Invariant holonomic systems for symmetric spaces.

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(joint with Gwyn Bellamy, Thierry Levasseur and Tom Nevins)

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Write V' for the regular semisimple elements in V; thus

$$V' = (\delta \neq 0) \quad \text{for } \delta \text{ the discriminant.}$$

Thus δ is the coordinate function for the product of reflecting hyperplanes in $\mathfrak{h}$.

Harish-Chandra's Theory:

We are interested in the Harish-Chandra modules, notably

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where $\mathrm{Dist}(V_0)$ denotes the distributions on V_0.

Theorem 1 (HC 1965): (1) $\mathcal{G}_\lambda$ has no nonzero δ-torsion factor module.
(2) If $T \in \Omega$, then T cannot be supported on $V_0 \smallsetminus V_0'$.

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The next application of $\mathcal{G}_\lambda$ is:

Theorem 3 (Hotta-Kashiwara 1984) $\mathcal{G}_\lambda$ is a semi-simple $\mathcal{D}(V)$-module with specified irreducible summands.

This has deep consequences for the geometric theory of $\mathfrak{g}$-representations.

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By Chevalley's Theorem, $\mathbb{C}[V]^G \cong \mathbb{C}[\mathfrak{h}]^W$. So $\exists$ a homomorphism

$$\phi : \mathcal{D}(V)^G \longrightarrow \mathcal{D}(V/\!/G) = \mathcal{D}(\mathfrak{h}/\!/W) := \mathcal{D}(\mathbb{C}[\mathfrak{h}]^W).$$

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Theorem 4 (Wallach, Levasseur-S 1993-5) $\mathrm{Im}(\phi) \cong \mathcal{D}(\mathfrak{h})^W$.

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Theorem 4 (Wallach, Levasseur-S 1993-5) $\mathrm{Im}(\phi) \cong \mathcal{D}(\mathfrak{h})^W$.

Warning: as Harish-Chandra showed, the image of ϕ does not lie in $\mathcal{D}(h)^W$.

As Wallach showed, this result is important since it allows one to reduce questions about Ω and $\mathcal{G}_\lambda$ to questions about $\mathcal{D}(\mathfrak{h})^W$, which are much easier to handle.

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Definitions: Let $\widetilde{G}$ be a reductive Lie group with involution θ. Then $\widetilde{g} = Lie(\widetilde{G}) = \mathfrak{g} \oplus V$, where $\mathfrak{g} = \widetilde{g}^{\theta}$ and $V = \mathfrak{p}$ is the (-1)-eigenspace. Set $G = \widetilde{G}^{\theta}$. Here $(\widetilde{G}, \theta)$ is called a symmetric pair and V the corr. symmetric space. Symmetric spaces are classified (see the book by Helgason) and include the adjoint action of G on $\mathfrak{g}$ (take $\widetilde{G} = G \oplus G$ with θ swapping terms).

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As before G acts on V and $\mathcal{D}(V)$ and there is $\mu : \mathfrak{g} \to \mathrm{Der}(V) \subset \mathcal{D}(V)$. Also:

Chevalley Theorem: $\mathbb{C}[V]^G \cong \mathbb{C}[\mathfrak{h}]^W$, and $\mathrm{Sym}(V)^G \cong \mathrm{Sym}(\mathfrak{h})^W$ where $\mathfrak{h} \subseteq V$ is an abelian subalgebra with associated complex reflection group W.

Again there is a discriminant δ and we can define Harish-Chandra modules $\mathcal{G}_\lambda$ as before: for $\lambda \in \mathfrak{h}^*$ with maximal ideal $\mathfrak{m}_\lambda \subset Sym(V)^G \cong Sym(\mathfrak{h})^W$,

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Running Example: Take $\widetilde{G} = SL_2$ and θ conjugation by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then $G = \mathbb{C}^*$, with V the off-diagonal matrices. Set $\mathbb{C}[V] = \mathbb{C}[x_1, x_2]$; thus G acts with weight 1 on x_1 and -1 on x_2. Write

$$\mathcal{D} = \mathcal{D}(V) = \mathbb{C}\langle x_1, x_2, \partial_1, \partial_2 \rangle \quad \text{for} \quad \partial_i = \frac{\partial}{\partial x_i}.$$

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Here $z = x_1x_2$ is the discriminant and $\mu(\mathfrak{g}) = \mathbb{C}\nabla$ for $\nabla = x_1\partial_1 - x_2\partial_2$. Moreover

$$\mathbb{C}[V]^G = \mathbb{C}[z] \cong \mathbb{C}[\mathfrak{h}]^W, \qquad \text{and} \qquad \text{Sym}(V)^G = \mathbb{C}[\partial_1\partial_2].$$

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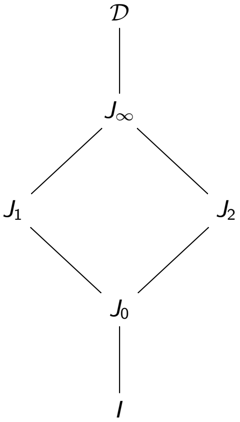
So $\mathcal{G}_0 = \mathcal{D}/\mathcal{D}I$ where $I = \mathcal{D}\nabla + \mathcal{D}\partial_1\partial_2$. Then $\mathcal{G}_0$ has the following lattice of submodules:

Running Example (cont):



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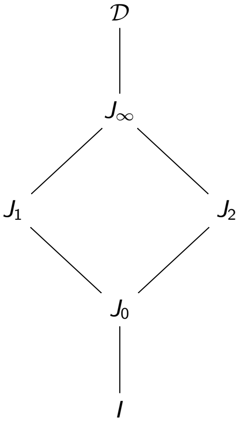

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Conclusion: $\mathcal{G}_0$ has simple top $\mathcal{D}/J_\infty \cong \mathbb{C}[V]$ and simple socle $J_0/I \cong \mathbb{C}[V]$ but the "middle" terms are all δ-torsion. For example, $J_2/J_0 \cong \mathcal{D}(V)/(x_1, \partial_2)$.

The Main Results for a Symmetric Space:

Fix a symmetric space V with $G, \mathfrak{g}$ as before and $\mathbb{C}[V]^G \cong \mathbb{C}[\mathfrak{h}]^W$, etc. You again have a quantum Hamiltonian reduction

$$\phi : \mathcal{D}(V)^G \longrightarrow \mathcal{D}(\mathfrak{h}/\!/W) := \mathcal{D}(\mathbb{C}[\mathfrak{h}]^W).$$

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However, now in general $\mathrm{Im}(\phi) \ncong \mathcal{D}(\mathfrak{h})^W$. Instead we use:

Defn/Theorem (Etingof-Ginzburg): Associated to $(\mathfrak{h}, W)$ one has a spherical algebra (or spherical subalgebra of a Cherednik algebra) A_κ. This is a deformation of $\mathcal{D}(\mathfrak{h})^W$ for some parameter κ and it contains copies of $\mathbb{C}[\mathfrak{h}]^W$ and $\mathrm{Sym}(\mathfrak{h})^W$.

Note: All infinite dimensional primitive factors of $U(\mathfrak{sl}_2)$ appear among the A_κ.

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Theorem 5 (BLNS): (i) $\mathrm{Im}(\phi) \cong A_\kappa$ for some such spherical algebra A_κ.

(ii) If A_κ is a simple algebra then $Ker(\phi) = [\mathcal{D}(V)\mu(\mathfrak{g})]^G$.

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For the Running Example, A_κ is a simple factor ring of $U(\mathfrak{sl}_2)$.

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Remark: To be more precise A_κ is simple provided *either* κ is an integer *or* V is a "nice symmetric space" in the sense of Sekiguchi. The cases when κ is an integer are precisely the examples that appear in work of Berest-Etingof-Ginzburg on rings of quasi-invariants. The nice symmetric spaces are defined as follows:

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Analogous to the situation for Lie algebras, given a symmetric pair $(\widetilde{\mathfrak{g}}, \theta)$ one has

- a *restricted root system* Σ associated to $(\mathfrak{g}, \mathfrak{h})$ and a
- *weight space decomposition* of $\mathfrak{g}$ into weight spaces $\{g_\alpha : \alpha \in \Sigma\}$.

Then $(\widetilde{\mathfrak{g}}, \theta)$ or the corresponding symmetric space V is nice if

$$\dim_{\mathbb{C}} \mathfrak{g}_\alpha + \dim_{\mathbb{C}} \mathfrak{g}_{2\alpha} \leq 2 \qquad \text{for all } \alpha \in \Sigma.$$

A standard fact for semisimple Lie algebras shows that the adjoint symmetric spaces (where $\widetilde{g} = \mathfrak{g} \times \mathfrak{g}$) are indeed nice.

The Main Results for a symmetric space V (cont.):

Theorem 6 (BNS): (1) If $A_\kappa = \mathrm{Im}(\phi)$ is a simple algebra then

$$\mathcal{G}_\lambda = \mathcal{D}(V)/\mathcal{D}(V)\mu(\mathfrak{g}) + \mathcal{D}(V)\mathfrak{m}_\lambda$$

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So Harish-Chandra's Theorem 1 generalises quite nicely.

The Main Results for a symmetric space V (cont.):

Theorem 6 (BNS): (1) If $A_\kappa = \mathrm{Im}(\phi)$ is a simple algebra then

$$\mathcal{G}_\lambda = \mathcal{D}(V)/\mathcal{D}(V)\mu(\mathfrak{g}) + \mathcal{D}(V)\mathfrak{m}_\lambda$$

has no nonzero δ-torsion factor module nor δ-torsion submodule (where δ is the discriminant).

(2) If A_κ is not simple, then $\mathcal{G}_0$ has a nonzero δ-torsion factor module.

Corollary: Assume that A_κ is simple. Then:

(1) $\mathcal{G}_\lambda$ is the minimal extension of its restriction $\mathcal{L} = \mathcal{G}_\lambda|_{V'}$ to the regular locus $V' = (\delta \neq 0)$.

(2) If T is an equiv. eigendist. supported on a real form of $V \smallsetminus V'$, then $T = 0$.

So Harish-Chandra's Theorem 1 generalises quite nicely.

Remarks: For nice symm. spaces, Theorem 6(1) was conjectured by Sekiguchi.

For nice symmetric spaces, the result about δ-torsion factors (and hence part (2) of the corollary) was proved by Sekiguchi and Galina-Laurent.

Generalising the Hotta-Kashiwara Theorem:

Recall that Hotta-Kashiwara proved that $\mathcal{G}_\lambda$ is semisimple in the adjoint case $V = \mathfrak{g}$. And that the analogous result fails for the Running Example.

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Theorem 7 (BNS): If V is any symmetric space then

$$\mathcal{G}_0 = \mathcal{D}(V)/\mathcal{D}(V)\mu(\mathfrak{g}) + \mathcal{D}(V)Sym(V)^G_+$$

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Corollary (BNS): Assume $H(W)$ is semisimple (and hence A_κ is simple). Then

$$\mathcal{G}_0 = \bigoplus \{\mathcal{G}_{0,\rho} \otimes_{\mathbb{C}} \rho^* \mid \rho \in \mathrm{Irr}(H(W))\},$$

as a $(\mathcal{D}(V), H(W))$-bimodule. Each $\mathcal{G}_{0,\rho}$ is irreducible as a $\mathcal{D}(V)$-module and they are non-isomorphic for distinct ρ.

Generalising the Hotta-Kashiwara Theorem (cont):

Corollary (BNS): ($\mathfrak{g}$ a simple Lie algebra.) Over a symmetric space V, the Harish-Chandra module $\mathcal{G}_0$ is semisimple for precisely the following families of symmetric spaces:

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(1) $(\widetilde{\mathfrak{g}}, \mathfrak{g}) = (\mathfrak{sl}(2n), \mathfrak{sp}(2n))$ $n \geq 2$;

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For A_κ the category is $\mathcal{O}^{\mathrm{sph}}$; the category of finitely generated left A_κ-modules on which $\mathrm{Sym}(\mathfrak{h})^W$ acts locally finitely.

For $\mathcal{D}(V)$ one has the category $\mathcal{C}$ of admissible modules; finitely generated, G-equivariant left $\mathcal{D}(V)$-modules on which $\mathrm{Sym}(V)^G \cong \mathrm{Sym}(\mathfrak{h})^W$ acts locally finitely. (This category contains the $\mathcal{G}_\lambda$.)

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The key to these results is to understand the relationship between these two categories. For this we use the $\big(\mathcal{D}(V), A_\kappa\big)$-bimodule

$$\mathcal{M} = \mathcal{D}(V)/\mathcal{D}(V)\mu(\mathfrak{g}).$$

Notice that $A_\kappa = \mathcal{M}^G$ while $\mathcal{G}_\lambda = \mathcal{M} \otimes_{A_\kappa} \mathcal{N}_\lambda$ for $\mathcal{N}_\lambda = A_\kappa / A_\kappa \mathfrak{m}_\lambda$.

Idea behind the proof cont:

Set $A = A_\kappa$, $\mathcal{D} = \mathcal{D}(V)$ and $\mathcal{N}_\lambda = A/A\mathfrak{m}_\lambda$. Set

$$\mathbb{D}_{\mathcal{D}}(-) = \mathrm{Ext}^{n+m}_{\mathcal{D}}(-, \mathcal{D}) \qquad \text{for } n + m = \dim V = \tfrac{1}{2}\mathrm{GKdim}(\mathcal{D}).$$

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Fact: (A) $\mathbb{D}_{\mathcal{D}}$ gives a contravariant equivalence between left and right holonomic $\mathcal{D}$-modules that maps admissible left modules to admissible right modules and δ-torsion modules to δ-torsion modules.

(B) If $Z \in \mathcal{O}^{\mathrm{sph}}$ then $\mathrm{GKdim}(Z) = n$ and (hence) Z cannot have δ-torsion. (Here we identify $\delta \in \mathbb{C}[\mathfrak{h}]^W \subset A$.)

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Intertwining Theorem 8 (BNS). Assume that A is simple. Then

$$\mathbb{D}_{\mathcal{D}}(\mathcal{G}_\lambda) = \mathbb{D}_{\mathcal{D}}(\mathcal{M} \otimes_A \mathcal{N}_\lambda) = \mathbb{D}_A(\mathcal{N}_\lambda) \otimes_{\mathcal{D}} \mathcal{M}', \qquad \text{for } \mathcal{M}' = \mathrm{Ext}^m_{\mathcal{D}}(\mathcal{M}, \mathcal{D}).$$

Idea behind the proof of Theorem 6 cont:

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This contradicts Fact B. □

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Generalisation. *Everything we stated for the $\mathcal{G}_\lambda$ also holds for $\mathcal{G} = \mathcal{M} \otimes_{A_\kappa} \mathcal{P}$, where $\mathcal{P}$ a projective object in $\mathcal{O}^{\mathrm{sph}}$. This class includes the* $\mathcal{G}_\lambda = \mathcal{M} \otimes_{A_\kappa} A_\kappa / A_\kappa \mathfrak{m}_\lambda$.

Further generalisations:

The correct context for these results is for the polar representations (G, V) of Dadok-Kac, since these are perhaps the most general class of representations for which one has an analogue $\mathbb{C}[V]^G \cong \mathbb{C}[\mathfrak{h}]^W$ of the Chevalley isomorphism.

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A nice example of a polar rep. is the following. Let Q_ℓ denote the cyclic quiver with ℓ nodes. Set $V = V_{\ell,n}$ for representation space $V = \mathrm{Rep}(Q_\ell, n\mathfrak{d})$ for dimension vector $n\mathfrak{d} = (n, \ldots, n)$, regarded as a representation for $G = GL(n)^\ell$. Note that the Running Example can also be regarded as $V_{2,1}$.

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All our earlier results apply to $V_{\ell,n}$. In particular, Theorem 5 (saying that $[\mathcal{D}(V_{\ell,n})/\mathcal{D}(V_{\ell,n})\mu(\mathfrak{g})]^G \cong A_\kappa$ for some κ) is a result of Oblomkov and Gordon. The corresponding Harish-Chandra module $\mathcal{G}_0$ is quite striking. For $n = 1$, $\mathcal{G}_0$ still has simple socle and top but has roughly $2^{\ell-1}$ δ-torsion subfactors.

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The earlier results generalise to suitable polar reps. However those results are not as clean as the results for symmetric spaces, so I will skip them.

Thank you.