ e \cdot z = 1 \quad f \cdot z = -qz^2 \end{array} \right.$	$A_1 \times A_1$
* $u_q(gl_2)$	$A_1 \times A_1$
* two Drinfeld $\left\{ \begin{array}{l} u_q(gln) \\ u_q(sln) \end{array} \right.$ twists of $n \geq 2$	$A_{n-1} \times A_{n-1}$
* $2^{\text{rank}(g)-1}$ twists of $u_q^{\geq 0}(g)$ finit dim'l simple Lie algebras	Same type as g .

② The following are non-examples of findim'l pointed GT Hopf algs of finite Cartan type

<u>Non-GT Hopf alg</u>	<u>finite Cartan type</u>
generalized Taft algebras $T(n, m, \alpha) \neq T(n)$	A_1
$h(p, p)$ for $p \neq 1$	$A_1 \times A_1$
$gr(u_q(sl_2))$	$A \times A_1$

Preview of Part II of paper: These ^{H in ①} are essentially it in some sense ---

(6)

Some questions to consider. (in the H-ame setting)

① Quasiclassical analogue of $H \xrightarrow{\quad} A$
 H finite dim A commutative domain

G = algebraic group w/ Poisson bracket (not nec. zero).

X = irreducible algebraic variety w/ zero Poisson bracket.

Determine which G can act on X & classify all such $G \curvearrowright X$.

* This includes the problem of classifying 'Poisson homog. spaces'
 $X = G/G'$ (G' = closed subgroup of G)

② (Alexandru Chirvasitu ← postdoc at UW next year, UC Berkeley student)
Compact Quantum Groups (CQG)

Conjecture (Das-Goswami-Jordan)

If H CQG acts faithfully on a connected compact ^{smooth} manifold,
 then H a classical compact group