

Invariant Theory of Twisted Generalized Weyl Algebras

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This talk focuses primarily on several recent projects:

- *Fixed rings of generalized Weyl algebras*, J. Algebra 536 (2019). Joint with Rob Won.
- *Fixed rings of quantum generalized Weyl algebras*, Communications in Algebra, 2020, 48 (9), 4051-4064. Joint with Phuong Ho (undergraduate).
- *Fixed rings of twisted generalized Weyl algebras*, arXiv:2011.13029. Joint with Daniele Rosso.
- *Pointed Hopf actions on quantum generalized Weyl algebras*, forthcoming. Joint with Rob Won.
- *Reflexive hull discriminants and applications*, forthcoming. Joint with Kenneth Chan, Rob Won, and James Zhang. (More on this at JMM.)

Throughout, $\mathbb{k}$ is an algebraically closed field of characteristic zero.

Theorem (Shephard-Todd-Chevalley)

Let G be a finite group of graded automorphisms on $A = \mathbb{k}[x_1, \dots, x_n]$. The fixed ring A^G is a polynomial ring if and only if G is generated by reflections.

A major goal in noncommutative invariant theory is to generalize the STC theorem to noncommutative analogues of polynomial rings.

- Artin-Schelter regular algebras (Kirkman, Kuzmanovich, Zhang)
- Preprojective algebras (Weispfenning)
- Quadratic Poisson algebras (-, Veerapen, Wang)

To further generalize STC, we can relax the grading requirement and consider Calabi-Yau algebras.

- An algebra A is *homologically smooth* if it has a finitely generated projective resolution of finite length in $A^e - \text{MOD}$.
- The algebra A is *twisted Calabi-Yau* of dimension d if it is homologically smooth and there exists an invertible bimodule U of A such that $\text{Ext}_{A^e}^i(A, A^e) = \delta_{id} U$.
- If $U = A$, then A is said to be *Calabi-Yau*.

Question

When is the fixed ring of a (twisted) Calabi-Yau algebra again (twisted) Calabi-Yau?

This talk will focus on certain $\mathbb{Z}$ -graded twisted Calabi-Yau algebras and their generalizations.

Definition

Let R be a unital $\mathbb{k}$ -algebra and n a positive integer. Let $\sigma = (\sigma_1, \dots, \sigma_n)$ be an n -tuple of commuting automorphisms of R and $t = (t_1, \dots, t_n)$ be an n -tuple of nonzero central elements of R such that $\sigma_i(t_j) = t_j$ for $i \neq j$. The *rank n generalized Weyl algebra* (GWA) $R(\sigma, t)$ is generated over R by $x_1, \dots, x_n$ and $y_1, \dots, y_n$ subject to the relations

$$\begin{aligned}x_i r &= \sigma_i(r) x_i, & y_i r &= \sigma_i^{-1}(r) y_i \text{ for all } r \in R, \\y_i x_i &= t_i, & x_i y_i &= \sigma_i(t_i), & [x_i, x_j] &= [y_i, y_j] = [x_i, y_j] = 0 \text{ for } i \neq j.\end{aligned}$$

Proposition

Let $R(\sigma, t)$ be a rank n GWA.

- The algebra $R(\sigma, t)$ is $\mathbb{Z}^n$ -graded by setting $\deg(x_i) = \mathbf{e}_i$, $\deg(y_i) = -\mathbf{e}_i$, and $\deg(r) = \mathbf{0}$ for all $r \in R$.
- (Bavula) If R is (left/right) noetherian, then so is $R(\sigma, t)$.
- (Bavula) If R is a domain, then so is $R(\sigma, t)$.
- (Ebrahim) If each σ_i is locally algebraic, then $\text{GKdim } R(\sigma, t) = \text{GKdim}(R) + n$.
- (Liu) A rank one GWA $\mathbb{k}[h](\sigma, t)$ is twisted Calabi-Yau if and only if it has finite global dimension.

Theorem (Jordan, Wells)

Let $A = R(\sigma, t)$ be a rank one GWA and let α be a primitive ℓ th root of unity. Define the automorphism Θ_α of R by

$$\Theta_\alpha(x) = \alpha x, \quad \Theta_\alpha(y) = \alpha^{-1}y, \quad \text{and } \Theta_\alpha(r) = r \text{ for all } r \in R.$$

The fixed ring $A^{\langle \Theta_\alpha \rangle}$ is the GWA $R(\sigma^\ell, T)$, generated over R by x^ℓ and y^ℓ with defining polynomial $T = \prod_{i=0}^{\ell-1} \sigma^{-i}(t)$.

The following proposition was inspired by work of Kirkman and Kuzmanovich.

Proposition (-, Won)

Let g be a finite filtered automorphism of $A_1(\mathbb{k})$. Then g acts diagonally on a generating set $\{X, Y\}$ of $A_1(\mathbb{k})$ such that $XY - YX = 1$. Thus, $A_1(\mathbb{k})^{\langle g \rangle}$ is a (classical) GWA.

The classical case - automorphisms

A rank one GWA $A = \mathbb{k}[h](\sigma, t)$ is *classical* if $\sigma(h) = h - 1$. Examples of classical GWAs include the Weyl algebra, primitive quotients of $U(\mathfrak{sl}_2)$, and noncommutative deformations of type A Kleinian singularities. Automorphisms of classical GWAs were classified by Bavula and Jordan.

Let $n = \deg_h(t)$, $\lambda \in \mathbb{k}$, $\alpha \in \mathbb{k}^\times$, $m \in \mathbb{N}$, and let $\Delta_m : \mathbb{k}[h] \rightarrow \mathbb{k}[h]$ be the linear map given by $\sigma^m - 1$. Generically, $\text{Aut}(A)$ is generated by the following maps:

$$\Theta_\alpha : x \mapsto \alpha x, y \mapsto \alpha^{-1}y, h \mapsto h,$$

$$\Psi_{m,\lambda} : x \mapsto x, y \mapsto y + \sum_{i=1}^n \frac{\lambda^i}{i!} \Delta_m^i(t) x^{im-1}, h \mapsto h - m\lambda x^m,$$

$$\Phi_{m,\lambda} : x \mapsto x + \sum_{i=1}^n \frac{(-\lambda)^i}{i!} y^{im-1} \Delta_m^i(t), y \mapsto y, h \mapsto h + m\lambda y^m.$$

If t is *reflective* ($t(\rho - h) = (-1)^n t(h)$ for some $\rho \in \mathbb{k}$) there is an additional generator Ω given by

$$\Omega(x) = y, \quad \Omega(y) = (-1)^n x, \quad \Omega(h) = 1 + \rho - h.$$

The classical case - invariants

In the classical case, $A = \mathbb{k}[h](\sigma, t)$ is $\mathbb{N}$ -filtered by setting $\deg(h) = 2$ and $\deg(x) = \deg(y) = \deg_h(t)$.

Generically, in the case $\deg_h(t) > 2$, the only filtered maps are the Θ_α .

In the case t is quadratic, g is a filtered automorphism of R if and only if

- $g = \tau_{\lambda, \mu, \alpha} = \psi_{1, \lambda} \circ \phi_{1, \mu} \circ \Theta_\alpha$ or
- $g = \tau_{\lambda, \mu, \alpha} \circ \Omega$.

Theorem (-, Won)

Let $A = \mathbb{k}[h](\sigma, t)$ be a classical GWA with t quadratic. Let g be a finite filtered automorphism of R . There exists a generating set $\{X, Y\}$ over $\mathbb{k}[h]$ with GWA structure such that g acts diagonally. Hence, $A^{\langle g \rangle}$ is again a classical GWA.

The classical case - global dimension of invariants

Let $A = \mathbb{k}[h](\sigma, t)$ be a classical GWA. Two roots c_1, c_2 of a are said to be *congruent* if there exists an $i \in \mathbb{Z}$ such that, as ideals of $\mathbb{k}[h]$, $(\sigma^i(h - c_1)) = (h - c_2)$. Work of Bavula, Hodges, Jordan, and Stafford implies that

$$\text{gldim } A = \begin{cases} \infty & \text{if } t \text{ has a multiple root} \\ 2 & \text{if } t \text{ has a congruent root and no multiple roots} \\ 1 & \text{if } t \text{ has no congruent roots and no multiple roots.} \end{cases}$$

Corollary

Let $A = \mathbb{k}[h](\sigma, h(h - k))$ be a classical GWA. Let g be a finite filtered automorphism of A .

- If $\text{gldim}(A) = 1$ (resp. ∞), then $\text{gldim}(A^{\langle g \rangle}) = 1$ (resp. ∞).
- If $\text{gldim}(A) = 2$, then $k \in \mathbb{Z}$ and

$$\text{gldim}(A^{\langle g \rangle}) = \begin{cases} 2 & \text{if } |k| \geq |g| \\ \infty & \text{otherwise.} \end{cases}$$

To consider automorphisms on GWAs that do not act trivially on the base ring, we need the following generalization of Jordan and Wells' result.

Theorem $(-, \text{Ho})$

Let R be an integral domain, let $A = R(\sigma, t)$ be a rank one GWA, and let $\phi \in \text{Aut}(A)$ with $\text{ord}(\phi) < \infty$. Suppose

- ϕ restricts to an automorphism of R with $n = \text{ord}(\phi|_R)$,*
- $\phi(x) = \alpha x$ and $\phi(y) = \alpha^{-1}y$ for $\alpha \in \mathbb{k}^\times$ with $m = \text{ord}(\alpha)$, and*
- $\gcd(n, m) = 1$.*

Then $A^{\langle \phi \rangle} = R^{\langle \phi \rangle}(\sigma^m, T)$ with $T = \prod_{i=0}^{m-1} \sigma^{-i}(t)$.

The quantum case - automorphisms

A rank one GWA $R(\sigma, t)$ is *quantum* if $R = \mathbb{k}[h]$ or $\mathbb{k}[h^{\pm 1}]$ and $\sigma(h) = qh$ for $q \in \mathbb{k} \setminus \{0, 1\}$. Examples of quantum GWAs include quantum Weyl algebras, quantum planes, and primitive quotients of $U_q(\mathfrak{sl}_2)$. Suárez-Alvarez and Vivas fully determined the automorphism group for quantum GWAs $A = R(\sigma, t)$:

- Write $t = \sum_{i \in I} t_i h^i$ where $I = \{i : t_i \neq 0\}$.
- Let $g = \gcd\{i - j : t_i t_j \neq 0\}$.
- If t is a monomial then let $C_g = \mathbb{k}^\times$ and otherwise let C_g be the subgroup of $\mathbb{k}^\times$ consisting of g th roots of unity.

For $(\gamma, \alpha) \in C_g \times \mathbb{k}^\times$, define $\eta_{\gamma, \alpha} \in \text{Aut}(R)$ by

$$\eta_{\gamma, \alpha}(h) = \gamma h, \quad \eta_{\gamma, \alpha}(y) = \alpha^{-1} y, \quad \eta(x) = \gamma^{i_0} \alpha x, \quad i_0 \in I.$$

Corollary

Let $A = \mathbb{k}[h](\sigma, t)$ be a quantum GWA and $\eta = \eta_{\gamma, \alpha} \in \text{Aut}(A)$. Set $n = \text{ord}(\gamma)$ and $m = \text{ord}(\alpha)$ with $n, m < \infty$. If $\gcd(n, m) = 1$ and $n \mid i_0$, then $A^{\langle \eta \rangle}$ again a quantum GWA.

Global dimension for quantum GWAs is much more sensitive. The following criteria is due to Bavula and Jordan.

Let $A = R(\sigma, t)$ be a quantum GWA.

- A has infinite global dimension if and only if t has multiple roots.
- A has global dimension 2 if and only if t has no multiple roots and one of the following hold:
 - $R = \mathbb{k}[h]$,
 - q is a root of unity, or
 - t has a pair of congruent roots.
- A has global dimension 1 otherwise.

Suppose $A = R(\sigma, t)$ is a quantum GWA. Let $\eta = \eta_{\gamma, \alpha} \in \text{Aut}(R)$ with $n = \text{ord}(\gamma) < \infty$ and $m = \text{ord}(\alpha) < \infty$ such that $\gcd(n, m) = 1$.

Theorem $(-, \text{Ho})$

- If $\text{gldim } A = 1$, then $\text{gldim } A^{\langle \eta \rangle} = 1$.
- If $\text{gldim } A = \infty$, then $\text{gldim } A^{\langle \eta \rangle} = 2$ if and only if $m = 1$ and 0 is a root of t with multiplicity $k = n$. Otherwise $\text{gldim } A^{\langle \eta \rangle} = \infty$.
- If $\text{gldim } A = 2$ and q is not a root of unity, then $\text{gldim } A^{\langle \eta \rangle} = \infty$ if and only if there exists roots c_i, c_j of t such that $c_i = q^k c_j$ for some k with $0 \leq k \leq m - 1$. Otherwise $\text{gldim } A^{\langle \eta \rangle} = 2$.
- If $\text{gldim } A = 2$ and q is a root of unity, $q \neq 1$, then $\text{gldim } A^{\langle \eta \rangle} = \infty$ if and only if one of the following conditions is satisfied:
 - t has multiple roots,
 - t has congruent roots c_i, c_j such that $c_i = q^k c_j$ with $0 \leq k \leq m - 1$, or
 - there exists k such that $0 < k \leq m - 1$ and $\text{ord}(q)$ divides nk .Otherwise $\text{gldim } A^{\langle \eta \rangle} = 2$.

Definition

Let R be a unital $\mathbb{k}$ -algebra and n a positive integer.

- A *twisted generalized Weyl datum (TGWD)* of rank n is the triple (R, σ, t) where $\sigma = (\sigma_1, \dots, \sigma_n)$ is an n -tuple of commuting automorphisms of R , and $t = (t_1, \dots, t_n)$ is an n -tuple of nonzero central elements of R .
- Given a TGWD (R, σ, t) and $\mu = (\mu_{ij}) \in M_n(\mathbb{k}^\times)$, the associated *twisted generalized Weyl construction (TGWC)*, $\mathcal{C}_\mu(R, \sigma, t)$, is the $\mathbb{k}$ -algebra generated over R by the $2n$ indeterminates $X_1^\pm, \dots, X_n^\pm$ subject to the relations

$$\begin{aligned} X_i^\pm r - \sigma_i^{\pm 1}(r) X_i^\pm & && \text{for all } r \in R \text{ and all } i, \\ X_i^- X_i^+ - t_i, \quad X_i^+ X_i^- - \sigma_i(t_i) & && \text{for all } i, \\ X_i^+ X_j^- - \mu_{ij} X_j^- X_i^+ & && \text{for all } i \neq j. \end{aligned}$$

There is a natural $\mathbb{Z}^n$ -grading on $\mathcal{C}_\mu(R, \sigma, t)$ obtained by setting $\deg(r) = \mathbf{0}$ for all $r \in R$ and $\deg(X_i^\pm) = \pm \mathbf{e}_i$ for all i .

- The associated *twisted generalized Weyl algebra (TGWA)*, $A = \mathcal{A}_\mu(R, \sigma, t)$, is the quotient $\mathcal{C}_\mu(R, \sigma, t)/\mathcal{I}$ where $\mathcal{I}$ is the sum of all graded ideals $J = \bigoplus_{g \in \mathbb{Z}^n} J_g$ such that $J_0 = \{0\}$.

Twisted Generalized Weyl algebras

Examples of TGWAs include

- GWAs of rank n ,
- multiparameter quantized Weyl algebras, and
- certain primitive quotients of enveloping algebras of simple Lie algebras.

Definition

- A TGWD (R, σ, t) is *regular* if t_i is regular in R for all i .
- A TGWD is μ -consistent if the canonical map $R \rightarrow A_\mu(R, \sigma, t)$ is injective.
- A TGWA $A_\mu(R, \sigma, t)$ is *regular* (resp. μ -consistent) if the underlying TGWD is regular (resp. μ -consistent).

If (R, σ, t) is μ -consistent for some parameter matrix μ , then $A_\mu(R, \sigma, t)$ is necessarily non-trivial. If (R, σ, t) is a regular TGWD, then (R, σ, t) is μ -consistent if and only if the following equations hold:

$$\sigma_i \sigma_j(t_i t_j) = \mu_{ij} \mu_{ji} \sigma_i(t_i) \sigma_j(t_j) \text{ for all distinct } i, j, \quad (1)$$

$$t_j \sigma_i \sigma_k(t_j) = \sigma_i(t_j) \sigma_k(t_j) \text{ for all pairwise distinct } i, j, k. \quad (2)$$

Theorem (Futorny, Hartwig)

Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a regular, μ -consistent TGWA. Then A is a domain if and only if R is a domain.

Theorem (-, Rosso)

Let $A = A_\mu(R, \sigma, t)$ (resp. $A' = A_{\mu'}(R', \sigma', t')$) be a regular, μ -consistent (resp. μ' -consistent) TGWAs of rank n (resp. m). Define $\eta = (\eta_{ij}) \in M_{m+n}(\mathbb{k}^\times)$ by

$$\eta_{ij} = \begin{cases} \mu_{ij} & \text{if } 1 \leq i, j \leq m \\ \mu'_{(i-m)(j-m)} & \text{if } m < i, j \leq m+n \\ 1 & \text{otherwise.} \end{cases}$$

Then $A \otimes A'$ is a regular, η -consistent TGWA of rank $m+n$

Question

Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a regular, μ -consistent TGWA. If R is (left/right) noetherian, is A also (left/right) noetherian?

Let $C = \mathcal{C}_\mu(R, \sigma, t)$ be a TGWC of rank n . Assume $\phi \in \text{Aut}(C)$ satisfies

- $\phi|_R$ is an automorphism of R with $\ell = \text{ord}(\phi|_R) < \infty$;
- for each i , $\phi(X_i^\pm) = \alpha_i^{\pm 1} X_i^\pm$ for some $\alpha_i \in \mathbb{k}^\times$ with $m_i = \text{ord}(\alpha_i) < \infty$;
- the integers $\ell, m_1, \dots, m_n$ are pairwise relatively prime;
- either R is a commutative domain or $\phi|_R = \text{id}_R$.

We will call automorphisms satisfying the above *diagonal*.

Lemma

Let $C = \mathcal{C}_\mu(R, \sigma, t)$ be a TGWC and suppose $\phi \in \text{Aut}(C)$ is diagonal. Then $\phi(\mathcal{J}) = \mathcal{J}$. Hence, ϕ descends to an automorphism of $\mathcal{A}_\mu(R, \sigma, t)$ (which we will denote again by ϕ).

The next theorem is the most general version of the Jordan-Wells theorem (thus far).

Theorem

Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a TGWA of rank n and let $\phi \in \text{Aut}(A)$ be diagonal. Let B be the subalgebra of $A^{\langle \phi \rangle}$ generated by $R^{\langle \phi \rangle}$ and the $(X_i^\pm)^{m_i}$. Then B is a TGWA of rank n . Moreover, B is regular and satisfies the first consistency equation if A does.

Let $C = \mathcal{C}_\mu(R, \sigma, t)$ be the TGWC inducing A . Let D be the subalgebra of $C^{\langle \phi \rangle}$ generated by $R^{\langle \phi \rangle}$ and the $(X_i^\pm)^{m_i}$. Then D is a TGWC. Let $\mathcal{J}'$ be canonical ideal associated to D so that $D/\mathcal{J}'$ is a TGWA. The proof requires that we show $B \cong D/\mathcal{J}'$ so that there is a commutative diagram:

$$\begin{array}{ccc} D & \xhookrightarrow{\iota} & C \\ \downarrow \wr & & \downarrow \pi \\ B & \xhookrightarrow{\quad} & A \end{array}$$

Question

When is $A^{\langle \phi \rangle} = B$ so that $A^{\langle \phi \rangle}$ is a TGWA?

Definition

Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a regular, μ -consistent TGWA. For each i, j , define

$$V_{ij} = \text{Span}_{\mathbb{k}}\{\sigma_i^k(t_j) : k \in \mathbb{Z}\}.$$

Then A is $\mathbb{k}$ -finitistic if $\dim_{\mathbb{k}} V_{ij} < \infty$ for all i, j .

Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a $\mathbb{k}$ -finitistic TGWA. For each i, j , let $p_{ij} \in \mathbb{k}[x]$ denote the minimal polynomial for σ_i acting on V_{ij} . Associated to this data we define the (generalized Cartan) matrix $C_A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 1 - \deg p_{ij} & \text{if } i \neq j. \end{cases}$$

Let $A = A_\mu(R, \sigma, t)$ be a $\mathbb{k}$ -finitistic TGWA of (Cartan) type $(A_1)^n$. Then for all $i \neq j$, $p_{ij}(x) = x - \gamma_{ij}$ for some $\gamma_{ij} \in \mathbb{k}^\times$. This condition is equivalent to

$$\sigma_i(t_j) = \gamma_{ij} t_j \quad \text{for all } i \neq j.$$

In this case, A is (isomorphic to) the $\mathbb{k}$ -algebra generated over R by $X_1^\pm, \dots, X_n^\pm$ with the following relations for all $r \in R$ and all i, j with $i \neq j$,

$$\begin{aligned} X_i^\pm r - \sigma_i^{\pm 1}(r) X_i^\pm, \quad X_i^- X_i^+ - t_i, \quad X_i^+ X_i^- - \sigma_i(t_i), \\ X_i^+ X_j^- - \mu_{ij} X_j^- X_i^+, \quad X_i^+ X_j^+ - \gamma_{ij} \mu_{ij}^{-1} X_j^+ X_i^+, \quad X_j^- X_i^- - \gamma_{ij} \mu_{ji}^{-1} X_i^- X_j^-. \end{aligned}$$

Theorem (-, Rosso)

Let $A = A_\mu(R, \sigma, t)$ be a $\mathbb{k}$ -finitistic TGWA of type $(A_1)^n$.

- If R is (left/right) noetherian, then so is A .
- If R is an Auslander-Gorenstein domain, then so is A .
- We have $\text{GKdim}(A) \geq \text{GKdim}(R) + n$.
- If $\text{lgld } R < \infty$ and $\text{lgld } A < \infty$, then $\text{lgld } R \leq \text{lgld } A \leq \text{lgld } R + n$.

Theorem (-, Rosso)

Let R be a domain and let $A = A_\mu(R, \sigma, t)$ be a $\mathbb{k}$ -finitistic TGWA of type $(A_1)^n$. Let ϕ be diagonal. Set

$$s_i = \prod_{k=0}^{m_i-1} \sigma_i^{-k}(t_i),$$

$\tau = (\tau_1, \dots, \tau_n) = (\sigma_1^{m_1}, \dots, \sigma_n^{m_n})$, and $\nu = (\nu_{ij}) = (\mu_{ij}^{m_i m_j})$ for $i \neq j$.

Then $A^{\langle \phi \rangle} = A_\nu(R^{\langle \phi \rangle}, \tau, s)$ is a TGWA of type $(A_1)^n$.

Corollary

Let $A = A_\mu(R, \sigma, t)$ be a $\mathbb{k}$ -finitistic TGWA of type $(A_1)^n$. Suppose ϕ is a diagonal automorphism.

- If R is (left/right) noetherian, then so is $A^{\langle \phi \rangle}$.
- If $R^{\langle \phi \rangle}$ is an AG domain, then so is $A^{\langle \phi \rangle}$.

Let q and β be indeterminates and let $a \in \mathbb{N}$. Define $S_a(q, \beta) \in \mathbb{Z}[q, \beta]$ by

$$S_a(q, \beta) := \sum_{i=0}^{\lfloor \frac{a}{2} \rfloor} (-1)^i \binom{a-i}{i} \beta^i q^{a-2i}.$$

The polynomials $S_a = S_a(q, \beta)$ can also be defined by the recurrence relation

$$S_{a+1} = qS_a - \beta S_{a-1}, \quad S_0 = 1, \quad S_1 = q.$$

They are related to the Chebyshev polynomials of the second kind by

$$S_a(q, \beta) = \beta^{a/2} U_a \left(\frac{q}{2\beta^{1/2}} \right)$$

and they satisfy the identity

$$\beta S_{c-2} S_{a-1} + S_{a+c-1} = S_a S_{c-1}$$

for all $a \geq 1, c \geq 2$.

Let A be a $\mathbb{k}$ -finitistic TGWA of type A_2 . Fix $\lambda_1, \lambda_2, \eta_1, \eta_2 \in \mathbb{k}$ such that

$$p_{12}(x) = x^2 + \lambda_1 x + \lambda_2, \quad p_{21}(x) = x^2 + \eta_1 x + \eta_2.$$

Then A is generated by $R, X_1^\pm, X_2^\pm$ with the usual TGWA relations plus

$$\begin{aligned} (X_1^+)^2 X_2^+ + \lambda_1 \mu_{12}^{-1} X_1^+ X_2^+ X_1^+ + \lambda_2 \mu_{12}^{-2} X_2^+ (X_1^+)^2 &= 0, \\ X_2^- (X_1^-)^2 + \lambda_1 \mu_{21}^{-1} X_1^- X_2^- X_1^- + \lambda_2 \mu_{21}^{-2} (X_1^-)^2 X_2^- &= 0, \\ (X_2^+)^2 X_1^+ + \eta_1 \mu_{21}^{-1} X_2^+ X_1^+ X_2^+ + \eta_2 \mu_{21}^{-2} X_1^+ (X_2^+)^2 &= 0, \\ X_1^- (X_2^-)^2 + \eta_1 \mu_{12}^{-1} X_2^- X_1^- X_2^- + \eta_2 \mu_{12}^{-2} (X_2^-)^2 X_1^- &= 0. \end{aligned}$$

Lemma

Let A be a $\mathbb{k}$ -finitistic TGWA of type A_2 .

- If $S_a(-\lambda_1, \lambda_2) \neq 0$ for all $a \geq 0$, then the monomials

$$(X_2^+)^a (X_1^+)^b (X_2^+)^c, \quad (X_1^+)^a (X_2^-)^b, \quad (X_1^-)^a (X_2^+)^b, \quad (X_2^-)^a (X_1^-)^b (X_2^-)^c$$

with $a, b, c \geq 0$ generate A as a left (and as a right) R -module.

- If $S_a(-\eta_1, \eta_2) \neq 0$ for all $a \geq 0$, then the monomials

$$(X_1^+)^a (X_2^+)^b (X_1^+)^c, \quad (X_1^+)^a (X_2^-)^b, \quad (X_1^-)^a (X_2^+)^b, \quad (X_1^-)^a (X_2^-)^b (X_1^-)^c$$

with $a, b, c \geq 0$ generate A as a left (and as a right) R -module.

Theorem

Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a $\mathbb{k}$ -finitistic TGWA of type A_2 .

- Let ϕ_1 be a diagonal automorphism with $\alpha_1 = 1$. If $S_a(-\eta_1, \eta_2) \neq 0$ for all $a \geq 0$, then $A^{\langle \phi_1 \rangle}$ is a regular, ν -consistent rank 2 TGWA.
- Let ϕ_2 be a diagonal automorphism with $\alpha_2 = 1$. If $S_a(-\lambda_1, \lambda_2) \neq 0$ for all $a \geq 0$, then $A^{\langle \phi_2 \rangle}$ is a regular, ν -consistent rank 2 TGWA.

In general, the fixed rings $A^{\langle \phi_1 \rangle}$ and $A^{\langle \phi_2 \rangle}$ will no longer be of type A_2 .

Thank You!