Invariants under Permutation Automorphisms

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Setting – Noncommutative Invariant Theory

- A a noncommutative algebra.
  Here A an AS regular algebra.

- H a Hopf algebra acting on A. Here $H = G$ a finite group of graded automorphisms of A.

- Find the structure of A^G.

- Today G = a group of automorphisms given by permutations of generators.

Invariants under S_n – Permutations of $x_1, \cdots, x_n$.

(Painter: Christian Albrecht Jensen) (Wikepedia)

The subring of invariants under S_n is a polynomial ring

$$\mathbb{C}[x_1, \cdots, x_n]^{S_n} = \mathbb{C}[\sigma_1, \cdots, \sigma_n]$$

where σ_ℓ are the n elementary symmetric functions for $\ell = 1, \ldots, n$,

$$\sigma_\ell = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1} x_{i_2} \cdots x_{i_\ell}$$

$$= \mathcal{O}_{S_n}(x_1 x_2 \cdots x_\ell).$$

or the n power sums:

$$P_\ell = x_1^\ell + \cdots + x_i^\ell + \cdots + x_n^\ell = \mathcal{O}_{S_n}(x_1^\ell).$$

Invariants under the Alternating Group A_n

$\mathbb{C}[x_1, \ldots, x_n]^{A_n}$ is generated by the symmetric polynomials (or power functions) and

$$D = D(x_1, \cdots, x_n) = \prod_{i<j} (x_i - x_j),$$

which has degree $\binom{n}{2}$.

$\mathbb{C}[x_1, \cdots, x_n]^{A_n}$ is a complete intersection:

$$\mathbb{C}[x_1, \cdots, x_n]^{A_n} \cong \frac{\mathbb{C}[\sigma_1, \cdots, \sigma_n][y]}{(y^2 - D^2)}$$

under the map that associates y to D (and the ith symmetric polynomial in the x_j to σ_i).

For a general G group of permutations of $x_1, \ldots, x_n$,

$\mathbb{C}[x_1, \cdots, x_n]^G$ need not be a complete intersection, or even Gorenstein,

for example $\mathbb{C}[x_1, x_2, x_3, x_4]^{\langle(x_1,x_2,x_3,x_4)\rangle}$ has Hilbert series

$$\frac{t^3 + t^2 - t + 1}{(1-t)^4(1+t)^2(1+t^2)}.$$

To find generators of $\mathbb{C}[x_1, \cdots, x_n]^G$ one can take orbit sums of monomials. Bounds on the degrees of generators are useful.

Noether's Bound (1916):
For k of characteristic zero, generators of $k[x_1, \cdots, x_n]^G$ can be chosen of degree $\leq |G|$.

Göbel's Bound (1995):
For subgroups G of permutations in S_n, generators of $k[x_1, \cdots, x_n]^G$ can be chosen of degree $\leq \max\{n, \binom{n}{2}\}$.

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Skew-polynomials with $q_{i,j} = -1$:

$$A = \mathbb{C}_{-1}[x_1, \cdots, x_n]$$

$$x_j x_i = -x_i x_j$$

Noncommutative Gauss' Theorem?

Example: $S_2 = \langle g \rangle$, for $g : x \mapsto y$ and $y \mapsto x$ acts on $A = \mathbb{C}_{-1}[x, y]$:

$$\begin{aligned} yx &= qxy \\ g(yx) &= g(qxy) \\ xy &= qyx \\ xy &= q^2xy \\ q^2 &= 1. \end{aligned}$$

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A^{S_2} is generated by

$$P_1 = x + y \text{ and } P_3 = x^3 + y^3$$

$(x^2 + y^2 = (x + y)^2$ and $g \cdot xy = yx = -xy$ so no generators in degree 2). The generators are NOT algebraically independent.
A^{S_2} is NOT AS regular (but it is a hyperplane in an AS regular algebra).

The transposition $(1, 2)$ is NOT a “reflection".

Generating sets

$P_1 = x + y = \mathcal{O}_{S_2}(x)$ and $P_3 = x^3 + y^3 = \mathcal{O}_{S_2}(x^3)$

or

$s_1 = x + y = \mathcal{O}_{S_2}(x)$ and $s_2 = x^2y + xy^2 = \mathcal{O}_{S_2}(x^2y)$.

$$3 > |S_2| = 2 = \max\{2, \binom{2}{2}\}$$

so both upper bounds on the degrees of generators fail for A^{S_2}.

To get an analogue of Gauss's Theorem (more generally the Shephard-Todd-Chevalley Theorem) we use permutations with a twist e.g.

$$g \cdot x_1 = -x_2 \quad \text{and} \quad g \cdot x_2 = x_1,$$

which turn out to be the analogues of reflections. Then we have

$$g \cdot x_1 x_2 = -x_2 x_1 = x_1 x_2$$

so that $x_1 x_2$ is invariant.

$$Tr_A(g,t) = \sum_{k=0}^{\infty} trace(g|A_k)t^k.$$

For a permutation:

when $A = \mathbb{C}[x_1, \ldots, x_n]$

$$Tr_A(g,t) = \frac{1}{\text{Det}(I - gt)};$$

when $A = \mathbb{C}_{-1}[x_1, \ldots, x_n]$

$$Tr_A(g,t) = \frac{1}{\text{Perm}(I - gt)}.$$

Reflection

An element g is a *reflection* of A if

$$Tr_A(g,t) = \frac{p(t)}{(1-t)^{n-1}q(t)}$$

for $q(1) \neq 0$ and $n = \mathrm{GKdim}\, A$.

For A AS regular, G finite, if A^G is AS regular it is necessary that G contain a reflection.

For $A = \mathbb{C}_{-1}[x_1, \ldots, x_n]$ there are no permutations that are reflections; all permutation subgroups of S_n are “small".

When A is AS regular of dimension n, then when the trace is written as a Laurent series in t^{-1}

$$Tr_A(g,t) = (-1)^n(\text{hdet } g)^{-1}t^{-\ell} + \text{higher terms}$$

Generalized Watanabe's Theorem: A^G is AS Gorenstein when all elements of G have homological determinant 1.

If g is a 2-cycle and $A = \mathbb{C}_{-1}[x_1 \ldots, x_n]$ then

$$Tr_A(g) = \frac{1}{(1+t^2)(1-t)^{n-2}} = (-1)^n \frac{1}{t^n} + \text{ higher terms}$$

so hdet $g = 1$, and for ALL groups G of $n \times n$ permutation matrices, A^G is AS Gorenstein. Not true for commutative polynomial ring – e.g.

$$\mathbb{C}[x_1, x_2, x_3, x_4]^{\langle(1,2,3,4)\rangle}$$

is not Gorenstein, while

$$\mathbb{C}_{-1}[x_1, x_2, x_3, x_4]^{\langle(1,2,3,4)\rangle}$$

is AS Gorenstein.

Invariants of $\mathbb{C}_{-1}[x_1, \ldots, x_n]$ under the full Symmetric Group S_n

Invariants are generated by sums over S_n-orbits
$\mathcal{O}_{S_n}(X^I) =$ the sum of the S_n-orbit of a monomial X^I.
$\mathcal{O}_{S_n}(X^I)$ can be represented by X^I, where I is a partition;
X^I is the leading term of $\mathcal{O}_{S_n}(X^I)$ under the lexicographic order,
$x_1 > x_2 > \ldots > x_n$.

$\mathcal{O}_{S_n}(X^I) = 0$ if and only if it corresponds to a partition with no repeated odd parts (e.g. $\mathcal{O}_{S_n}(x_1^2 x_2 x_3) = 0$).

A^{S_n} is generated by the n odd power sums $P_1, \ldots, P_{2n-1}$
or the n invariants $s_k = \mathcal{O}_{S_n}(x_1^2 \ldots x_{k-1}^2 x_k)$.

Bound on degrees of generators of A^{S_n} is $2n - 1$.

A^{S_n} contains $R = \mathbb{C}_{-1}[x_1^2, \ldots, x_n^2]^{S_n}$, which is a commutative polynomial ring (in $p_1 = P_2, \ldots, p_n = P_{2n}$).

$$A^{S_n} \cong \frac{R[y_1 : \tau_1, \delta_1] \cdots [y_n : \tau_n, \delta_n]}{\langle y_1^2 - p_1, \ldots, y_i^2 - p_i, \ldots, y_n^2 - p_n \rangle}$$

where $y_i \mapsto P_{2i-1}$.

A^{S_n} is a *classical complete intersection*.
Used the Hilbert series:

$$\frac{(1-t^2)(1-t^6)(1-t^{10})\cdots(1-t^{4n-2})}{(1-t)(1-t^2)(1-t^3)\cdots(1-t^{2n-1})(1-t^{2n})}.$$

Invariants of $A = \mathbb{C}_{-1}[x_1, \ldots, x_n]$ under the Alternating Group:

A^{A_n} is generated by $\mathcal{O}_{A_n}(x_1 x_2 \cdots x_{n-1})$,

and the n-1 polynomials $s_1, \ldots, s_{n-1}$

(or the power functions $P_1, \ldots, P_{2n-3}$),

bound on the degrees of generators of A^{A_n} is $2n - 3$.

Let $R = \mathbb{C}[p_1, \cdots, p_n]$ be a commutative polynomial ring.

$$A^{A_n} \cong \frac{R[y_1 : \tau_1, \delta_1] \cdots [y_{n+1} : \tau_{n+1}, \delta_{n+1}]}{\langle y_1^2 - p_1, \dots, y_i^2 - p_i, \dots, y_{n-1}^2 - p_{n-1}, y_n^2 - r_1, y_{n+1}^2 - r_2 \rangle}$$

where $r_i \in \mathbb{C}[p_1, \cdots, p_n]$, $p_i \mapsto P_{2i}$, $y_i \mapsto P_{2i-1}$ for $i \leq n-1$,
$y_n \mapsto \mathcal{O}_{A_n}(x_1 \cdots x_n) = x_1 x_2 \cdots x_n$ and
$y_{n+1} \mapsto \mathcal{O}_{A_n}(x_1 \cdots x_{n-1})$.

A^{A_n} is a *classical complete intersection*.

Used the Hilbert series:

$$\frac{(1+t)(1+t^3)\cdots(1+t^{2n-3})(1+t^n)(1+t^{n-1})}{(1-t^2)(1-t^4)\cdots(1-t^{2n})}.$$

Bounds on degrees of generators

Broer's Bound: Let A be a quantum polynomial algebra of dimension n and C an iterated Ore extension $k[f_1][f_2; \tau_2, \delta_2] \cdots [f_n; \tau_n, \delta_n]$. Assume that

1. $B = A^H$ where H is a semisimple Hopf algebra acting on A,
2. $C \subset B \subset A$ and A_C is finitely generated, and
3. $\deg f_i > 1$ for at least two distinct i's.

Then degrees of generators of A^H

$$\leq \ell_C - \ell_A = \sum_{i=1}^{n} \deg f_i - n.$$

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Broer's bound for permutations of skew polynomials

Taking $A = k_{-1}[x_1, \ldots, x_n]$,

$C := k[P_4, P_8, \ldots, P_{4\lfloor \frac{n}{2} \rfloor}][P_1][P_3; \tau_3, \delta_3] \cdots [P_{n'}; \tau_{n'}, \delta_{n'}]$

where $n' = 2\lfloor \frac{n-1}{2} \rfloor + 1$, gives:

Theorem. Let G be a subgroup of S_n acting on $k_{-1}[x_1, \ldots, x_n]$ as permutations. Suppose $|G|$ does not divide char k. Then degrees of generators of A^G

$$\leq \frac{1}{2}n(n-1) + \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1) \sim \frac{3}{4}n^2.$$

Bireflection

An element g is a *bireflection* of A if

$$Tr_A(g,t) = \frac{p(t)}{(1-t)^{n-2}q(t)}$$

for $q(1) \neq 0$ and $n = \mathrm{GKdim}\, A$.

For example $(1,2)(3,4)$ is a bireflection of $A = \mathbb{C}[x_1, x_2, x_3, x_4]$.

Kac-Watanabe-Gordeev Theorem

KWG's Theorem states that for $\mathbb{C}[x_1, \ldots, x_n]^G$ to be a complete intersection it is *necessary* that G be generated by bireflections of $\mathbb{C}[x_1, \ldots, x_n]$.

A permutation g is a bireflection of $\mathbb{C}_{-1}[x_1, \ldots, x_n]$ if and only if it is a 2-cycle or a 3-cycle.

Invariants A^G are examples of AS Gorenstein rings – some will not be “complete intersections" (e.g. $\mathbb{C}_{-1}[x_1, x_2, x_3, x_4]^{\langle(1,2,3,4)\rangle}$ is not a “complete intersection").

Question: Is KWG Theorem true for $A = \mathbb{C}_{-1}[x_1, \cdots, x_n]$?

True for $n \leq 4$.

For $A = \mathbb{C}_{-1}[x_1, \ldots, x_n]$:

Theorem. If G is generated by bireflections then A^G is a classical complete intersection.

Lemma: Let G be a subgroup of S_n.

1. If G is generated by 3-cycles, then G is an internal direct product of alternating groups.
2. If G is generated by 3-cycles and 2-cycles, then G is an internal direct product of alternating and symmetric groups.

The converse of KWG Theorem is NOT true for $A = \mathbb{C}[x_1, \ldots, x_n]$.
Let $G = \langle (1,2)(3,4), (2,3)(4,5) \rangle$ act on $\mathbb{C}[x_1, x_2, x_3, x_4, x_5]$.

$$H_{A^G}(t) = \frac{t^6 - t^5 + 2t^3 - t + 1}{(1-t)^2(1-t^2)^2(1-t^5)}.$$

The numerator is not cyclotomic, hence A^G cannot be a complete intersection.

Auslander's Theorem

Let G be a finite subgroup of $GL_n(k)$ that contains no reflections, and let $A = k[x_1, \ldots, x_n]$. Then the skew-group ring $A\#G$ is isomorphic to $\text{End}_{A^G}(A)$ as rings.

Question: Does Auslander's Theorem generalize to our context?

Mori-Ueyama:

Theorem: A an AS regular domain of dimension $d \geq 2$, generated in degree 1. Take r such that $r|\ell$, and let $G = \langle \sigma_r \rangle$ where $\sigma_r(a) = \omega^{\deg(a)}a$, where ω is a primitive rth root of 1. Then $A^G = A^{(r)}$, the rth Veronese, and $A\#G \cong \operatorname{End}_{A^G}(A)$.

Case: $A = \mathbb{C}_{-1}[x_1, \ldots, x_n]$ and
$G = \langle (x_1, \ldots, x_n) \rangle$:

For n= 2,3: $A \# G \cong \text{End}_{A^G}(A)$.

Question: Does Auslander's Theorem hold for
$A = \mathbb{C}_{-1}[x_1, \ldots, x_n]$ all permutation subgroups of
S_n?

Statements about A^G	when $A = k[\underline{x}]$	when $A = k_{-1}[\underline{x}]$
Being AS Gorenstein	Not always	Always
Being AS regular	Sometimes	Never
Bound on degrees of generators	$\max\{n, \binom{n}{2}\}$	$\binom{n}{2} + \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1)$
KWG theorem holds	Yes	Conjecture
Converse of KWG holds	No	Yes
All subgroups small	No	Yes

Case: 3-dimensional Skylanin Algebra
(with S.Sierra and C. Walton)

$$axy + byx + cz^2 = 0$$

$$ayz + bzy + cx^2 = 0$$

$$azx + bxz + cy^2 = 0$$

and $g : x \mapsto y, y \mapsto z, z \mapsto x$.

In the generic case A has no refections, so all finite groups of graded automorphisms are “small".

Trace of g:

$$\mathrm{Tr}_A(g,t) = \frac{1}{1-t^3} = \frac{1}{(1-t)(1+t+t^2)}.$$

g is a bireflection. Homological determinant of g is 1, so $A^{\langle g \rangle}$ is AS-Gorenstein.

Let X, Y and Z be the associated eigenvectors of g, the 3×3 cyclic permutation matrix

$$X = x + \omega y + \omega^2 z$$

$$Y = x + \omega^2 y + \omega z$$

$$Z = x + y + z$$

Then

$$AXY + BYX + CZ^2 = 0$$
$$AYZ + BZY + CX^2 = 0$$
$$AZX + BXZ + CY^2 = 0$$

Hence WLOG we can assume that
g is diagonal$(\omega^2, \omega, 1)$.

Theorem: (Generic case):
$A^{\langle g \rangle}$ is a complete intersection.

Questions:

- Does KWG hold for A?
- Does the converse of KWG hold for A?
- Is $A \# G \cong \text{End}_{A^G}(A)$?
- What if A is PI?

Case: Down-up algebras $A(\alpha, \beta)$

$$y^2x = \alpha yxy + \beta xy^2$$

$$yx^2 = \alpha xyx + \beta x^2y.$$

$A(\alpha, \beta)$ have no reflections, so all finite groups are “small".

Transposition $g : x \mapsto y, y \mapsto x$ acts on $A(0,1)$ and $A(\alpha,-1)$, and

$$Tr_A(g,t) = \frac{1}{(1-t)(1+t)(1+t^2)}$$

so g is a bireflection of hdet(g) = 1, and A^g is AS Gorenstein. Further A^g is a “complete intersection of GK-type"