Support Varieties for Finite Tensor Categories

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Finite tensor categories

Definition A **finite tensor category** $\mathcal{C}$ is a locally finite k-linear abelian category with finitely many simple objects, enough projectives, and a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying some conditions. There is a **unit object** $\mathbf{1}$ that is simple, and every object has both left and right duals. Assume $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1}) = k$, a field.

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(3) Benson-Etingof-Ostrik symmetric tensor categories

Module categories

Definition An **exact module category** $\mathcal{M}$ over $\mathcal{C}$ is a locally finite k-linear abelian category with a bifunctor $*$: $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$, exact in the first argument and compatible with the structures of $\mathcal{C}$ and $\mathcal{M}$, for which $P * M$ is projective whenever P is projective. Assume $\mathcal{M}$ has finitely many simple objects.

Examples

(1) $\mathcal{M} = \mathcal{C}$ and $*$ is $\otimes$

(2) $\mathcal{M}$ - finite tensor category, $\mathcal{C} = Z(\mathcal{M})$ (the center of M),
$*$ is the forgetful functor $Z(\mathcal{M}) \to \mathcal{M}$ followed by $\otimes$

(3) More generally, any $\mathcal{C}$, $\mathcal{M}$ for which $\exists$ exact functor $\mathcal{C} \to \mathcal{M}$,
e.g. restriction functor from representation category of Hopf algebra to Hopf subalgebra

Hopf algebras

A **Hopf algebra** is an algebra A over a field k together with algebra homs. $\Delta : A \to A \otimes A$, $\varepsilon : A \to k$ and an algebra anti-hom. $S : A \to A$ satisfying some conditions.

Hopf algebras include:
- group algebras kG $(\Delta(g) = g \otimes g$ for all $g \in G)$
- universal enveloping algebras of Lie algebras $U(\mathfrak{g})$,
  restricted Lie algebras $(\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g})$
- quantum groups $U_q(\mathfrak{g})$, small quantum groups $u_q(\mathfrak{g})$

Their categories of modules are examples of tensor categories:
If M, N are A-modules, then $M \otimes N$ is an A-module via Δ;
set $\mathbf{1} = k$, an A-module via the augmentation map $\varepsilon : A \to k$

Cohomology

$\mathcal{C}$ - finite tensor category with tensor product $\otimes$ and unit object $\mathbf{1}$

Notation $\mathrm{H}^*(\mathcal{C}) := \mathrm{Ext}^*_{\mathcal{C}}(\mathbf{1},\mathbf{1}) = \bigoplus_{n\geq 0} \mathrm{Ext}^n_{\mathcal{C}}(\mathbf{1},\mathbf{1})$

where $\mathrm{Ext}^n_{\mathcal{C}}(\mathbf{1},\mathbf{1})$ consists of equivalence classes of n**-extensions**

$$\mathbf{1} \to X_{n-1} \to \cdots \to X_1 \to X_0 \to \mathbf{1}$$

of objects in $\mathcal{C}$.

Similarly denote $\mathrm{H}^*(M) := \mathrm{Ext}^*_{\mathcal{M}}(M,M)$ for objects M of $\mathcal{M}$, defined in terms of equivalence classes of n-extensions of M by M.

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Conjecture
If $\mathcal{C}$ is a finite tensor category, then $\mathsf{H}^*(\mathcal{C})$ is finitely generated, and $\mathsf{H}^*(X)$ is a finitely generated $\mathsf{H}^*(\mathcal{C})$-module for all $X \in \mathcal{C}$.

For those $\mathcal{C}$ for which the conjecture holds, it follows, by a theorem of Negron-Plavnik (2022), that $\mathsf{H}^*(\mathcal{M})$ is a finitely generated $\mathsf{H}^*(\mathcal{C})$-module.

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Caution Analogous Hochschild cohomology of finite dimensional algebra <u>not</u> finitely generated in general.

Status of the conjecture

$\mathrm{H}^*(\mathcal{C})$ is known to be finitely generated etc. in case:

• $\mathcal{C} = A$-mod for a fin. dim. cocommutative Hopf algebra A in positive characteristic (Friedlander-Suslin 1997, generalizing work of Golod 1959, Venkov 1959, Evens 1960, Friedlander-Parshall 1983)

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- $\mathcal{C} = A$-mod for many other classes of Hopf algebras (Gordon 2000, Mastnak-Pevtsova-Schauenburg-W 2010, Vay-Ştefan 2016, Drupieski 2016, Erdmann-Solberg-Wang 2018, Nguyen-Wang-W 2018, Friedlander-Negron 2018, Negron-Plavnik 2018, Negron 2021, Angiono-Andruskiewitsch-Pevtsova-W 2022, $\ldots$)

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Object M of $\mathcal{M}$ $\rightsquigarrow$ $\mathsf{H}^*(\mathcal{C})$-module $\mathsf{H}^*(M) := \mathsf{Ext}^*_{\mathcal{M}}(M, M)$
(and the maximal ideal spectrum of the quotient of $\mathsf{H}^*(\mathcal{C})$ by its annihilator)

Varieties for tensor categories: details

From now on let $\mathcal{C}$ be a finite tensor category for which $\mathrm{H}^*(\mathcal{C}) := \mathrm{Ext}^*_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$ is a finitely generated graded commutative algebra over the field k, and $\mathrm{H}^*(M) := \mathrm{Ext}^*_{\mathcal{M}}(M, M)$ is a finitely generated $\mathrm{H}^*(\mathcal{C})$-module for each object M of $\mathcal{M}$.

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See also Buan-Krause-Snashall-Solberg 2020, Nakano-Vashaw-Yakimov arXiv 2019, 2020, 2021, for tensor triangulated categories; Maurer 2019 for Lie superalgebras via relative cohomology

Varieties for tensor categories: example

$\mathcal{C} = \mathcal{M} = kG$-mod, where G is a cyclic group of prime order p and k has characteristic p:

$\mathsf{H}^*(\mathcal{C})$ is essentially $k[x]$, so $\mathcal{V}(k)$ is a line.

More generally if M is the indecomposable kG-module with $\dim_k(M) = n$ and $n < p$, then $\mathcal{V}(M)$ is a line.

Complexity

The **complexity** $\mathrm{cx}(M)$ of an object M is the rate of growth of a minimal projective resolution P of M, $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$, as measured by length of the P_n.

Theorem (Bergh-Plavnik-W) Let M be an object of $\mathcal{M}$. Then $\mathrm{cx}(M) = \dim \mathcal{V}(M)$.

Carlson's L_ζ-objects

Let $\Omega^n(\mathbf{1})$ be the nth syzygy of $\mathbf{1}$, and $\zeta \in \mathsf{H}^n(\mathcal{C}) \cong \mathsf{Hom}_{\mathcal{C}}(\Omega^n(\mathbf{1}), \mathbf{1})$, nonzero. Since $\mathbf{1}$ is simple, there is an object L_ζ and a short exact sequence

$$0 \to L_\zeta \longrightarrow \Omega^n(\mathbf{1}) \xrightarrow{\zeta} \mathbf{1} \to 0.$$

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Theorem (Bergh-Plavnik-W) For every object M of $\mathcal{M}$,

$$\mathcal{V}(L_\zeta * M) = \mathcal{V}(L_\zeta) \cap \mathcal{V}(M),$$

and $\mathcal{V}(L_\zeta) = Z(\zeta) := \mathrm{Max}(\mathrm{H}^*(\mathcal{C})/(\zeta))$.

Module product property

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Remark This is known to be an equality when $\mathcal{C} = \mathcal{M} = A$-mod for a cocommutative Hopf algebra A (Friedlander-Pevtsova 2005) and for some quantum groups A (Negron-Pevtsova arXiv 2020, Nakano-Vashaw-Yakimov arXiv 2020).

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In general, it is only known to be an equality when $X = L_\zeta$ for some ζ.

It is known **not** to be an equality for some modules of some noncocommutative Hopf algebras (Benson-W 2014, Plavnik-W 2018).

Module product property: reduction to complexity 1

Theorem (Bergh-Plavnik-W) Let $\mathcal{C}$ be a *braided* finite tensor category with finitely generated cohomology etc., $\mathcal{M}$ an exact module category. TFAE:

(i) $\mathcal{V}(X * M) = \mathcal{V}(X) \cap \mathcal{V}(M)$ for all objects X, M.

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Remark
$\mathcal{V}(X * M) \subseteq \mathcal{V}(X) \cap \mathcal{V}(M)$ follows from defns of actions and braiding.

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Thm TFAE: (i) $\mathcal{V}(X * M) = \mathcal{V}(X) \cap \mathcal{V}(M)$ for all X, M.
(ii) $\mathcal{V}(X * M) = \mathcal{V}(X) \cap \mathcal{V}(M)$ for all X, M of complexity 1.

Idea of proof that (ii) implies (i): Assume cx$(X) \geq 1$, cx$(M) \geq 1$.
Induction on cx(X) + cx(M); (ii) is case cx(X) + cx$(M) = 2$.

Case cx$(M) \geq 2$: Reduce to case $\mathcal{V}(M) = Z(\mathfrak{p})$ for a minimal prime $\mathfrak{p}$.
Reduce complexity: $\exists\, \zeta_i \in \mathrm{H}^*(\mathcal{C})$ with
cx$(L_{\zeta_i} * M) =$ cx$(M) - 1$ and (a) $\mathcal{V}(M) = \cup_i \mathcal{V}(L_{\zeta_i} * M)$.
By induction, (b) $\mathcal{V}(X * (L_{\zeta_i} * M)) = \mathcal{V}(X) \cap \mathcal{V}(L_{\zeta_i} * M)$.

Combining (a) and (b): $\mathcal{V}(X) \cap \mathcal{V}(M) \overset{(a)}{=}$

$\mathcal{V}(X) \cap (\cup_i \mathcal{V}(L_{\zeta_i} * M)) \overset{(b)}{=} \cup_i \mathcal{V}(X * (L_{\zeta_i} * M)) \overset{(c)}{\subseteq} \mathcal{V}(X * M)$,
(c) by properties of $*$ and varieties.

Summary of open questions

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(2) Is $\mathcal{V}(X * M) = \mathcal{V}(X) \cap \mathcal{V}(M)$?
(Known to be true in some settings; known not to be true in others; unknown in general; for $\mathcal{C}$ braided, reduced to objects of complexity 1.)