Defining a Notion of Non-commutative Complete Intersection via Base-Point Modules
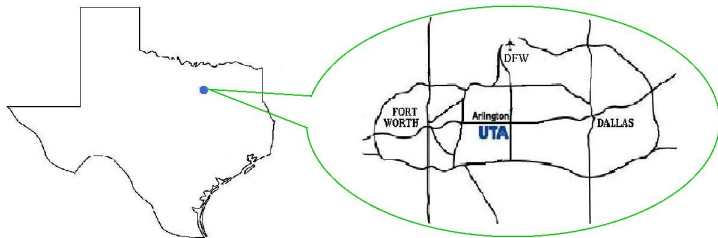
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Main Goal

... generalize the following well-known commutative result to certain non-commutative algebras satisfying good homological properties:

Theorem ([standard])

Let R denote the (commutative) polynomial ring on n variables over an algebraically closed field $\Bbbk$. Homogeneous elements $f_1, \ldots, f_m \in R_{>0}$ define/yield a complete intersection if one of the following equivalent conditions holds:

(a) *$\{f_1, \ldots, f_m\}$ is a regular sequence in R (i.e., for all i, f_i is not a zero divisor in $R/\langle f_1, \ldots, f_{i-1}\rangle$);*

(b) *for all $k = 1, \ldots, m$, $GKdim(R/\langle f_1, \ldots, f_k\rangle) = n - k$.*

Moreover, if $m = n$, then there exist another 2 equiv conditions equiv to (a) and (b):

(c) *$\dim_{\Bbbk}(R/\langle f_1, \ldots, f_n\rangle) < \infty$;*

(d) *$\mathcal{V}(f_1, \ldots, f_n) \subset \mathbb{P}^{n-1}$ is empty.*

Assumptions & Notation

- $\Bbbk$ = algebraically closed field.

- $A = \bigoplus_{i=0}^{\infty} A_i$ = an $\mathbb{N}$-graded, connected ($A_0 = \Bbbk$), finitely generated $\Bbbk$-algebra, generated by A_1.

- $\mathrm{GKdim}(A) = n \in \mathbb{N}$.

- A has a normalizing sequence $F = \{f_1, \ldots, f_m\} \subset A \setminus \Bbbk^{\times}$ consisting of homogeneous elements (i.e., for all i, f_i is normal in $A/\langle f_1, \ldots, f_{i-1} \rangle$).

- $I = \langle F \rangle$.

The Desired Equivalent Conditions

In order to address the notion of complete intersection in the context of A and F, we will be interested in 4 (possibly equivalent) conditions on A and F as follows:

I. F is a regular sequence in A;

II. for each $k = 1, \ldots, m$, $\mathrm{GKdim}(A/\langle f_1, \ldots, f_k \rangle) = n - k$;
   (last slide: $\mathrm{GKdim}(A) = n$)

III. $\dim_{\Bbbk}(A/\langle F \rangle) < \infty$;

IV. $\widehat{\mathcal{V}(F)}$ is empty (to be defined below).

Base-Point Modules

In past work with T. Cassidy on skew poly rings S, he & I initially thought $\widehat{\mathcal{V}(F)}$ should be the scheme of point modules over $S/\langle F \rangle$. However, J. T. Stafford led us to realise that, even for skew poly rings, one needs to consider **both** point modules **&** fat point modules. The same is true for the algebra A in this talk....

Definition ([CV2])

A right **base-point module** over A is any 1-critical (wrt GK-dimension) graded right A-module that is generated by its homogeneous degree-zero elements & which has Hilbert series $H(t) = c/(1-t)$ for some $c \in \mathbb{N}$.

So, $c = 1 \Leftrightarrow$ point module, and $c \geq 2 \Leftrightarrow$ fat point module.

Notation: $\widehat{\mathcal{V}(F)}$ is the collection of (right) base-point modules over $A/\langle F \rangle$.

Example (See [ATV1], [CV2])

If $A =$ poly ring & $\deg(f_i) = 2$ for all i, then F is a quadric system and there is a one-to-one corresp between $\widehat{\mathcal{V}(F)}$ and the base points of the quadric system.

Proposed Definition

Definition

Let A be as above and suppose conditions I-IV are equivalent for all normalizing sequences F in A of the above form that have length n. For such an F, if the equivalent conditions I-IV hold, call $A/\langle F \rangle$ a **complete intersection**.

Are there many algebras A to which this definition applies?

Examples

- Regular skew poly rings, i.e., $S = \Bbbk$-algebra on generators $z_1, \ldots, z_n$ where $z_j z_i = \mu_{ij} z_i z_j$ for all $i \neq j$ such that $\mu_{ij} \in \Bbbk^{\times}$ & $\mu_{ij}\mu_{ji} = 1$.
- Regular graded Clifford algebras (with char($\Bbbk$) $\neq 2$).
- Regular graded skew Clifford algebras (with char($\Bbbk$) $\neq 2$).
- Coordinate ring of quantum $m \times \ell$ matrices, where $n = m\ell$.
- Homogenization of the universal enveloping algebra of any Lie algebra of dimension $n - 1 < \infty$.

Main Theorem

Initial generalization to non-commutative setting was to the case of skew poly rings with T. Cassidy in [CV1] & [CV2].

Later, I generalized it further as follows:

Theorem

Let $A = \bigoplus_{i=0}^{\infty} A_i$ denote a connected, $\mathbb{N}$-graded $\mathbb{k}$-algebra that is generated by A_1. Suppose that A is (noetherian) Auslander-Gorenstein of finite injective dimension & satisfies the Cohen-Macaulay property, & that there exists a normalizing sequence $\{y_1, \ldots, y_\nu\} \subset A \setminus \mathbb{k}^\times$ ($\nu < \infty$) consisting of homog elements such that $GKdim(A/\langle y_1, \ldots, y_\nu\rangle) = 1$. If $GKdim(A) = n \in \mathbb{N}$, then conditions I-IV are equivalent for any normalizing sequence $F \subset A \setminus \mathbb{k}^\times$ that consists of n homogeneous elements.

Remarks

- Auslander-Gorenstein includes assumption A is noetherian.

- Auslander-Gorenstein & Cohen-Macaulay property allow application of results of T. Levasseur ([L]).

- Most of the result is straightforward to prove & follows from work in [CV1];   the part that needs work is III $\Leftrightarrow$ IV.

Next several slides $=$ outline of proof of III $\Leftrightarrow$ IV.

Recall   III. $\dim_{\Bbbk}(A/\langle F \rangle) < \infty$;

                IV. $\widehat{\mathcal{V}(F)}$ is empty.

Outline of Proof of III ⟺ IV

IV′ ⇒ III′

Suppose $M = \bigoplus_{i=0}^{\infty} M_i \in \widehat{\mathcal{V}(F)}$. In particular, $M = M_0(A/\langle F \rangle)$ and $\dim_{\Bbbk}(M_0) < \infty$, while $\dim_{\Bbbk}(M) = \infty$; thus, $\dim_{\Bbbk}(A/\langle F \rangle) = \infty$.

III′ ⇒ IV′

Let $I = \langle F \rangle$ & suppose $\dim_{\Bbbk}(A/I) = \infty$. Setting $y_0 = 0 \in A$, $\exists\, \theta \in \{0, \ldots, \nu\}$ such that $\quad \mathrm{GKdim}(A/(I + \sum_{i=0}^{\theta} y_i A)) = 1$.

Let k denote the smallest such θ, and let $\quad \mathcal{A} = A/(I + \sum_{i=0}^{k} y_i A)$, which is a $\Bbbk$-algebra since $\{y_1, \ldots, y_\nu\}$ is a normalizing sequence in A. By construction, $\mathrm{GKdim}(\mathcal{A}) = 1$.

We may now apply Example 5.5 of [AZ] to $\mathcal{A}$ as follows.

$$\text{III. } \dim_{\Bbbk}(A/\langle F \rangle) < \infty; \qquad \text{IV. } \widehat{\mathcal{V}(F)} = \emptyset$$

III$'$ $\Rightarrow$ IV$'$ (cont'd)

By [SSW], $\mathcal{A}$ is P.I. and finitely generated over its center & contains a

regular homogeneous central element z of positive degree $d \in \mathbb{N}$.

Let $B = \mathcal{A}[z^{-1}]$ and $R = B_0$. The algebras $\mathcal{A}$ and B are locally finite, so

$\dim_{\Bbbk}(R) < \infty$ & so R has a finite-dimensional simple module N.

Goal: build a BPM over $\mathcal{A}$ from N.

Corollary I.3.26 in [NV] $\Rightarrow B \cong R[x, x^{-1}; \sigma]$ for some $x \in B_1$, $\sigma \in \operatorname{Aut}(R)$.

Still following Example 5.5 of [AZ], we can now apply Example 5.4 of [AZ], to obtain that $\mathrm{spec}(R) \cong \mathrm{proj}(\mathcal{A})$, where $\mathrm{proj}(\mathcal{A})$ is the category of fin gen graded $\mathcal{A}$-modules mod the subcat of fin gen graded torsion $\mathcal{A}$-modules.

In particular, by §7 of [ATV2], we set
$\hat{N} = \bigoplus_{i=0}^{\infty} (N \otimes_R B)_i = \bigoplus_{i=0}^{\infty} (N \otimes_R B_i)$, which is a fin gen graded $\mathcal{A}$-module, generated by $\hat{N}_0 = N \otimes \mathbb{k}$. As vector spaces, $\hat{N}_i = N \otimes \mathbb{k} x^i \quad \forall i$.

 Goal: prove $\hat{N}$ is a BPM over $\mathcal{A}$.

It suffices to prove $\hat{N}$ is 1-critical. ($\Rightarrow \hat{N} \in \widehat{\mathcal{V}(F)}$ & so IV′ holds.)

III′ ⇒ IV′ (cont'd) $(B = \mathcal{A}[z^{-1}], \; R = B_0, \; B \cong R[x, x^{-1}; \sigma])$

By definition, $\hat{N}_i = N \otimes_R B_i$, so is an R-module.

Let $e \in \hat{N}_i^{\times}$. $e = v \otimes x^i$ for some $v \in N^{\times}$.

x is normal & regular in B, so $\quad eR = (v \otimes x^i)R = v \otimes Rx^i = vR \otimes \Bbbk x^i$

$N =$ simple R-module $\Rightarrow vR = N$, so $\quad eR = N \otimes \Bbbk x^i = N \otimes_R B_i = \hat{N}_i$.

Hence, $\hat{N}_i$ is a simple R-module for all i.

Let $0 \neq M = \bigoplus_{i=r}^{\infty} M_i$ be a graded $\mathcal{A}$-submodule of $\hat{N}$, where $M_r \neq 0$.

We must prove $M_j = \hat{N}_j$ for all $j \gg 0$.

By definition of $\hat{N}$, the central regular homog element z acts faithfully on $\hat{N}$; so z acts faithfully on M. Hence, $M_i \neq 0$ for all $i \geq r$ (as $\mathcal{A}$ generated by $\mathcal{A}_1$) &, for each $i \in \{r, \ldots, r + d - 1\}$, $\{\dim_\Bbbk(M_{i+td})\}_{t \geq 0}$ is a nondecreasing sequence. However, $\dim_\Bbbk(\hat{N}_j) = \dim_\Bbbk(N)$ for all j, so $\dim_\Bbbk(M_j) = \dim_\Bbbk(M_{j+d})$, for all $j \gg 0$. Thus, using the faithful action of z, M_j is an R-module for all $j \gg 0$; i.e., M_j is a nonzero R-submodule of the simple R-module $\hat{N}_j$ for all $j \gg 0$; whence, $M_j = \hat{N}_j$ for all $j \gg 0$. □

Examples

Examples (revisited)

- Regular skew poly rings, i.e., $S = \mathbb{k}$-algebra on generators $z_1, \ldots, z_n$ where $z_j z_i = \mu_{ij} z_i z_j$ for all $i \neq j$ such that $\mu_{ij} \in \mathbb{k}^\times$ & $\mu_{ij}\mu_{ji} = 1$. In this case, $z_i = y_i$ from the theorem.

- Regular graded Clifford algebras (with $\operatorname{char}(\mathbb{k}) \neq 2$): generators $x_1, \ldots, x_n, Y_1, \ldots, Y_n$ where $\deg(x_i) = 1$ & $\deg(Y_i) = 2$ for all i, with defining relations given by
  $$x_i Y_j = Y_j x_i \quad \& \quad Y_i Y_j = Y_j Y_i \quad \text{for all } i, j, \quad \&$$
  $$\forall\, i,\, j:\ x_i x_j + x_j x_i = \sum_{k=1}^n \alpha_{ijk} Y_k, \qquad \alpha_{ijk} \in \mathbb{k} \text{ for all } i, j, k$$
  where each scalar matrix $M_k = (\alpha_{ijk})$ is symmetric. When the quadric system determined by the M_k is base-point free, the algebra is quadratic & regular, & $\{Y_1, \ldots, Y_n\}$ is a regular sequence, so the result applies to the algebra with $Y_i = y_i$ from the theorem.

- Regular graded skew Clifford algebras (with $\operatorname{char}(\mathbb{k}) \neq 2$). Generalization of previous example.

Examples

- Coordinate ring of quantum $m \times \ell$ matrices, where $n = m\ell$.
  Results of [GL], [LS] & [L] show that such an algebra satisfies the
  hypotheses of the theorem.
  E.g., case $m = 2 = \ell$: generators a, b, c, d of degree one, with
  defining relations

$$ab = qba, \qquad\qquad bd = qdb,$$
$$ac = qca, \qquad\qquad cd = qdc,$$
$$bc = cb, \qquad ad - da = (q - q^{-1})bc,$$

  where $q \in \mathbb{k}^{\times}$ and $q^2 \neq 1$ [FRT]. Here, $\{b, c, d\}$ is normalizing, &
  factoring out $\langle b, c, d \rangle$ yields $\mathbb{k}[a]$.

- Homogenization of the universal enveloping algebra of any Lie algebra
  of dimension $n - 1 < \infty$.
  [LV] $\Rightarrow$ the homogenizing element z is central & regular of degree
  one; factoring out $\langle z \rangle$ yields the poly ring. Hence, the theorem
  applies.

Conclusion

Definition (revisited)

Let A be as in the main theorem. If F is a normalizing sequence in A that consists of n homogeneous elements of positive degree, call $A/\langle F \rangle$ a **complete intersection** if the equivalent conditions I-IV hold.

Remarks

- All the above examples are Auslander-regular and Artin-Schelter regular. So, for those of us who view a regular algebra as being a non-commutative analogue of the poly ring, this is perhaps further support of that viewpoint.

- Since, in general, not all algebras have normalizing elements (or enough of them), other notions of complete intersection for non-commutative algebras are being investigated by researchers, such as the recent work in [KKZ] that uses a more homological approach than that used here.

Question

Can the result (or a modified version thereof) be extended to AS-regular algebras without the assumption of Auslander-Gorenstein?

Question

Let $I_k = \langle f_1, \ldots, f_k \rangle$ for all $k \leq m \leq n$. If A is commutative, then, for each k, $\widehat{\mathcal{V}(I_k)}$ is a (projective) scheme, and so has a dimension. In particular, if A is the polynomial ring, then F is regular if and only if $\dim(\widehat{\mathcal{V}(I_k)}) = n - k - 1$, for all $k \leq m$. However, if A is not commutative, is there an analogous statement and under what hypotheses on A could it hold?

References

[ATV1] M. ARTIN, J. TATE & M. VAN DEN BERGH, Some Algebras Associated to Automorphisms of Elliptic Curves, *The Grothendieck Festschrift* **1**, 33-85, Eds. P. Cartier et al., Birkhäuser (Boston, 1990).

[ATV2] M. ARTIN, J. TATE AND M. VAN DEN BERGH, Modules over Regular Algebras of Dimension 3, *Invent. Math.* **106** (1991), 335-388.

[AZ] M. ARTIN AND J. J. ZHANG, Noncommutative Projective Schemes, *Adv. Math.* **109** No. 2 (1994), 228-287.

[CV1] T. CASSIDY & M. VANCLIFF, Generalizations of Graded Clifford Algebras and of Complete Intersections, *J. Lond. Math. Soc.* **81** (2010), 91-112.

[CV2] T. CASSIDY & M. VANCLIFF, Corrigendum to "Generalizations of Graded Clifford Algebras and of Complete Intersections" *J. London Math. Soc.*, to appear; www.uta.edu/math/vancliff/R/scliff-cor.pdf

[FRT] L. D. FADDEEV, N. YU. RESHETIKHIN AND L. A. TAKHTADZHYAN, Quantization of Lie Groups and Lie Algebras, *Leningrad Math. J.* **1** No. 1 (1990), 193-225.

[GL] K. R. GOODEARL AND E. S. LETZTER, Prime Factor Algebras of the Coordinate Ring of Quantum Matrices, *Proc. Amer. Math. Soc.* **121** No. 4 (1994), 1017-1025.

[KKZ] E. KIRKMAN, J. KUZMANOVICH AND J. J. ZHANG, Noncommutative Complete Intersections, preprint (2013).

[LV] L. LE BRUYN AND M. VAN DEN BERGH, On Quantum Spaces of Lie Algebras, *Proc. Amer. Math. Soc.* **119** No. 2 (1993), 407-414.

[L] T. LEVASSEUR, Some Properties of Non-commutative Regular Graded Rings, *Glasgow Math. J.* **34** (1992), 277-300.

[LS] T. LEVASSEUR AND J. T. STAFFORD, The Quantum Coordinate Ring of the Special Linear Group, *J. Pure & Appl. Algebra* **86** (1993), 181-186.

[NV] C. NĂSTĂCESCU AND F. VAN OYSTAEYEN, *Graded Ring Theory*, North-Holland, Amsterdam, 1982.

[SSW] L. W. SMALL, J. T. STAFFORD AND R. B. WARFIELD, Affine Algebras of Gelfand-Kirillov Dimension One are PI, *Math. Proc. Cambridge Philos. Soc.* **97** (1985), 407-414.