

Degree Bounds for Hopf Actions on Artin-Schelter Regular Algebras arXiv 2008.05047

Ellen Kirkman

kirkman@wfu.edu



WAKE FOREST
UNIVERSITY

Department of Mathematics

Joint with Robert Won and James Zhang
Seattle Noncommutative Algebra Day

December 19, 2020

Classical and Noncommutative Invariant Theory

Classical Invariant Theory:

Group G acting linearly on the algebra $\mathbb{k}[x_1, \dots, x_n]$ and study $\mathbb{k}[x_1, \dots, x_n]^G$.

Noncommutative Invariant Theory:

Replace:

$\mathbb{k}[x_1, \dots, x_n]$ with appropriate noncommutative algebra A

G with a group (or Hopf algebra) that acts on A

to extend classical results.

Classical Invariant Theory

$\mathbb{k}$ a field of characteristic 0

G a finite group, represented as $n \times n$ matrices,
acting on a polynomial ring $A = \mathbb{k}[x_1, \dots, x_n]$:

$$g \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ on } \mathbb{k}[x, y]$$

$$\text{as } g.x = ax + cy \quad g.y = bx + dy$$

$$\text{Invariants} = A^G = \{a \in A : g.a = a \text{ for all } g \in G\}$$

$$\beta(A^G) = \min\{k : A^G \text{ generated by elements} \\ \text{of degree } \leq k\}$$

Theorem (Noether (1916))

Let $A = \mathbb{k}[x_1, \dots, x_n]$. Then $\beta(A^G) \leq |G|$
if $\mathbb{k}$ has characteristic zero or
 $|G| < \text{characteristic } \mathbb{k}$.

Constructing Invariants

“Orbit sum” of a monomial $m \in A$ is an invariant where

$$\mathcal{O}(m) = \sum_{g \in \mathcal{O}(g)} g.m.$$

Invariants are linear combinations of orbit sums.

Fleischmann (2000) and Fogarty (2001)
extended to non-modular case ($|G|^{-1} \in \mathbb{k}$).

Derksen and Sidman (2002):
Subspace arrangement approach using
Castelnuovo-Mumford regularity of the
vanishing ideal of the subspace arrangement.

Noether's bounds improvements

Noether bound can be sharp $\beta(A^G) = |G|$:
 $\epsilon = e^{2\pi i/n}$, $A = \mathbb{k}[x]$, $G = (\epsilon)$, $A^G = \mathbb{k}[x^n]$.

Theorem (Domokos and Hegedüs (2000))
If G non-cyclic then A^G can be generated by
polynomials of total degree:

$$\begin{aligned} &\leq 3|G|/4 \text{ if } |G| \text{ is even, and} \\ &\leq 5|G|/8 \text{ if } |G| \text{ is odd.} \end{aligned}$$

Sezer (2002): Extended to non-modular case.

Bounds on degrees of relations:

Derksen (2004): $A := \mathbb{k}[x_1, \dots, x_n]$

$f_1, \dots, f_r$ = a minimal set of generators of $A^G_{\geq 1}$;

$$I := (f_1, \dots, f_r)A$$

$$\tau_G(A) := \min\{d : I_d = A_d\} \leq |G|$$

$S := \mathbb{k}[y_1, \dots, y_r]$ with degree $y_i = \text{degree } f_i$

$$S/J \cong A^G.$$

Then J is generated in degree $\leq 2\tau_G(A) \leq 2|G|$.

Bounds on Degrees of $\mathrm{Tor}_i^S(\mathbb{k}, \mathbb{k})$:

For connected graded algebra S :

$$\beta(S) = \deg \mathrm{Tor}_1^S(\mathbb{k}, \mathbb{k})$$

$$\beta_i(S) = \deg \mathrm{Tor}_i^S(\mathbb{k}, \mathbb{k}) \text{ for } i \geq 2$$

For $R = \mathbb{k}[x_1, \dots, x_n]^G$ Derksen (2004) proved:

$$\beta_G^i(R) \leq (i + n - 1)|G| - n$$

and conjectured:

$$\beta_G^i(R) \leq (i)|G|.$$

Chardin and Symonds (2016):
counterexample to Derksen conjecture, and

$$\beta_G^i(R) \leq (i)|G| + (i - 2) \text{ for } i \geq 2.$$

Theorem (Göbel (1995)) In any characteristic, G a group of permutations of x_i :

$$\beta(\mathbb{k}[x_1, x_2, \dots, x_n]^G) \leq \max\left\{n, \binom{n}{2}\right\}.$$

Example: Bound is sharp. $\mathbb{k}[x_1, x_2, \dots, x_n]^{A_n}$ is generated by $\sigma_1, \dots, \sigma_n, \nabla$, where

$$\nabla = \prod_{i < j} (x_i - x_j).$$

(largest degree relation = $n(n - 1)$) = $2\tau_G(A)$

Modular case: $\beta(A^G)$ depends upon dimension of representation

The Noether bound can fail: e.g. in char 2
 $|G| = 2$ acting on polynomials in 6 variables with
 $\beta(A^G) = 3 > |G| = 2$.

Theorem (Symonds (2011))

If G is a finite group of order $|G| > 1$ acting linearly on $A := \mathbb{k}[x_1, \dots, x_n]$ with $n \geq 2$ then
 $\beta(A^G) \leq n(|G| - 1)$.

Relations among generators in degrees
 $\leq 2n(|G| - 1)$.

Artin-Schelter regular algebras. ($\mathbb{k} = \mathbb{C}$)

A connected graded noetherian with

- $\text{gldim } A = d$
- $\text{GKdim } A = d$
- $\text{Ext}_A^i(\mathbb{k}, A) = 0$ for $i \neq d$
- $\text{Ext}_A^d(\mathbb{k}, A) = \mathbb{k}(\ell)$

Examples:

- $\mathbb{k}[x_1, \dots, x_d]$
- $R[x : \sigma, \delta]$ for R AS regular
- Various rings defined by generators and relations (e.g. **Down-up**, **Sklyanin algebras**).

The action of a Hopf algebra H on A :

- (a) H is a semisimple, hence finite-dimensional, Hopf algebra,
- (b) A is a left H -module algebra and for each i , A_i is a left H -submodule of A .

$$A^H = \{a \in A \mid h.a = \epsilon(h)a \text{ for all } h \in H\}.$$

Noncommutative Case is Different

If any finite group G acts nontrivially on $A = \mathbb{k}[x_1, \dots, x_n]$, then $\beta(\mathbb{k}[x_1, \dots, x_n]^G) > 1$.

When A is noncommutative, $\beta(A^G)$ can be 1 even if A is a Koszul AS regular algebra:

Noncommutative Case is Different

Example: Take A generated by x , y , and z with relations

$$xy - yx = z^2, \quad z \text{ is central.}$$

Let σ be the automorphism of A determined by

$$\sigma(x) = x, \quad \sigma(y) = y, \quad \text{and} \quad \sigma(z) = -z.$$

$A^{\langle \sigma \rangle}$ is generated in degree 1 and

$$\beta(A^{\langle \sigma \rangle}) = 1.$$

Noncommutative Case is Different

Noether Bound fails:

When A is noncommutative $\beta(A^G)$ can be strictly larger than $|G|$.

In fact $\beta(A^G) - |G|$ can be arbitrarily large.

Noncommutative Setting – Noether Bound fails– $\beta(A^G) > |G|$

A an Artin Schelter regular algebra, and
 G acting linearly on generators of A .
Invariants are linear combinations of orbit sums.

Example: $A = \mathbb{k}_{-1}[x, y]$, $yx = -xy$ and

$g : x \leftrightarrow y$ acts on A . $\mathcal{O}(x) = x + y$

$$\mathcal{O}(x^2) = x^2 + y^2 = (x + y)^2$$

$$\mathcal{O}(xy) = xy + yx = 0$$

A set of generators of A^G is $x + y$ and $x^3 + y^3$
(or $x^2y + xy^2$). $\beta(A^G) = 3 > |G| = 2$,

Difference $\beta(A^G) - |G|$ is arbitrarily large

Ferraro, K, Moore, Peng: arXiv: 1907.06761

$A = \mathbb{k}_{-1}[x, y]$, $\epsilon = e^{2\pi i/n}$, and $G = \langle g \rangle$

$$g = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}$$

$$g^2 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \text{ so } |G| = 2n.$$

Invariants must have total degree a multiple of n
and are linear combinations of

$$\mathcal{O}(x^{kn-i}y^i) := x^{kn-i}y^i + (-1)^{(kn-i)i}\epsilon^i x^i y^{kn-i}.$$

Difference $\beta(A^G) - |G|$ is arbitrarily large

$$A = \mathbb{k}_{-1}[x, y], \epsilon = e^{2\pi/n}, \text{ and } G = \langle g \rangle$$

$$g = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix} \quad |G| = 2n$$

$$n \equiv 0 \pmod{4} : \beta(A^G) = 2n$$

$$n \equiv 2 \pmod{4} : \beta(A^G) = n$$

$$n \equiv 1 \pmod{2} : \beta(A^G) = 3n$$

n odd: Noether's bound fails (for 2 dim rep).

What happens for $A = \mathbb{k}_{-1}[x_1, \dots, x_m]$?

Example: There is no uniform bound for $\beta(A^G)$ depending only on $|G|$ for all noetherian AS regular algebras A .

$\mathfrak{g} :=$ free Lie algebra generated by x and y .
 $\mathfrak{g}$ is a graded Lie algebra with universal enveloping algebra $S := U(\mathfrak{g}) \cong \mathbb{k}\langle x, y \rangle$ as graded algebras by assigning $\deg x = \deg y = 1$.

No uniform bound involving only $|G|$

Let $m > 2$ and consider the graded quotient Lie algebra

$$\mathfrak{g}_m = \mathfrak{g}/\mathfrak{g}_{\geq m},$$

$\mathfrak{g}_m$ is a graded Lie algebra generated by x, y and subject to relations

$$[a_1, [a_2, [\cdots, [a_{m-1}, a_m] \cdots]]] = 0$$

for all a_i being x or y . Then $\mathfrak{g}_m$ is a finite-dimensional graded Lie algebra. Its universal enveloping algebra $A = U(\mathfrak{g}_m)$ is a noncommutative noetherian AS regular algebra.

No uniform bound involving only $|G|$

Let σ be the automorphism of $\mathfrak{g}$ sending x to $-x$ and y to y . Then σ has order 2, and it induces an order 2 graded algebra automorphism on A . Let $G = \langle \sigma \rangle$. We show that $\beta(A^G) \geq m - 1$.

Since m can be arbitrarily large, there is no uniform bound for $\beta(A^G)$ which depends only on $|G|$ when we consider all noetherian AS regular algebras A .

Fogarty's saturation

$$R = A^H$$

The *left Hilbert ideal* of the H -action on A is the left ideal of A

$$J_H(A) := AR_{\geq 1}.$$

The *left τ -saturation degree* of the H -action on A is defined to be

$$\tau_H(A) = \min\{d : J_H(A)_d = A_d\}$$

Example: The m th Veronese is $A^{\langle\sigma\rangle}$ of A , where $\omega \in \mathbb{k}$ is a primitive m th root of unity, A is generated in degree 1, $\sigma(a) = \omega a$ for all $a \in A_1$ and extended to an automorphism of A , so

$$\begin{aligned} J_{\langle\sigma\rangle}(A) &= 0 \text{ in degrees } < m \\ &= A \text{ in degrees } \geq m \end{aligned}$$

$$\text{so } \tau_{\langle\sigma\rangle}(A) = m.$$

If A is a domain, then $\beta(A^{\langle\sigma\rangle}) = m$.

Question: Is $\tau(A^{\text{op}}) = \tau(A)$?

Relating saturation τ to bounds of generators β

Proposition

Let A be a connected graded algebra and let H be a semisimple Hopf algebra acting on A homogeneously. Then $\beta(A^H) \leq \tau_H(A)$.

Theorem

When A is an AS regular domain generated in degree 1, A^H has finite global dimension and $A \# H$ is prime then $\beta(A^H) \leq \dim H$.

Using Central Subalgebras

Proposition

Let A be a connected graded domain and let H be a semisimple Hopf algebra acting on A homogeneously. Suppose that

- (a) A is a finitely generated module over a central subalgebra Z ;
- (b) Z is stable under the H -action;
- (c) Z is generated as an algebra by elements of degree $\leq d$;
- (d) A is generated as a Z -module by elements of degree $\leq m$; and
- (e) either $\mathbb{k}$ is an algebraically closed field of characteristic zero or H is a group algebra.

Then $\beta(A^H) \leq \tau_H(A) \leq d \dim H + m$.

Corollary

Let G be a finite group acting as graded automorphisms on $\mathbb{k}_{-1}[x_1, \dots, x_n]$ and suppose that $|G|$ is invertible in $\mathbb{k}$. Then

$$\beta(\mathbb{k}_{-1}[x_1, \dots, x_n]^G) \leq 2|G| + n.$$

Proposition

$$\beta_i(\mathbb{k}_{-1}[x_1, \dots, x_n]^G) \leq i(2|G| + n + 1) - 2$$

for all $i \geq 2$

Broer bound for commutative polynomials

$$A = \mathbb{k}[x_1, \dots, x_n] \text{ char } \mathbb{k} = 0$$

$$A^G \text{ Cohen-Macaulay: } A^G = \bigoplus_{j=1}^m \eta_j \mathbb{k}[f_1, \dots, f_n].$$

$$\beta(A^G) \leq \max\{\deg(\eta_j), \deg(f_p)\}$$

The Hilbert series of a graded algebra S :

$$H_S(t) = \sum_j \dim(S_j) t^j.$$

$$H_{A^G}(t) = \sum_j \frac{\deg(\eta_j)}{(1-t^{\deg(f_1)}) \cdots (1-t^{\deg(f_n)})}$$

$$\text{Molien's Theorem: } H_{A^G}(t) = \frac{1}{|G|} \sum_g \frac{1}{(1-\lambda_1 t) \cdots (1-\lambda_n t)}$$

$$\deg(\eta_j) - \sum_{j=1}^n \deg(f_j) \leq -n$$

$$\beta(A^G) \leq \sum_{j=1}^n \deg(f_j) - n$$

Extended to CM-case (any char) by Broer, to general $\mathbb{k}$ by Symonds.

Theorem (K, Kuzmanovich, Zhang (2014))

A quantum polynomial algebra of dimension n ,
 H semisimple Hopf algebra, and $C \subset A^H \subset A$,
 C a graded iterated Ore extension
 $C = \mathbb{k}[f_1][f_2 : \tau_2, \delta_2] \dots [f_n : \tau_n, \delta_n]$,
 A_C finitely generated, $\deg f_i > 1$ for at least 2
distinct i 's.

$$\beta(A^H) \leq \sum \deg f_i - n.$$

$$A = \mathbb{k}_{-1}[x_1, \dots, x_n]$$

$$P_u = \sum x_i^u$$

$$P'_u = \begin{cases} P_u & \text{if } u \text{ odd} \\ P_{2u} & \text{if } u \text{ even} \end{cases}$$

$$P_u P_v = P_v P_u \text{ when } u \text{ even}$$

$$P_u P_v + P_v P_u = 2P_{u+v} \text{ when } u, v \text{ odd}$$

C subalgebra generated by $P'_1, P'_2, \dots, P'_n$

$$C \cong$$

$$\mathbb{k}[P_4, P_8, \dots, P_{4\lfloor \frac{n}{2} \rfloor}][P_1][P_3 : \tau_3, \delta_3] \dots [P_m : \tau_m, \delta_m]$$

for $m = 2\lfloor \frac{n-1}{2} \rfloor + 1$.

Theorem (K, Kuzmanovich, Zhang (2014)):

$$A = \mathbb{k}_{-1}[x_1, \dots, x_n]$$

- $\beta(A^{S_n}) = 2n - 1$
relations in degree $\leq 4n - 2$
- $\beta(A^{A_n}) = 2n - 3$
relations in degree $\leq 4n - 6$
- $\beta(A^G) \leq 3n^2/4$ for G permutations.
- $\beta(A^G) \leq n^2$ for G a subgroup of $S_n \rtimes \{\pm 1\}$

S is Koszul if the trivial graded S -module $\mathbb{k}$ has a minimal free resolution that is linear, or equivalently, $\mathrm{Tor}_i^S(\mathbb{k}, \mathbb{k})_j = 0$ for all $j \neq i$.

Measure of growth of degrees of generators in a resolution

S a connected graded algebra.

M a graded module.

The *Tor-regularity* of M is $\text{Torreg}(M) :=$

$$\sup\{j - i \mid \text{Tor}_i^S(\mathbb{k}, M)_j \neq 0 \text{ for } i, j \in \mathbb{Z}\}.$$

S is Koszul if and only if $\text{Torreg}(\mathbb{k}) = 0$

Noncommutative version studied by Jørgensen and Dong-Wu.

Another measure of growth of degrees of generators in a resolution

Let M be a nonzero graded left S -module.

The *Castelnuovo–Mumford regularity* of M is

$$\begin{aligned}\mathrm{CMreg}(M) &:= \inf\{p \in \mathbb{Z} \mid H_{\mathfrak{m}}^i(M)_{>p-i} = 0, \forall i \in \mathbb{Z}\} \\ &= \sup\{i + \deg(H_{\mathfrak{m}}^i(M)) \mid i \in \mathbb{Z}\}.\end{aligned}$$

Noncommutative S studied by Jørgensen,
Dong-Wu.

Let A be an AS regular algebra of type (d, ℓ) .
Then

$$\text{CMreg}(A) = d - \ell = \text{gldim } A + \deg H_A(t).$$

A is Koszul if and only if $\text{CMreg}(A) = 0$

(P. Jørgensen, Dong-Wu) A is a Koszul AS regular algebra if and only if

$$\text{Torreg}(M) = \text{CMreg}(M) \text{ for all } M.$$

For S commutative, generated in degree 1

$$\text{Torreg}(\mathbb{k}) = 0, \infty.$$

If A is AS regular then $\text{Torreg}(\mathbb{k}) = -\text{CMreg}(A)$.

If A is a non-Koszul AS regular algebra of type $(3, 4)$ then $T = \bigotimes^n A$ is AS regular of type $(3n, 4n)$ and

$$\text{Torreg}(\mathbb{k}) = -\text{CMreg}(T) = -(3n - 4n) = n.$$

Generalization of a result of Symonds:

Suppose there is a graded algebra map $S \rightarrow A^H$ where S is a noetherian AS regular algebra such that A^H is finitely generated over S on both sides. Let

$\delta(A/S) = \text{CMreg}(A) - \text{CMreg}(S)$. Then

$$(1) \quad \beta(A^H) \leq \max\{\beta(S), \delta(A/S)\}, \text{ and}$$

$$(2) \quad \beta_2(A^H) \leq \max\{2\delta(A/S), \delta(A/S) + \beta(S), \beta_2(S)\}.$$

Noncommutative version of Theorem of Derksen

A generated in degree 1, S a noetherian AS regular algebra such that the minimal generating vector spaces of S and $R := A^H$ have the same dimension and there exists a graded algebra surjection $S \rightarrow R$.

- (1) $\beta_2(R) \leq 2 - 2 \operatorname{CMreg}(S) + \operatorname{CMreg}(A)$.
- (2) Suppose that $\operatorname{Tor}_1^S(\mathbb{k}, R) \otimes_R \mathbb{k} \cong \operatorname{Tor}_1^S(\mathbb{k}, R)$. Let K be the kernel of the algebra map $S \rightarrow R$. Then, as a left ideal of S , K is generated in degree at most

$$2 - 2 \operatorname{CMreg}(S) + \operatorname{CMreg}(A).$$

Work in progress with R. Won and J. Zhang

$\xi = (\xi_0, \xi_1)$, $\xi_0 \neq 0$, $X \neq 0$ object in $D_{fg}^b(A \text{ Gr})$.

The ξ -*Castelnuovo-Mumford regularity* of X is

$$\text{CMreg}_{\xi}(X) = \sup\{\xi_0 \deg(H_{\mathfrak{m}}^i(X)) + \xi_1 i \mid \forall i \in \mathbb{Z}\}.$$

The ξ -*Tor-regularity* of X

$$\text{Torreg}_{\xi}(X) = \sup\{\xi_0 \deg(\text{Tor}_i^A(\mathbb{k}, X)) - \xi_1 i \mid \forall i \in \mathbb{Z}\}.$$

Questions for A noncommutative:

- Is $\beta(A^H)$ bounded by a function of the numerical invariants

$$\dim_{\mathbb{k}} H, \quad \text{gldim } A, \quad \text{CMreg}(A)?$$

- Is $\beta(A^G)$ biggest for cyclic groups?
- Does the dimension of H play the same role as $|G|$?
- Can $\beta(A^H)$ be larger when H is not semisimple or characteristic $\mathbb{k} = p$?

Thanks