Local Forms of Noncommutative Functions

Michael Wemyss

Paul Smith Conference, Seattle, 24th June 2022.

(joint with Gavin Brown)

Plan of Talk

1. Singularity Theory: commutative and noncommutative
2. The proposal, dimension and numerics
3. Main results, and ADE
4. Geometric Consequences

What is Singularity Theory?

To classify (even smooth points), you need to work suitably locally.

...so work in the commutative power series ring $\mathbb{C}[\![x_1, \dots, x_d]\!]$.
Similar to the polynomial ring, but allow infinite sums

$$f = \lambda_1 + \lambda_2 x + \lambda_3 y + \lambda_4 xy + \dots$$

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Singularity Theory (à la Arnold)

Classify all $f \in \mathbb{C}[\![x_1, \dots, x_d]\!]$, up to specified isomorphism, satisfying some fixed numerical criteria, and produce theory for when classification is not possible.

Key Example: Simple Singularities

Define $f \in \mathbb{C}[\![x_1, \ldots, x_d]\!] = \mathbb{C}[\![\mathbf{x}]\!]$ to be a simple singularity if

$$\#\{I \mid I \text{ proper ideal of } \mathbb{C}[\![\mathbf{x}]\!] \text{ with } f \in I^2\} < \infty.$$

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Theorem

If f is a simple singularity, then up to relabelling the variables $z_1, \ldots z_{d-2}, x, y$, and up to isomorphism, f is one of

A_n	$z_1^2 + \ldots + z_{d-2}^2 + x^2 + y^{n+1}$	$n \geq 1$
D_n	$z_1^2 + \ldots + z_{d-2}^2 + x^2y + y^{n-1}$	$n \geq 4$
E_6	$\mathbf{z}^2 + x^3 + y^4$	
E_7	$\mathbf{z}^2 + x^3 + xy^3$	
E_8	$\mathbf{z}^2 + x^3 + y^5$	

NC Change in View

Consider instead the formal *noncommutative* power series ring $\mathbb{C}\langle\langle x_1, \ldots, x_d \rangle\rangle$. Basically the same as the free algebra in d variables, except now allow infinite sums

$$f = \lambda_1 + \lambda_2 x + \lambda_3 y + \lambda_4 xy + \lambda_5 yx + \ldots$$

(the complete path algebra of the d-loop quiver)

But: when are $f, g \in \mathbb{C}\langle\langle x_1, \ldots, x_d \rangle\rangle$ isomorphic? And what are the numerical criteria?

Jacobi Algebras

Given any $f \in \mathbb{C}\langle\langle x_1, \ldots, x_d \rangle\rangle$, we can cyclically differentiate it with respect to any variable. For example

$$\delta_x(x^3y) = xxy + xyx + yxx, \text{ and } \delta_y(x^3y) = xxx.$$

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Definition

Given any $f \in \mathbb{C}\langle\langle x_1, \dots, x_d \rangle\rangle$, the Jacobi algebra is

$$\mathrm{Jac}(f) = \frac{\mathbb{C}\langle\langle x_1, \dots, x_d \rangle\rangle}{(\!(\delta_1 f, \dots, \delta_d f)\!)}.$$

Favourite Example: $f = x^4 + xy^2 \in \mathbb{C}\langle\langle x, y \rangle\rangle$ gives

$$\frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{(\!(4x^3 + y^2, xy + yx)\!)}.$$

Small Digression: what is dimension?

Gelfand–Kirillov dimension is problematic: $\operatorname{GKdim} \mathbb{C}[\![x]\!] = \infty$.

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Gelfand–Kirillov dimension is problematic: $\operatorname{GKdim} \mathbb{C}[\![x]\!] = \infty$. But $\mathfrak{Jac}(f)$ is local, with Jacobson radical $\mathfrak{J}$, giving intrinsic filtration

$$\mathfrak{Jac}(f) \supseteq \mathfrak{J} \supseteq \mathfrak{J}^2 \supseteq \dots .$$

Definition

$\operatorname{JRdim} \mathfrak{Jac}(f)$ is the growth rate of this chain, namely

$$\inf \{ r \in \mathbb{R} \mid \text{for some } c \in \mathbb{R},\ \dim \mathfrak{Jac}(f)/\mathfrak{J}^n \leq cn^r \text{ for every } n \in \mathbb{N} \} .$$

Calibration: $\operatorname{JRdim} \mathbb{C}[\![x]\!] = 1$, and further $\operatorname{JRdim} \mathfrak{Jac}(f) = 0$ is equivalent to $\mathfrak{Jac}(f)$ being a finite dimensional algebra.

Noncommutative Singularity Theory

Write $f \cong g$ to mean $\mathfrak{J}\mathrm{ac}(f) \cong \mathfrak{J}\mathrm{ac}(g)$. With ring $\mathbb{C}\langle\langle x_1, \ldots, x_d \rangle\rangle$ and equivalence relation $\cong$ fixed, aim of singularity theory remains: classify all elements f satisfying some numerical criteria.

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Problem

For every $n \geq 0$, produce a set of elements $\mathcal{S}_n$ which realise *every* Jacobi algebra of JR-dimension n, up to isomorphism.

Key: We insist that the elements of $\mathcal{S}_n$ should be a *normal form*, namely if $f, g \in \mathcal{S}_n$ with $f \neq g$, then $\mathcal{J}\mathrm{ac}(f) \ncong \mathcal{J}\mathrm{ac}(g)$.

Calibration: $\mathcal{S}_0$ gives complete list of finite dimensional Jacobi algebras, with no repetitions.

The Proposal

...for small n we propose that such a classification is desirable, and:

1. ...a classification is in fact possible! (c.f. Arnold)
2. ...there are no moduli. Just very few countable families.
3. ...the classification is ADE.
4. ...this algebraic classification *is* (and implies) the classification of flops, and of crepant divisorial contractions to curves.

Numerical Criteria

For $m \geq 1$, the *mth corank of f* is defined to be

$$\mathcal{C}\mathrm{rk}_m(f) = \dim_{\mathbb{C}} \left(\frac{\mathfrak{J}^m}{\mathfrak{J}^{m+1}} \right),$$

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Theorem (Iyudu–Shkarin, Brown–W)

If JRdim $\mathcal{J}\mathrm{ac}(f) \leq 1$, then one of the following holds

$\mathcal{C}\mathrm{rk}(f)$	$\mathcal{C}\mathrm{rk}_2(f)$
1	1
2	2
2	3

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Type	$\mathcal{C}\mathrm{rk}(f)$	$\mathcal{C}\mathrm{rk}_2(f)$
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E	2	3

Type A

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Proposition (really just the Splitting Lemma)

Suppose $f \in \mathbb{C}\langle\langle z_1, \ldots, z_{d-2}, x, y \rangle\rangle$ with $\mathcal{C}\mathrm{rk}(f) \leq 1$. Then either

$$f \cong \begin{cases} z_1^2 + \ldots + z_{d-2}^2 + x^2 \\ z_1^2 + \ldots + z_{d-2}^2 + x^2 + y^n & \text{for some } n \geq 2. \end{cases}$$

In all cases, $\mathrm{JRdim}\,\mathcal{J}\mathrm{ac}(f) \leq 1$.

Type D

Theorem (Brown–W)

Suppose that $f \in \mathbb{C}\langle\langle z_1, \dots z_{d-2}, x, y \rangle\rangle$ with $\mathcal{C}\mathrm{rk}(f) = 2$ and $\mathcal{C}\mathrm{rk}_2(f) = 2$. Then either

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These are normal forms. All satisfy $\mathrm{JRdim}\,\mathcal{J}\mathrm{ac}(f) \leq 1$.

...there are no moduli!

Summary of families: JRdim 0 case

Theorem (Brown–W)

If $\dim_{\mathbb{C}} \mathcal{J}\mathrm{ac}(f) < \infty$, then $f \cong$ to one of the following

		Normal form	Conditions
A	A_n	$\mathbf{z}^2 + x^2 + y^n$	$n \geq 2$
D	$D_{n,m}$	$\mathbf{z}^2 + xy^2 + x^{2n} + x^{2m-1}$	$n, m \geq 2, m \leq 2n-1$
	$D_{n,\infty}$	$\mathbf{z}^2 + xy^2 + x^{2n}$	$n \geq 2$
E	$E_{6,n}$	$\mathbf{z}^2 + x^3 + xy^3 + y^n$	$n \geq 4$
		$\mathbf{z}^2 + x^3 + \mathcal{O}_4$	(various cases)

Summary of families: JRdim 1 case

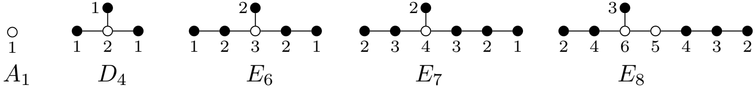
Theorem (Brown–W)

If JRdim $\mathcal{J}\mathrm{ac}(f) = 1$, then $f \cong$ to one of the following

		Normal form	Conditions
A	A_∞	$\mathbf{z}^2 + x^2$	
D	$D_{\infty,m}$	$\mathbf{z}^2 + xy^2 + x^{2m-1}$	$m \geq 2$
	$D_{\infty,\infty}$	$\mathbf{z}^2 + xy^2$	
E	$E_{6,\infty}$	$\mathbf{z}^2 + x^3 + xy^3$	
		$\mathbf{z}^2 + x^3 + \mathcal{O}_4$	

Extracting ADE

Consider the ADE Dynkin diagrams, with their highest roots.



To each, can associate the preprojective algebra Π.

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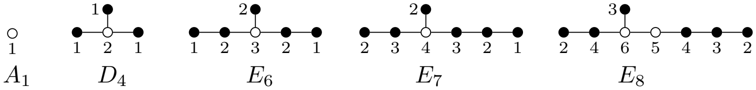

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Definition

Write $\mathcal{Z} = Z(\mathrm{Jac}(f))$. We say that $\mathrm{Jac}(f)$ has Type X if for all finite dimensional vector spaces $V \subset \mathfrak{m}_{\mathcal{Z}}$ such that $V \twoheadrightarrow \mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$, there exists a Zariski open subset U of V such that $\mathrm{Jac}(f)/(u) \cong e\Pi e$ for all $u \in U$, where Π is the preprojective algebra of Type X, and e is an idempotent marked $\circ$ above.

...a general hyperplane section $u \in \mathcal{Z}$ satisfies $\mathrm{Jac}(f)/(u) \cong e\Pi e$.

Theorem (Brown–W)

Consider the previous normal forms, and define s as follows

Type	Normal form	Conditions	s
A	$\mathbf{z}^2 + x^2 + \varepsilon_1 y^n$	$n \in \mathbb{N}_{\geq 2} \cup \{\infty\}$	y
D	$\mathbf{z}^2 + xy^2 + \varepsilon_2 x^{2n} + \varepsilon_3 x^{2m-1}$	$m, n \in \mathbb{N}_{\geq 2} \cup \{\infty\}$	x^2
E	$\mathbf{z}^2 + x^3 + xy^3 + \varepsilon_4 y^n$	$n \in \mathbb{N}_{\geq 4}$	$g_{6,n}$

where $g_{6,n}$ some explicit element.

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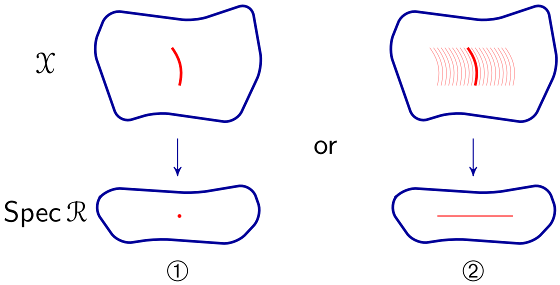
where $g_{6,n}$ some explicit element.

1. s is central in $\mathcal{J}\mathrm{ac}(f)$, with $\mathcal{J}\mathrm{ac}(f)/(s) \cong e\Pi e$, where Π has Type A_1, D_4, or E_6, and e is the idempotent marked $\circ$.
2. In Type A and D, a *generic* central element g satisfies $\mathcal{J}\mathrm{ac}(f)/(g) \cong e\Pi e$.

Applications: why do we care?

Contraction algebras arise in the birational geometry.

Today: crepant contractions of two types:



Assumptions: $\mathcal{X}$ is smooth, and only one curve above the origin.

To this data associate the contraction algebra $\mathrm{A}_{\mathrm{con}}$

Contraction Algebras

The contraction algebra $\mathrm{A}_{\mathrm{con}}$ is defined using (noncommutative) deformation theory of the reduced fibre above the origin.

Details are unimportant, the only facts we need today are:

1. Since only one curve, $\mathrm{A}_{\mathrm{con}}$ is a factor of $\mathbb{C}\langle\langle x, y\rangle\rangle$.
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Theorem (Donovan–W)

$$\text{Situation ① (flopping)} \iff \text{JRdim}\,\mathrm{A}_{\mathrm{con}} = 0.$$
$$\text{Situation ② (div}\to\text{curve)} \iff \text{JRdim}\,\mathrm{A}_{\mathrm{con}} = 1.$$

...motivates studying f such that JRdim $\mathcal{J}\mathrm{ac}(f) \leq 1$.

Two Conjectures

Classification Conjecture (Donovan–W)

Contraction algebras classify.

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Contraction algebras classify. If $\mathcal{X}_1 \to \operatorname{Spec}\mathcal{R}_1$ and $\mathcal{X}_2 \to \operatorname{Spec}\mathcal{R}_2$ be 3-fold irreducible crepant contractions, with one-dimensional fibres, where $\mathcal{X}_i$ are smooth, and $\mathcal{R}_i$ are complete local. Denote their corresponding contraction algebras by $\mathrm{A}_{\mathrm{con}}$ and $\mathrm{B}_{\mathrm{con}}$. Then

$$\mathcal{R}_1 \cong \mathcal{R}_2 \iff \mathrm{A}_{\mathrm{con}} \cong \mathrm{B}_{\mathrm{con}}.$$

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Realisation Conjecture (Brown–W)

Contraction algebras=Jacobi algebras.

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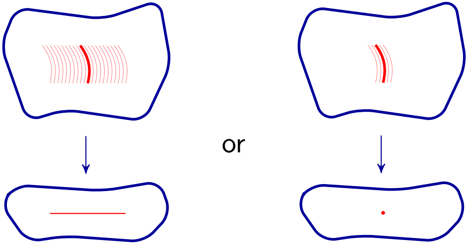
Realisation Conjecture (Brown–W)

Contraction algebras=Jacobi algebras. If $f \in \mathbb{C}\langle\langle x, y\rangle\rangle$ satisfies $\operatorname{JRdim} \mathcal{J}\mathrm{ac}(f) \leq 1$, then $\mathcal{J}\mathrm{ac}(f) \cong \mathrm{A}_{\mathrm{con}}$ for either a flopping contraction (JR zero), or div$\to$curve contraction (JR 1).

Type A

For Type A contractions, either:



$uv = s^2$ $\qquad$ $uv = s^2 + t^{2n}$

$\mathrm{A}_{\mathrm{con}}$ $\qquad$ $\mathbb{C}[\![y]\!]$ $\qquad$ $\mathbb{C}[\![y]\!]/y^n$

...these are precisely the Type A $\mathfrak{Jac}(f)$ from earlier.

All Type D are also geometric!

$$f \cong \begin{cases} xy^2 & \text{[Donovan–W] div}\to\text{curve} \\ xy^2 + x^{2m+1} & \text{[Brown–W] div}\to\text{curve} \\ xy^2 + x^{2n} & \text{[Aspinwall–Morrison] Laufer flops} \\ xy^2 + x^{2n} + x^{2m+1} & \text{[BW, van Garderen, Kawamata] flops} \\ xy^2 + x^{2m+1} + x^{2n} & \text{[van Garderen, Kawamata] flops} \end{cases}$$

...in all cases, in the corresponding geometric contraction, the elephant has type D singularities.

Corollary

The Realisation Conjecture is true, except possibly the only remaining case $f = x^3 +$ higher.

Classification of Type D

Theorem (Brown–W)

Suppose that $f\colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ is *any* smooth type D flop, or div$\to$curve contraction, one curve above the origin. Then

$$\mathrm{A}_{\mathrm{con}} \cong \mathcal{J}\mathrm{ac}(f)$$

for some f on the previous slide.

...so *all* possible contraction algebras in Type D are now classified.

GV invariants

To every flop is an associated tuple of numbers $(n_1, \ldots, n_6)$ called the Gopakumar–Vafa (GV) invariants.

..basically deform your flopping curve C into a disjoint union of $(-1, -1)$ curves, and count those. It is a bit more refined than this: n_j equals the number of such curves with curve class $j[C]$.

GV invariants

To every flop is an associated tuple of numbers $(n_1, \ldots, n_6)$ called the Gopakumar–Vafa (GV) invariants.

..basically deform your flopping curve C into a disjoint union of $(-1, -1)$ curves, and count those. It is a bit more refined than this: n_j equals the number of such curves with curve class $j[C]$.

Upshot

Type D flops have GV invariants $(a, b, 0, 0, 0, 0) = (a, b)$ for some $a, b \in \mathbb{N}$. Different flops can have the same GV invariants.

Question

What possible (a, b) can arise?

Gaps in GV

Corollary

For Type D flops, the only possible GV invariants (a, b) are:

(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)					
	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)
	(7,2)				
		(8,3)	(8,4)	(8,5)	(8,6)
		(9,3)			
			(10,4)	(10,5)	(10,6)

Gaps in GV

Corollary

For Type D flops, the only possible GV invariants (a, b) are:

x^3+x^4	x^3+x^6	x^3+x^8	x^3+x^{10}	x^3+x^{12}	x^3+x^{14}
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)					
	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)
	(7,2)				
		(8,3)	(8,4)	(8,5)	(8,6)
		(9,3)			
			(10,4)	(10,5)	(10,6)

Gaps in GV

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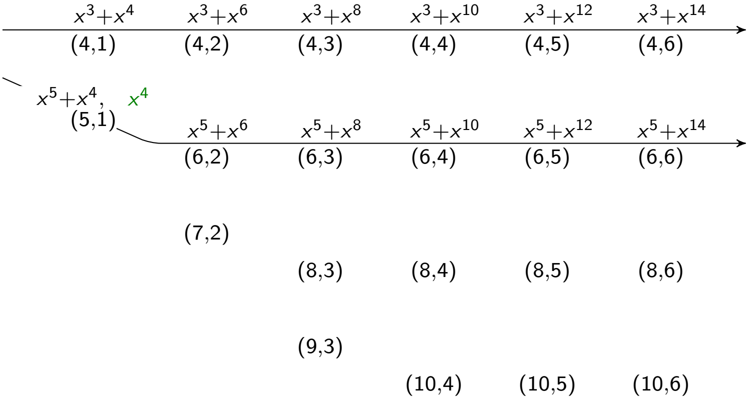

Gaps in GV

Corollary

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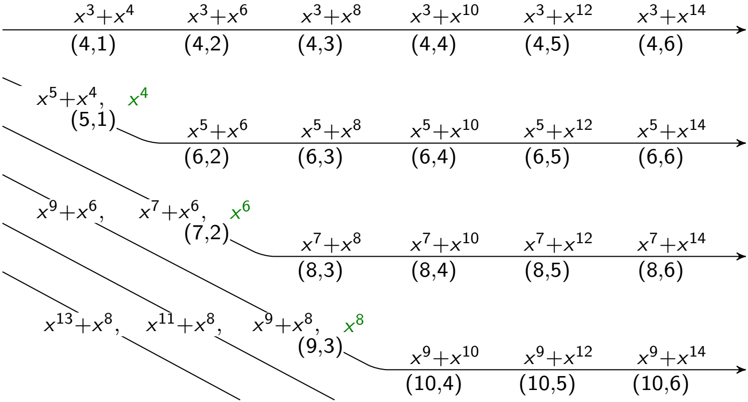

Gaps in GV

Corollary

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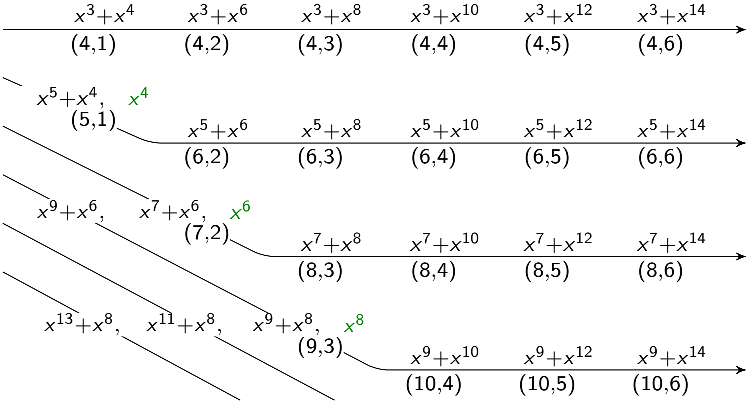

Key: The obstruction to e.g. $(5, 2)$ existing is noncommutative.