Enveloping algebras of ∞-dim'l L.A.'s

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(i) Algebras of derivations

$\frac{d}{dt}$

- W = Witt algebra $= \mathbb{C}[t,t^{-1}]\partial = \operatorname{Der}\mathbb{C}[t,t^{-1}]$
- Vir = Virasoro algebra $=_{vsp} W \oplus \mathbb{C}\cdot z$

z central, $[f\partial, g\partial] = (fg' - f'g)\partial + \underbrace{\operatorname{Res}_0(f''g''' - f'''g'')}_{\text{2-cocycle on } W} z$

- $\operatorname{Der} C$ for any comm. algebra C

Γ_1

(iii) Kac-Moody algebras

$A \in M_{n \times n}(\mathbb{Z}) \rightsquigarrow L(A)$, presented by generalised Serre relations

Γ_2

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affine: $L(A)$ "=" $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}] \oplus \mathbb{C}\cdot z =: \hat{\mathfrak{g}}$

for some (fin dim'l simple) $\mathfrak{g}$

Loop algebra of $\mathfrak{g}$: $[x\otimes f,\ y\otimes g] = [x,y]\otimes fg$

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Loop algebra of $\mathfrak{g}$: $[x \otimes f, y \otimes g] = [x, y] \otimes fg$

(iii) $-(\infty)$: a zoo!

<u>Thm</u> (O. Mathieu '92) Let L be $\mathbb{Z}$-graded simple, ∞-dim'l, polynomial growth. Then

$$L = \begin{cases} \mathfrak{g}[t, t^{-1}] \text{ for } \mathfrak{g} \text{ f. dim simple} \\ \mathrm{Der}\, \mathbb{C}[x_1, \ldots, x_n] \\ W \end{cases}$$

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$\mathfrak{g} \rightsquigarrow$ universal enveloping algebra

$$U(\mathfrak{g}) = T(\mathfrak{g}) \big/ {\scriptstyle xy - yx = [x,y]} \quad \text{for } x, y \in \mathfrak{g}$$

Q: Ring theory of $U(L)$ for $\dim L = \infty$?

1. 1-sided ideals (i.e. noetherianity)

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PBW theorem: $U(\mathfrak{g}) \cong_{vsp} S[\mathfrak{g}] = \mathrm{gr}\, U(\mathfrak{g})$

Cor: $\dim \mathfrak{g} < \infty \Rightarrow U(\mathfrak{g})$ $(L=R)$ noetherian

Q: (Amayo-Stewart '74) $\exists$? ∞-dim'l L with $U(L)$ noetherian?

(Dean-Small '90) Is $U(W)$ noetherian?

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(Dean-Small '90) Is $U(W)$ noetherian?

Thm (S.-Walton '13) (1) NO! $U(W)$ not noetherian

(2) L $\mathbb{Z}$-graded simple, ∞-dim'l, polynomial growth $\Rightarrow U(L)$ not noetherian

<u>Lemma</u> $\mathfrak{h} \leq \mathfrak{g}$, $U(\mathfrak{g})$ noeth $\Rightarrow$ $U(\mathfrak{h})$ noeth

$\llcorner$ on Mathieu's list $\supseteq$ $\left\{ \begin{array}{l} \mathfrak{a} \ \infty\text{-dim'l abelian} \\ \quad U(\mathfrak{a}) \text{ not noeth} \\ W_+ = t^2 \mathbb{C}[t]\partial \end{array} \right.$

STP $U(W_+)$ not noetherian

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STP $U(W_+)$ not noetherian

Pf 1: $U(W_+)$ is connected $\mathbb{N}$-graded. Point module?

Behave like point modules for non-noetherian rings which appear in classifying graded domains of GK dim 3 $\rightsquigarrow$ $U(W_+)$ not noetherian

Pf 2: $U(W_+)$

$$\begin{array}{ccc} U(W_+) & \xrightarrow{\varphi} & A_1 = \mathbb{C}[t, \partial] \\ f\partial & \mapsto & f\partial \end{array}$$

$\ker \varphi$ not fin gen as L or R ideal

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Pf 2:

$$\begin{tikzcd} U(W_+) \arrow[r, "\Phi"] \arrow[dr, "\varphi"] & A_1(\mathbb{C}[s]) = \mathbb{C}[s,t,\partial] \arrow[d, "s \mapsto 0"] \\ & A_1 = \mathbb{C}[t,\partial] \end{tikzcd}$$

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Thm $U(L)$ not noeth if L is

- (Buzaglo '21–'22) Der C, C commutative domain
- ∞-dim'l KM

- any other specific example

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<u>Thm</u> $U(L)$ not noeth if L is

- (Buzaglo '21-'22) $\mathrm{Der}\, C$, C commutative domain
- ∞-dim'l KM

<u>Pf</u>: $L \supseteq \begin{cases} \infty\text{-dim'l abelian (affine)} \\ \text{free Lie algebra (indefinite)} \end{cases}$ use lemma

- any other specific example

Thm (Topley 2018). char $k > 0$,

L a graded Lie alg / k of linear growth (∞-dim'l)

Then $U(L)$ is not noetherian

Pf: Show $U(L)$ has a big (i.e. non-noeth.) centre

NB: L = Mathieu's lie $\Rightarrow Z(U(L)) = \mathbb{C}$.

2. 2-sided ideals

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<u>Prop</u> (SW '15)

(1) $\ker \varphi$, $\ker \Phi$ are <u>principal</u> 2-sided ideals

(2) $\operatorname{Im} \Phi$ has ACC on 2-sided ideals

(3) $\operatorname{Im} \Phi$, $\operatorname{Im} \varphi$ have polynomial growth

<u>Conj</u> $\mathcal{U}(W_+)$, $\mathcal{U}(W)$ have ACC on 2-sided ideals [Petukhov–S. 17]

<u>Def:</u> R has <u>just-infinite growth</u> if GKdim $R = \infty$ but $\forall\ 0 \neq I \trianglelefteq R$, GKdim $R/I < \infty$

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Thm [Iyudu-S. '19] $U(W_+)$ has J.I. growth.
So does $U(Vir)/(z-\lambda)$ for any $\lambda \in \mathbb{C}$.
(includes $U(W) = U(Vir)/(z)$)

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<u>Thm</u> (Biswal - S. '22) $\mathfrak{g}$ f. dim simple

(1) $U(\mathfrak{g}[t, t^{-1}])$ has $\underline{\mathrm{JI}}$ growth

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(2) $\lambda \neq 0 \Rightarrow U(\mathfrak{g}) / (z - \lambda)$ is simple (!)

Thm (Biswal–S. '22) $\mathfrak{g}$ f. dim simple

(1) $U(\mathfrak{g}[t, t^{-1}])$ has JI growth

(2) $\lambda \neq 0 \Rightarrow U(\hat{\mathfrak{g}}) / (z - \lambda)$ is simple (!)

Cor: $M \in \mathrm{Rep}\,\hat{\mathfrak{g}}$ has central character $\lambda \neq 0$

$\Rightarrow \mathrm{Ann}_{U(\hat{\mathfrak{g}})} M = (z - \lambda)$

(Annihilators previously known just

for $M =$ Verma (Chari '85))

Q: Rep-theoretic consequences?

Q: For which L does $U(L)$ have JI growth?

Q: Does $U(\mathfrak{g}[t,t^{-1}])$ have ACC on 2-sided ideals?

Q: Is $U(Vir)/(z-\lambda)$ simple for $\lambda \neq 0$?

(Rep theory related to $\hat{\mathfrak{g}}$ via Sugawara construction)

3. 3-sided ideals

3. ~~3-sided ideals~~ Poisson ideals

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3. ~~3-sided~~ ideals Poisson ideals

L any Lie alg, w/ basis $\{x_i\}_{i\in I}$.

The <u>symmetric</u> <u>algebra</u> of L is $S(L) = \mathbb{C}[x_i]_{i\in I}$

$S(L)$ is a Poisson algebra: $\{x_i, x_j\} = [x_i, x_j]$

Recall $S(L) = \operatorname{gr} U(L)$

So $J \trianglelefteq U(L) \implies \operatorname{gr} J$ is a Poisson ideal of $S(L)$
(write $\operatorname{gr} J \trianglelefteq_p S(L)$, ie $\operatorname{gr} J \trianglelefteq S(L)$ and a Lie ideal for $\{,\}$)

$S(L)$ has ACC on Poisson ideals
$\implies U(L)$ has ACC on 2-sided ideals

$\boxed{12}$

Let S be any Poisson algebra

Q1: What are the Poisson primitive Poisson ideals of S?

(Recall: $\exists$ $\mathfrak{m} \in \mathrm{MSpec}\, S$ so that

$P = \mathrm{PCore}(\mathfrak{m}) :=$ max'l Poisson ideal contained in $\mathfrak{m}$)

Q2: Does S satisfy the Poisson DME?

(Poisson locally closed $\Longleftrightarrow$

Poisson primitive $\Longleftrightarrow$ Poisson rational)

Def: $\mathcal{Z}(S/P)$ be trivial Poisson centre

$\boxed{13}$

Thm (Iyudu – S. '19) Let $\lambda \in \mathbb{C}$ and $0 \neq I \lhd_p S(\mathrm{Vir})/(z-\lambda)$. Then $\left(S(\mathrm{Vir})/(z-\lambda) \right)/I$ has polynomial growth

Thm (León-Sánchez – S. '20) If $L = \mathrm{Vir}$ or L is on Mathieu's list then $S(L)$ has ACC on radical Poisson ideals

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<u>Thm</u> (León-Sánchez – S. '20) If $L = \mathrm{Vir}$ or L is on Mathieu's list then $S(L)$ has ACC on <u>radical</u> Poisson ideals

$\hookrightarrow$ $0 \neq I \triangleleft_P S(\mathrm{Vir})/(z-\lambda)$ expect algebraic geometry on $V(I)$: finite dim'l, finitely many components.

Thm (Petukhov - S. '21)

(1) Classify the Poisson primitive ideals of $S(Vir)$

(2) $\lambda \neq 0 \Rightarrow (z-\lambda)$ is <u>maximal</u> Poisson ideal,

ie $S(Vir)/(z-\lambda)$ is Poisson simple

<u>Q:</u> Is $U(Vir)/(z-\lambda)$ simple for $\lambda \neq 0$?

(3) Parametrise Poisson primes of $S(W) = S(Vir)/(z)$

(4) The PDME holds for all Poisson primes of $S(Vir)$

<u>except</u> (z) which is primitive, rational, <u>not</u> locally closed.

(1) $\Rightarrow$

Thm (PS '21)

(a) $L \leq Vir$ finite codimension

$\Rightarrow z \in L' = [L, L]$

(b) $L \leq W$ finite codimension

$\Rightarrow \exists f \in \mathbb{C}[t, t^{-1}]$ so that $fW \subseteq L$.

$\neq 0$

(so $L \leq Vir \Rightarrow fW \oplus \mathbb{C}z \subseteq L$)

Q: Does $S(W)$ have ACC on Poisson ideals?

$\Updownarrow$

$S(\mathrm{Vir})$

$\searrow$ $U(\mathrm{Vir})$ has ACC on 2-sided ideals

Q: Primitive ideals of $U(\mathrm{Vir})$?

Q: "orbit method"? ie

Poisson primitives of $S(\mathrm{Vir}) \longrightarrow$ primitives of $U(\mathrm{Vir})$

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Happy birthday & congratulations!