On the Spectrum and Support Theory of a Finite Tensor Category

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jt work with Daniel Nakano (Univ Georgia) and Kent Vashaw (MIT)

Recent Advances and New Directions
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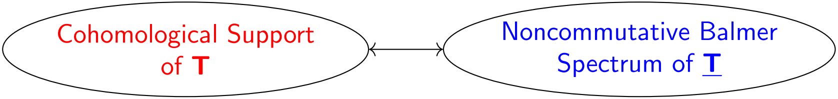
Goals of the Talk

$\mathbf{T}$= a Finite Tensor Category, $\underline{\mathbf{T}}$=its stable category



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- every object has finite length and $\dim_{\Bbbk} \mathrm{Hom}_{\mathbf{T}}(M, N) < \infty$;
- there are enough projectives;
- $- \otimes -$ is bilinear on spaces of morphisms;
- every object is dualizable;
- $\mathrm{End}_{\mathbf{T}}(\mathbf{1}) \cong \Bbbk$.

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- every object is dualizable;
- $\mathrm{End}_{\mathbf{T}}(\mathbf{1}) \cong \mathbb{k}$.

1. Background: Vast class, e.g., $\mathrm{mod}(H)$ for all finite dimensional Hopf algebras H.
2. Important symmetric/braided FTCs not coming from Hopf algebras.
3. Used in: Representation theory, Mathematical Physics, Quantum Computing, Low-dimensional Topology.

The stable category

- The tensor product is biexact.
- Finite tensor categories are Frobenius categories.
- We can form the stable category of $\mathbf{T}$, denoted $\underline{\mathbf{T}}$ having the same objects as $\mathbf{T}$ and morphisms

$$\mathrm{Hom}_{\underline{\mathbf{T}}}(M, N) := \mathrm{Hom}_{\mathbf{T}}(M, N) / \mathrm{PHom}_{\mathbf{T}}(M, N).$$

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- $- \otimes -$ descends to a biexact functor $\underline{\mathbf{T}} \times \underline{\mathbf{T}} \to \underline{\mathbf{T}}$.
- In particular, we have the isomorphisms

$$(\Sigma M) \otimes N \cong \Sigma(M \otimes N) \cong M \otimes (\Sigma N).$$

The cohomological support

Developed by Quillen (1971), Alperin-Evens (1981), Avrunin-Scott (1982), Carlson (1983), Benson-Carlson-Rickard (1996), ...

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The cohomology ring of an FTC $\mathbf{T}$ is the graded commutative ring

$$R^{\bullet}_{\mathbf{T}} := \bigoplus_{k \geq 0} \mathrm{Hom}_{\underline{\mathbf{T}}}(\mathbf{1}, \Sigma^k \mathbf{1})$$

with product

$$fg = (\Sigma^j f)g = f \otimes g \quad \text{for} \quad f \in \mathrm{Hom}_{\underline{\mathbf{T}}}(\mathbf{1}, \Sigma^k \mathbf{1}), g \in \mathrm{Hom}_{\underline{\mathbf{T}}}(\mathbf{1}, \Sigma^j \mathbf{1}).$$

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Then

$$\mathrm{Hom}^{\bullet}_{\underline{\mathbf{T}}}(M, N) := \bigoplus_{k \geq 0} \mathrm{Hom}_{\underline{\mathbf{T}}}(M, \Sigma^k N)$$

is an $R^{\bullet}_{\mathbf{T}}$-bimodule for $M, N \in \underline{\mathbf{T}}$. The cohomological support

$$W : \underline{\mathbf{T}} \to \mathcal{X}_{cl}(\mathrm{Proj}\, R^{\bullet}_{\mathbf{T}}) \quad \text{is} \quad W(M) := \{\mathfrak{p} \in \mathrm{Proj}\, R^{\bullet}_{\mathbf{T}} : \mathrm{Ann}(\mathrm{End}^{\bullet}_{\underline{\mathbf{T}}}(M)) \subseteq \mathfrak{p}\}.$$

The Tate cohomological support

The Tate cohomology ring of an FTC $\mathbf{T}$ is the graded commutative ring

$$\widehat{R}^{\bullet}_{\mathbf{T}} := \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_{\underline{\mathbf{T}}}(\mathbf{1}, \Sigma^k \mathbf{1})$$

with product defined in the same way. Then

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$\widehat{W} : \underline{\mathbf{T}} \to \mathrm{Spec}^h \widehat{R}^{\bullet}_{\mathbf{T}}$ is $\widehat{W}(M) := \{\mathfrak{p} \in \mathrm{Spec}^h \widehat{R}^{\bullet}_{\mathbf{T}} : \mathrm{Ann}(\widehat{\mathrm{End}}^{\bullet}_{\underline{\mathbf{T}}}(M)) \subseteq \mathfrak{p}\}.$

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- The (noncommutative) Balmer spectrum $\mathrm{Spc}(\underline{\mathbf{T}})$ is defined as the collection of prime ideals of $\underline{\mathbf{T}}$: thick ideals $\mathbf{P}$ such that if $\mathbf{I} \otimes \mathbf{J} \subseteq \mathbf{P}$, then $\mathbf{I}$ or $\mathbf{J}$ is contained in $\mathbf{P}$, over all thick ideals $\mathbf{I}$ and $\mathbf{J}$.
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- Nonempty: existence of maximal ideals.
- The topology of $\mathrm{Spc}\,\underline{\mathbf{T}}$ has closed sets

$$V(\mathcal{S}) := \{\mathbf{P} \in \mathrm{Spc}\,\underline{\mathbf{T}} \mid \mathcal{S} \cap \mathbf{P} = \varnothing\}; \quad V : \underline{\mathbf{T}} \to \mathcal{X}_{cl}(\mathrm{Spc}\,\underline{\mathbf{T}}) \text{ universal support.}$$

The Balmer spectrum

The universal support has the noncommutative tensor product property

$$\bigcup_{X \in \underline{\mathbf{T}}} V(M \otimes X \otimes N) = V(M) \cap V(N), \quad \forall M, N \in \underline{\mathbf{T}} \qquad (*)$$

Theorem [Nakano-Vashaw-Y]

(1) The universal support has the tensor product property

$$V(M \otimes N) = V(M) \cap V(N), \quad \forall M, N \in \underline{\mathbf{T}}.$$

if and only if every prime ideal $\mathbf{P}$ of $\mathbf{T}$ is completely prime, $\forall M, N \in \underline{\mathbf{T}}$ $M \otimes N \in \mathbf{P} \Rightarrow M \in \mathbf{P}$ or $N \in \mathbf{P}$.

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For part (1), think of [Dixmier]: For a fin dim Lie algebra $\mathfrak{g}$, every prime ideal of $U(\mathfrak{g})$ is completely prime if and only if $\mathfrak{g}$ is solvable.
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But nobody knows whether the cohomological support $W : \mathbf{T} \to \text{Proj}\, R^{\bullet}_{\mathbf{T}}$ has the property $(*)$.

Main problem

Problem

1. Describe the relationship between $\operatorname{Spc}\underline{\mathbf{T}}$ and $\operatorname{Proj} R^{\bullet}_{\mathbf{T}}$, respectively $\operatorname{Spc}\underline{\mathbf{T}}$ and $\operatorname{Spec}^{h}\widehat{R}^{\bullet}_{\mathbf{T}}$.

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2. Describe the relationship between the universal support $V : \underline{\mathbf{T}} \to \operatorname{Spc} \underline{\mathbf{T}}$ and the cohomological support $W : \underline{\mathbf{T}} \to \operatorname{Proj} R^\bullet_{\mathbf{T}}$,

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Theorem [Avrunin-Scott, Friedlander-Pevtsova, Balmer]

For every finite group scheme G,

$$\operatorname{Spc}(\underline{\operatorname{mod}}(\Bbbk G)) \cong \operatorname{Proj} R^{\bullet}_{\underline{\operatorname{mod}}(\Bbbk G)}$$

and V and W coincide under this identification.

Based on tensor product property for the cohomological support, in turn based on rank support [Carlson], π-support [Friedlander-Pevtsova].

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The categorical center of the cohomology ring

When the category $\mathbf{T}$ is not braided, there are known cases where the universal and cohomological support are not homeomorphic.

Definition (Nakano-Vashaw-Y)

The categorical center $\widehat{C}^{\bullet}_{\mathbf{T}}$ of the Tate cohomology ring $\widehat{R}^{\bullet}_{\mathbf{T}}$ of $\underline{\mathbf{T}}$ is the subalgebra generated by all homogenous $g \in \widehat{R}^{\bullet}_{\mathbf{T}}$ such that the diagram

$$\begin{array}{ccccc} \mathbf{1} \otimes M & \xrightarrow{\cong} & M & \xleftarrow[\cong]{} & M \otimes \mathbf{1} \\ \downarrow{\scriptstyle g \otimes \mathrm{id}_M} & & & & \downarrow{\scriptstyle \mathrm{id}_M \otimes g} \\ \Sigma^i \mathbf{1} \otimes M & \xrightarrow{\cong} & \Sigma^i M & \xleftarrow[\cong]{} & M \otimes \Sigma^i \mathbf{1} \end{array}$$

commutes for all simple objects M of $\mathbf{T}$.

The categorical center $C^{\bullet}_{\mathbf{T}}$ of the cohomology ring $R^{\bullet}_{\mathbf{T}}$ is defined to be

$$C^{\bullet}_{\mathbf{T}} = R^{\bullet}_{\mathbf{T}} \cap \widehat{C}^{\bullet}_{\mathbf{T}}.$$

The diagram

If $\mathbf{T}$ is braided, then

$$\widehat{C}^{\bullet}_{\mathbf{T}} = \widehat{R}^{\bullet}_{\mathbf{T}}, \quad C^{\bullet}_{\mathbf{T}} = R^{\bullet}_{\mathbf{T}}.$$

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In general, the categorical center is related to the following previously considered algebras:

$$\begin{array}{ccccccccc}
R^{\bullet}_{\mathbf{Z}(\mathbf{T})} & \longrightarrow & C^{\bullet}_{\mathbf{T}} & \hookrightarrow & R^{\bullet}_{\mathbf{T}} & & & & \\
 & \searrow & & \searrow & & \searrow & & & \\
 & & \widehat{R}^{\bullet}_{\mathbf{Z}(\mathbf{T})} & \longrightarrow & \widehat{C}^{\bullet}_{\mathbf{T}} & \hookrightarrow & \widehat{R}^{\bullet}_{\mathbf{T}} & \xhookrightarrow{\mathcal{R},\mathcal{L}} & Z^{\bullet}(\mathbf{T})
\end{array}$$

Drinfeld center (tensor cats) graded center (triangulated cats)

The Drinfeld center

The Drinfeld center $\mathbf{Z}(\mathbf{T})$ of a finite tensor category $\mathbf{T}$ is a braided finite tensor category.

- Objects: pairs (M, γ), $M \in \mathbf{C}$ and a natural isomorphism

$$\gamma_X : X \otimes M \xrightarrow{\cong} M \otimes X, \quad X \in \mathbf{T}$$

  called a half-braiding, satisfying usual braiding type axioms.
- Morphisms: $\mathrm{Hom}_{\mathbf{Z}(\mathbf{C})}((M, \gamma), (M', \gamma')) \subset \mathrm{Hom}_{\mathbf{C}}(M, M')$.

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Fact. The forgetful functor $\mathbf{Z}(\mathbf{T}) \to \mathbf{T}$ descends to $\underline{\mathbf{Z}(\mathbf{T})} \to \underline{\mathbf{T}}$

The graded center

The graded center $Z^\bullet(\underline{\mathbf{T}})$ of $\underline{\mathbf{T}}$ is a graded commutative algebra with degree n component consisting of natural transformations

$$\eta : \mathrm{id}_{\underline{\mathbf{T}}} \to \Sigma^n \quad \text{such that} \quad \eta\Sigma = (-1)^n \Sigma\eta.$$

Two injective homomorphisms $\mathcal{L}, \mathcal{R} : \widehat{R}^\bullet_{\underline{\mathbf{T}}} \hookrightarrow Z^\bullet(\underline{\mathbf{T}})$, which send $g \in \mathrm{Hom}_{\underline{\mathbf{T}}}(\mathbf{1}, \Sigma^n \mathbf{1})$ to

$$M \xrightarrow{\cong} \mathbf{1} \otimes M \xrightarrow{g \otimes \mathrm{id}_M} \Sigma^n \mathbf{1} \otimes M \xrightarrow{\cong} \Sigma^n M \quad \text{and}$$

$$M \xrightarrow{\cong} M \otimes \mathbf{1} \xrightarrow{\mathrm{id}_M \otimes g} M \otimes \Sigma^n \mathbf{1} \xrightarrow{\cong} \Sigma^n M,$$

respectively. The categorical center of the Tate cohomology ring $\widehat{R}^\bullet_{\underline{\mathbf{T}}}$ is

$$\widehat{C}^\bullet_{\underline{\mathbf{T}}} := \{ g \in \widehat{R}^\bullet_{\underline{\mathbf{T}}} \mid \mathcal{L}(g) = \mathcal{R}(g) \ \text{ on all simple objects } \ M \in \underline{\mathbf{T}} \}.$$

Groups of endotrivial objects

Definition. $M \in \mathbf{T}$ is endotrivial if $M^* \otimes M \cong \mathbf{1}$ in $\underline{\mathbf{T}}$. In some cases

$$C^\bullet_{\mathbf{T}} = (R^\bullet_{\mathbf{T}})^L,$$

L =a group of endotrivial objects (canonical action $L \curvearrowright \mathbf{T}$ and on $\widehat{R}^\bullet_{\mathbf{T}}$).

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Example. [Plavnik-Witherspoon Hopf algebras]. L a finite group acting on a finite dimensional Hopf algebra A:

$$A_L = (A^* \# \Bbbk L)^*$$

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Special case 1. [Benson-Witherspoon Hopf algebras]. $A = \Bbbk G$ and $H = (\Bbbk[G] \# \Bbbk L)^*$. So, $C^\bullet_{\mathrm{mod}(H)} \cong (R^\bullet_{\mathrm{mod}(\Bbbk G)})^L = H^\bullet(G, \Bbbk)^L$.

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Special case 2. [Coordinate rings of finite group schemes] $\Omega \cong \pi \ltimes \Omega_0$, Ω_0 infinitesimal group scheme, π a finite group. We have:

$$C^\bullet_{\mathrm{mod}(\Bbbk[\Omega])} \cong H^\bullet(\Bbbk[\Omega_0], \Bbbk)^\pi.$$

The weak fg condition

Conjecture (Etingof-Ostrik)

Every FTC $\mathbf{T}$ satisfies the (fg) condition that $R^{\bullet}_{\mathbf{T}}$ is a finitely generated algebra and for all $M \in \mathbf{T}$, $\mathrm{End}^{\bullet}_{\underline{\mathbf{T}}}(M)$ is a finitely generated $R^{\bullet}_{\mathbf{T}}$-module.

- Friedlander-Suslin [finite group schemes],
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Weak fg condition (wfg). For all $M \in \mathbf{T}$, $\mathrm{End}^\bullet_{\underline{\mathbf{T}}}(M)$ is a finitely generated $C^\bullet_{\mathbf{T}}$-module. Note: (fg) for $\mathbf{Z}(\mathbf{T})$ implies (wfg) for $\mathbf{T}$.

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Tate weak fg condition (Twfg). For all $M \in \mathbf{T}$, $\widehat{\mathrm{End}}^\bullet_{\underline{\mathbf{T}}}(M)$ is a finitely generated $\widehat{C}^\bullet_{\mathbf{T}}$-module.

Theorem: a bridge coh support $\leftrightarrow$ Balmer spec

Theorem (Nakano-Vashaw-Y)

1. There is a well-defined, continuous map

$$\rho : \mathrm{Spc}\, \underline{\mathbf{T}} \to \mathrm{Proj}\, C^{\bullet}_{\mathbf{T}},$$

$\rho(\mathbf{P}) = \mathrm{Span}\{g$ homogeneous in $C^{\bullet}_{\mathbf{T}}$: $\mathrm{cone}(g) \notin \mathbf{P}\}$. It satisfies

$$\rho^{-1}(W_C(A)) \subseteq V(A),$$

central coh support $W_C : \mathbf{T} \to \mathrm{Proj}\, C^{\bullet}_{\mathbf{T}}$ (restrict $R^{\bullet}_{\mathbf{T}}$-action to $C^{\bullet}_{\mathbf{T}}$).

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Theorem: a bridge coh support $\leftrightarrow$ Balmer spec

Theorem (Nakano-Vashaw-Y)

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$\rho(\mathbf{P}) = \mathrm{Span}\{g \text{ homogeneous in } C^\bullet_{\mathbf{T}} : \mathrm{cone}(g) \notin \mathbf{P}\}$. It satisfies
$$\rho^{-1}(W_C(A)) \subseteq V(A),$$
central coh support $W_C : \mathbf{T} \to \mathrm{Proj}\, C^\bullet_{\mathbf{T}}$ (restrict $R^\bullet_{\mathbf{T}}$-action to $C^\bullet_{\mathbf{T}}$).
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A conjecture

Conjecture (Nakano-Vashaw-Y)

The map $\rho : \operatorname{Spc} \underline{\mathbf{T}} \to \operatorname{Proj} C^{\bullet}_{\mathbf{T}}$ is always a homeomorphism.

Theorem: a right inverse of ρ

Recall: $\rho : \operatorname{Spc} \underline{\mathbf{T}} \to \operatorname{Proj} C^\bullet_{\mathbf{T}}$

Theorem (Nakano-Vashaw-Y)

Assume that the central cohomological support $W_C : \mathbf{T} \to \operatorname{Proj} C^\bullet_{\mathbf{T}}$ satisfies the noncommutative tensor product property

$$\bigcup_{X \in \mathbf{T}} V(M \otimes X \otimes N) = V(M) \cap V(N), \quad \forall M, N \in \mathbf{T}.$$

Then the following holds:

- There exists a continuous map $\eta : \operatorname{Proj} C^\bullet_{\mathbf{T}} \to \operatorname{Spc} \underline{\mathbf{T}}$ given by

$$\eta(\mathfrak{p}) = \{M \in \underline{\mathbf{T}} : \mathfrak{p} \notin W_C(M)\}.$$

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$+$ a similar theorem for $\widehat{\rho} : \operatorname{Spc} \underline{\mathbf{T}} \to \operatorname{Spec}^h \widehat{C}^\bullet_{\mathbf{T}}$.

Extended supports

To get that $\eta\rho = \mathrm{id}_{\mathrm{Spc}\,\underline{\mathbf{T}}}$, we need Rickard idempotent functors arising from Brown representability. This means that we need to work in a triangulated category with arbitrary coproducts and which is compactly generated.

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Definition. We say that W_C admits an extension to $\underline{\mathrm{Ind}(\mathbf{T})}$ if there is a support $\widetilde{W}_C$ on that category which restricts to W_C on the compact part $\underline{\mathbf{T}}$, satisfies the noncommutative tensor product property

$$\bigcup_{X \in \underline{\mathbf{T}}} \widetilde{W}_C(M \otimes X \otimes N) = \widetilde{W}(M) \cap \widetilde{W}(N)$$

for all $M \in \underline{\mathrm{Ind}(\mathbf{T})}$ and $N \in \underline{\mathbf{T}}$, and and is compatible with shifts, triangles, arbitrary coproducts, detects projectivity.

Theorem: an inverse of ρ

Theorem (Nakano-Vashaw-Y)

Suppose $\underline{\mathbf{T}}$ satisfies (wfg), $\operatorname{Proj} C^\bullet_{\mathbf{T}}$ is a Noetherian topological space, the central cohomological support satisfies the noncommutative tensor product property, and W_C has an extension to $\underline{\operatorname{Ind}(\mathbf{T})}$.

Then η and ρ are inverse homeomorphisms

$$\operatorname{Spc} \underline{\mathbf{T}} \underset{\eta}{\overset{\rho}{\rightleftarrows}} \operatorname{Proj} C^\bullet_{\mathbf{T}}.$$

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Note: We can prove that none of the conditions is overshooting, i.e., if the conjecture holds for $\mathbf{T}$, then $\underline{\mathbf{T}}$ satisfies all conditions in the theorem.

Examples

Recall the Plavnik-Witherspoon smash coproduct $A_L = (A^* \# \Bbbk L)^*$, based on a fin dim Hopf algebra A and a finite group L.

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2. Finite group scheme $\Omega = \Omega_0 \ltimes \pi$:

$$\text{Spc}(\underline{\text{mod}}(\Bbbk[\Omega])) \cong \text{Proj}(H^\bullet(\Bbbk[\Omega_0], \Bbbk)^\pi).$$

This recovers a theorem of Negron-Pevtsova, who used methods related to hypersurface support.

Thank you!