Deformations of gentle A_∞-algebras and some applications

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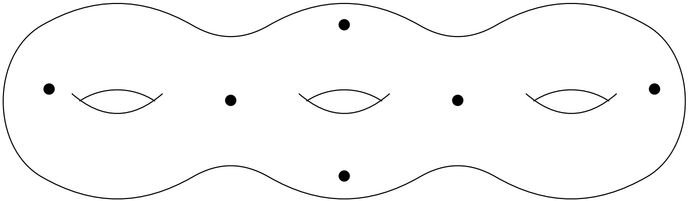
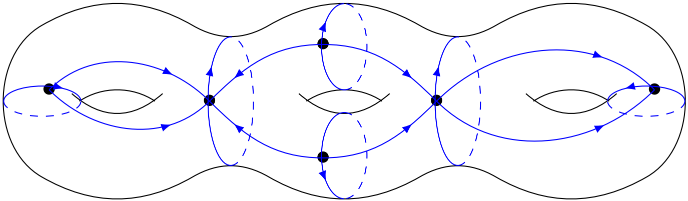
A surface with marked points (S, M)



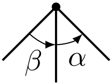
An arc collection $\mathcal{A}$



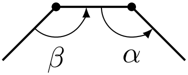
The gentle algebra $\mathrm{Gtl}_{\mathcal{A}}$

The gentle algebra $\mathrm{Gtl}_{\mathcal{A}}$ associated to an arc collection $\mathcal{A}$ is the path algebra of a quiver Q with relations

- The vertices of Q are 1-1 with the arcs.
- The arrows of Q are 1-1 with the angles of the polygons.
- The product of two consecutive angles in a polygon is zero.

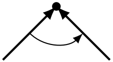


$\alpha\beta \neq 0$

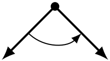


$\alpha\beta = 0$

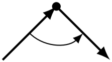
- Each arrow gets a $\mathbb{Z}_2$-degree.



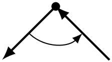
$|\alpha| = 0$



$|\alpha| = 0$

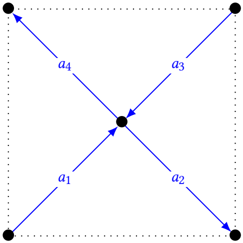


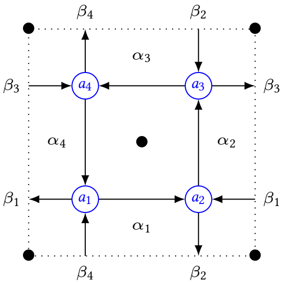
$|\alpha| = 1$



$|\alpha| = 1$

An example of a gentle algebra



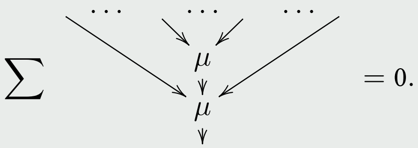


Definition

An A_∞-*algebra* is a graded vector space $A^\bullet$ with a collection of maps $\mu^i : A[1]^{\otimes i} \to A[1]$ of degree 1 with $i = 1, 2, \cdots$ such that the identities

$$[\mathrm{M_n}] \qquad \sum_{\substack{r+s+t=n \\ r,t\geq 0, s>0}} \mu^{r+t+1} \circ (\mathbf{1}^{\otimes r} \otimes \mu^s \otimes \mathbf{1}^{\otimes t}) = 0$$

hold for all $n \geq 1$. The formula can be graphically represented as

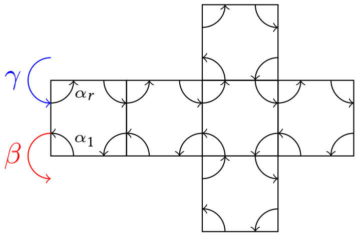


where the sum is over all rooted trees with n leaves and two nodes.

Just like for ordinary algebras there are notions of

- A_∞-modules.
- Perfect Complexes (a.k.a. twisted complexes).
- The (Derived) Category of Perfect complexes is a triangulated category.
- Derived Morita equivalences.

An A_∞-structure on the gentle algebra using 'tree-gons'



$$\mu(\beta\alpha_1, \ldots, \alpha_r) = \pm\beta$$

$$\mu(\alpha_1, \ldots, \alpha_r\gamma) = \pm\gamma$$

Gentle A_∞-algebras as surface invariants

Theorem (B., Haiden-Katzarkov-Kontsevich)

- *The products above turn $Gtl_{\mathcal{A}}$ into an A_∞-algebra.*
- *Two arc collections of the same marked surface have Morita equivalent gentle A_∞-algebras.*
- *The derived category $D(Gtl_{\mathcal{A}})$ is an algebraic model of the wrapped Fukaya category of the punctured surface.*

Question: How to fill the punctures?
Philosophy: Filling is deforming.

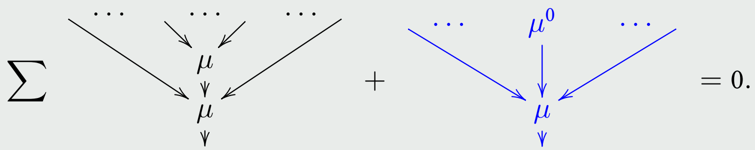
Curved A_∞-algebras

Definition

A *curved* A_∞*-algebra* is a graded vector space $A^\bullet$ with a collection of maps $\mu^i : A[1]^{\otimes i} \to A[1]$ of degree 1 with $i = 0, 1, 2, \cdots$ such that the identities

$$[\mathrm{M_n}] \qquad \sum_{\substack{r+s+t=n \\ r,t\geq 0,\, s\geq 0}} \mu^{r+t+1} \circ (\mathbf{1}^{\otimes r} \otimes \mu^s \otimes \mathbf{1}^{\otimes t}) = 0$$

hold for all $n \geq 0$. The formula can be graphically represented as



where the sum is over all rooted trees with n leaves and two nodes.

Representation theory of curved A_∞-algebras

You can make curved notions of modules and complexes but

- curved complexes have no homology theory $d^2 \neq 0$,
- curved modules do not give rise to a nice derived category,
- derived Morita equivalences are ill-defined.

We need curved A_∞-algebras in deformation theory

Morita-equivalent A_∞-algebras have the same deformation theory provided we allow curved deformations.

Example

$\mathbb{C}[\![X]\!]$ with $\deg X = 0$ and $\mathbb{C}[\xi]/(\xi)^2$ with $\deg \xi = 1$ are derived Morita equivalent.

1. The curved A_∞-deformations of $\mathbb{C}[\xi]/(\xi)^2$ are parametrized by $(a_i)_{i\in\mathbb{N}}$ such that
$$\mu^i(\xi, \ldots, \xi) = a_i \cdot 1$$
2. The curved A_∞-deformations of $\mathbb{C}[\![X]\!]$ are parametrized by $(a_i)_{i\in\mathbb{N}}$ such that
$$\mu^0(1) = \sum_i a_i X^i \text{ and } \mu^{>1} = 0$$

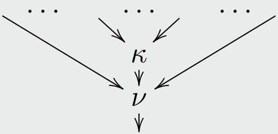
Definition

If (A, μ) is a $\mathbb{Z}$- or $\mathbb{Z}_2$-graded A_∞-algebra over a semisimple algebra k, we define the A_∞-Hochschild complex as

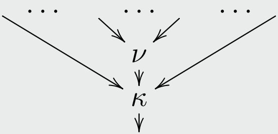
$$\mathrm{HC}^\bullet(A) = \mathrm{Hom}_k\left(\bigoplus_{i\geq 0} A[1]^{\otimes_k i}, A[1]\right)$$

On $\mathrm{HC}^\bullet(A)$ we have a bracket of degree 0 and a differential of degree 1:

$[\kappa, \nu] = \sum$



$\pm$



$$d := [\mu, -]$$

Deformation theory and the Hochschild complex

- The A_∞ Hochschild complex is a shift of the classical Hochschild complex such that $\mathrm{HC}^1(A) = \mathrm{HC}^2_{class}(A)$.
- The curved A_∞ deformations are solutions to the Maurer-Cartan equation

$$d\nu + [\nu, \nu] = 0 \text{ with } \nu \in \mathrm{HC}^1(A) \otimes R.$$

- Equivalent deformations are in the same orbit under the infinitesimal action

$$\kappa \cdot \nu = d\kappa + [\kappa, \nu] + [\nu, \kappa] \text{ with } \kappa \in \mathrm{HC}^0(A) \otimes R.$$

- Derived equivalent A_∞-algebras have quasi-isomorphic Hochschild complexes (Keller).

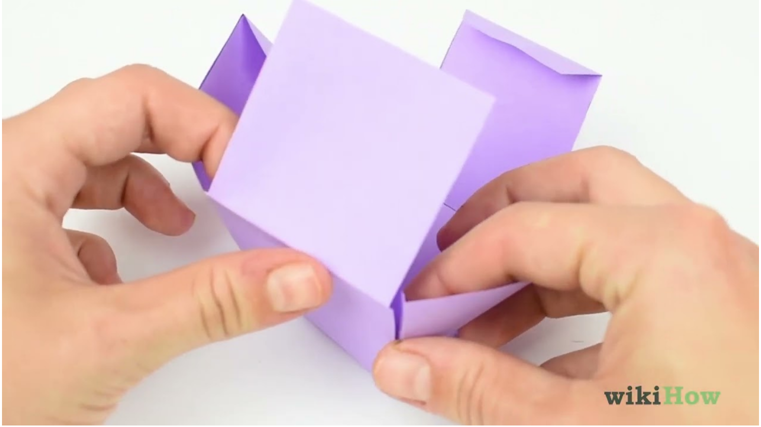
Theorem (B.-Van de Kreeke, Wong (1,2))

If the arc collection has no loops or 2-cycles then

1. $\mathrm{HH}^{\mathrm{odd}}(A) \cong \mathbb{C} \oplus \bigoplus_{p\in M, i\geq 1} \nu^o_{p,i}$
   $\cong Z(A) = \mathbb{C}[\ell_p | p \in M]/(\ell_p \ell_q | p \neq q \in M).$
2. $\mathrm{HH}^{\mathrm{even}}(A) \cong \bigoplus_{a\in\mathcal{A}\setminus T} \mathbb{C}\nu_a \bigoplus_{p\in M, i\geq 1} \mathbb{C}\nu^e_{p,i}$
3. *We have explicit representatives for each of the classes and formulas for the bracket and the cup product.*
4. *The Hochschild complex is formal (quasi-isomorphic to its cohomology).*

This implies that for each $r = r_0 + \sum r_{p,i}\nu^o_{p,i}$ there must be a deformation.

Orbigons and curved deformations



$$r_\psi = \prod_{\text{internal }(p,i)} r_{p,i}$$

$$\mu^\psi(\beta\alpha_1, \ldots, \alpha_r) = \pm\beta$$

$$\mu^\psi(\alpha_1, \ldots, \alpha_r\gamma) = \pm\gamma$$

Orbigons and curved deformations

Theorem (B.-Van de Kreeke)

For each $r = r_0 + \sum r_{p,i}\nu^o_{p,i}$ *there is a curved deformation* ν_r *satisfying*

- $\nu_r^0 = r_0 + \sum r_{p,i}\ell_p^i$
- $\nu_r^k = \sum_{|\psi|=k} r_\psi \mu^\psi$

Advantage: all products are explicit (algorithmically determinable)
Disadvantage: curved, so representation theory doesn't behave well.

Deforming complexes

1. We can transport the deformation ν_r to the category $D(A)$ by considering a complex of free modules as a 'complex' over the deformed algebra.
2. Problem: the complex becomes curved: $MC(\delta) \neq 0$.
3. Solution: uncurving

$$(P, \delta) \to (P, \delta + \dots)$$

   such that $MC(\delta + \dots) = 0$. But not every object in $D(A)$ can be uncurved.
4. Example: deform $A = \mathbb{C}[X]$ by adding curvature ϵX^n then
   - $\mathbb{C}[X] \xrightarrow{X} \mathbb{C}[X]$ can be uncurved as $\mathbb{C}[X] \underset{\epsilon X^{n-1}}{\overset{X}{\rightleftarrows}} \mathbb{C}[X]$
   - $\mathbb{C}[X] \xrightarrow{X-1} \mathbb{C}[X]$ cannot be uncurved.

Towards the relative Fukaya category

If all arrows have degree 1, we have a family of objects in $D(A)$

$$\mathcal{Z}_\lambda := \; A \xrightarrow{\sum_{\alpha \in Q_1} \lambda_i \alpha} A$$

Theorem (Van de Kreeke)

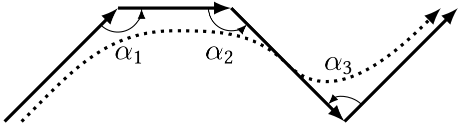
Fix the deformation $\sum_p r_p \ell_p^{n_p}$. If the arc system is consistent then

1. *$\mathcal{Z}_\lambda$ can be uncurved.*
2. *The minimal model of $\mathrm{End}_{\mathrm{D(A)}}(\mathcal{Z}_\lambda)$ has an uncurved deformation that matches the endomorphism ring of a certain generator of the relative Fukaya category.*

In general a Fukaya category has as objects Lagrangian submanifolds of a symplectic manifold and as morphisms linear combinations of intersection points.

- The wrapped Fukaya category describes the intersection theory of open curves on a punctured surface.
- The exact Fukaya category describes the intersection theory of closed curves on a punctured surface.
- The Fukaya category describes the intersection theory of closed curves on a closed surface (defined over a special field: the Novikov field).
- The relative Fukaya category describes the intersection theory of closed curves on a closed surface relative to a divisor (can be seen as a deformation of the exact Fukaya category).

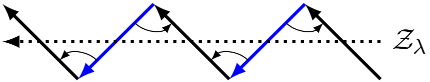
Basic idea: taking cones is stitching

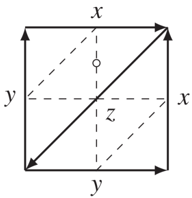


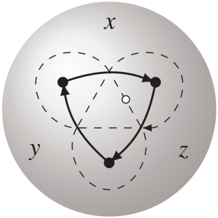
can be represented by a twisted complex

$$(L, \delta) := \left(\bigoplus a_j[i_j], \sum_{j=1}^{k-1} \alpha_u \right).$$

Basic idea: taking cones is stitching







Koszul duality

1. If $A = k\langle\langle X\rangle\rangle/I$ is an augmented completed algebra then the Koszul dual is defined as $A^! = \mathrm{Ext}^\bullet(k,k)$.

$$\text{E.g. } A = \mathbb{C}[\![X,Y]\!] \implies A^! = \mathbb{C}[\xi,\eta]/(\xi^2,\eta^2,\xi\eta+\eta\xi)$$

2. The degree n coefficients of the relations of A are encoded in the higher products of $A^!$ (Lu-Palmieri-Wu-Zhang)

$$r = \lambda X_{i_1}\dots X_{i_k} + \cdots \iff \mu_{A^!}(\xi_{i_k},\dots,\xi_{i_1}) = \lambda\rho + \dots$$

3. We can reconstruct a resolution for k as an A-module using $A^!$: $(A\otimes A^!, d)$ with

$$d(1\otimes u) = \lambda X_{i_1}\dots X_{i_k}\otimes v + \cdots \iff \mu_{A^!}(\xi_{i_k},\dots,\xi_{i_1},v) = \lambda u + \dots$$

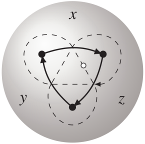
4. Starting from $A^!$ we can get A and a resolution for k.

'Koszul duality' with curvature (after Cho-Hong-Lau)

1. If $A^!$ is $\mathbb{Z}_2$-graded then $\mu(\dots) = \lambda 1 + \dots$.
2. Split $A^! = k \oplus [A^!]^1 \oplus [A^!]^2 \oplus k$
3. There is a relation dual to 1, keep it aside and call it W and group the relations dual to $[A^!]^2$ in R.
4. Construct $A = \widehat{T_k[A^!]^{1*}}/R$ then W is a central element in A and $(A \otimes A^!, d)$ satisfies $d^2 = W$.
5. Starting from $A^!$ we can get (A, W) and a matrix factorization for k.

‘Koszul duality’ for $\mathcal{Z}_\lambda$ over the bowling ball

For the $(\mathbb{S}^2, \{p_1, p_2, p_3\})$ and deformation $r = r_1\ell_1^3 + r_2\ell_2^3 + r_2\ell_3^3$.



1. $\mathcal{Z}_{-\mathbf{1}}$ over the undeformed (A, μ): $(\mathbb{C}[\![X, Y, Z]\!], XYZ)$
2. $\mathcal{Z}_\lambda$ over the undeformed (A, μ): $(\mathbb{C}_{q_\lambda}[\![X, Y, Z]\!], XYZ)$
3. $\mathcal{Z}_{-\mathbf{1}}^{uc}$ over the deformed $(A, \mu + \nu_r)$: $(\mathbb{C}[\![X, Y, Z]\!], XYZ + a_r X^3 + b_r Y^3 + c_r Z^3)$
4. $\mathcal{Z}_\lambda^{uc}$ over the deformed $(A, \mu + \nu_r)$: $(\widehat{\mathrm{Skl}_{\lambda,r}}, W_{\lambda,r})$

‘Koszul duality’ for $\mathcal{Z}_\lambda$ in general

Theorem (Cho-Hong-Lau)

$\mathcal{Z}_{-\mathbf{1}}$ over the undeformed $(Gtl_{\mathcal{A}}, \mu)$ gives the completed Jacobi algebra with its standard central element.

Theorem (Van de Kreeke)

If the arc collection is consistent then $\mathcal{Z}^{uc}_{-\mathbf{1}}$ ($\mathcal{Z}^{uc}_{\lambda}$) over the deformed $(Gtl_{\mathcal{A}}, \mu + \nu_r)$ gives a flat family of Calabi-Yau deformations of the completed (quantum) Jacobi algebra.

And finally...