Spectra of Quantum Algebras

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Recent Advances and New Directions in Noncommutative Algebra and Geometry
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For a noncommutative algebra A,

$\operatorname{Prim} A := \{$ primitive ideals of $A \}$ $\approx$ a NC affine variety

$\operatorname{Spec} A := \{$ prime ideals of $A \}$ $\approx$ a NC affine scheme.

Equip both with Zariski topologies.

Defn. Multiparameter quantum affine n-space.

Let $\mathbf{q} = (q_{ij}) \in M_n(K^*)$ with $q_{ji} = q_{ij}^{-1}$ and $q_{ii} = 1$ for all i, j.

$\mathcal{O}_{\mathbf{q}}(K^n) := K\langle x_1, \dots, x_n \mid x_i x_j = q_{ij} x_j x_i \ \forall\, i, j \rangle$.

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Single-parameter version: $q_{ij} =$ fixed $q \in K^*$ for all $i < j$;

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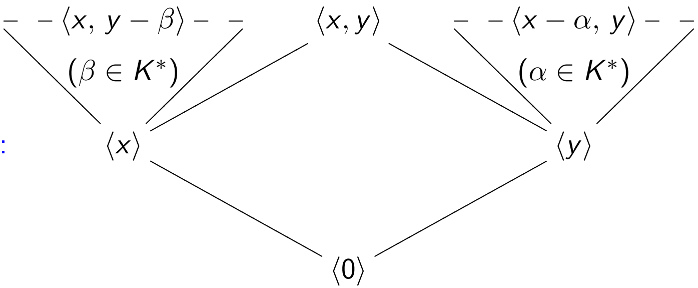
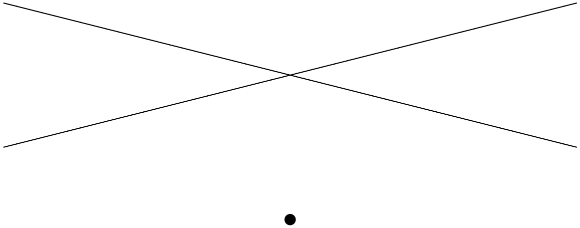
E.G. $A = \mathcal{O}_q(K^2)$, $q \neq \sqrt[\bullet]{1}$.

Spec A :



Prim A :



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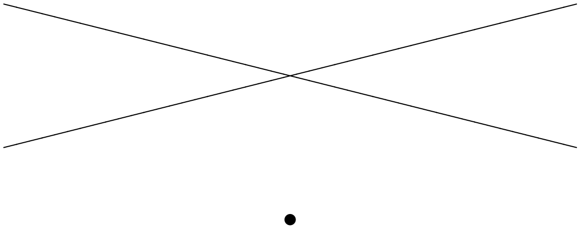

Thm. [Letzter-KG, 2000] $A = \mathcal{O}_{\mathbf{q}}(K^n)$. Assume $-1 \notin \langle q_{ij} \rangle$ or char $K = 2$. Then Prim A is a topological quotient of K^n and Spec A is a (compatible) topological quotient of Spec $\mathcal{O}(K^n)$.

These statements hold more generally for cocycle twists of commutative affine algebras graded by torsionfree abelian groups, and hence for quantum toric varieties.

Defn. $\mathcal{A}$ = a torsionfree $K[t^{\pm 1}]$-algebra such that $\mathcal{A}/(t-1)\mathcal{A}$ is commutative;

$\sim$ flat family of K-algebras $\big(\mathcal{A}/(t-q)\mathcal{A}\big)_{q \in K^*}$.

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$\mathcal{A}/(t-1)\mathcal{A} =$ the semiclassical limit of this family.

All commutators $[a,b]$ $(= ab - ba)$ in $\mathcal{A}$ are divisible by $t-1$.

$\therefore$ have a Lie bracket $\dfrac{1}{t-1}[-,-]$ on $\mathcal{A}$,

which induces a Lie bracket $\{-,-\}$ on $\mathcal{A}/(t-1)\mathcal{A}$.

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Additionally, there is a "product rule":

$$\{a, xy\} = \{a,x\}y + x\{a,y\} \;\; \forall\, a,x,y \in \mathcal{A}/(t-1)\mathcal{A}.$$

$\{-,-\}$ is a "Poisson bracket".

E.G. The Poisson bracket on $\mathcal{O}(SL_2(K))$ = semiclassical limit of the $\mathcal{O}_q(SL_2(K))$:

$$\{X_{11}, X_{12}\} = X_{11}X_{12} \qquad \{X_{21}, X_{22}\} = X_{21}X_{22}$$
$$\{X_{11}, X_{21}\} = X_{11}X_{21} \qquad \{X_{12}, X_{22}\} = X_{12}X_{22}$$
$$\{X_{12}, X_{21}\} = 0 \qquad \{X_{11}, X_{22}\} = 2X_{12}X_{21}$$

Defns. A Poisson algebra is an algebra R equipped with a Lie algebra bracket $\{-,-\} : R \times R \to R$ such that

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The Poisson-prime and Poisson-primitive spectra of R are

$\text{Pspec}\, R := \{$ Poisson-prime ideals of R $\}$

$\text{Pprim}\, R := \{$ Poisson-primitive ideals of R $\} \ \subseteq \ \text{Pspec}\, R,$

both with Zariski topologies.

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- Pspec R is a topological quotient of Spec R, via the map $\pi : P \longmapsto ($ largest Poisson ideal $\subseteq P$ $)$.

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- Pspec R is a topological quotient of Spec R, via the map $\pi : P \longmapsto$ (largest Poisson ideal $\subseteq P$).

- Assume R is affine over K and satisfies the Poisson Dixmier-Moeglin Equivalence; specifically: all $P \in$ Pprim R are locally closed in Pspec R.

Then Pprim R is a topological quotient of max R, via π.

Conjecture 1: Let $A =$ a generic quantized coordinate ring for an affine variety V, with semiclassical limit $\mathcal{O}(V)$.

Then $\exists$ compatible homeomorphisms $\operatorname{Spec} A \longrightarrow \operatorname{Pspec} \mathcal{O}(V)$ and $\operatorname{Prim} A \longrightarrow \operatorname{Pprim} \mathcal{O}(V)$.

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Known cases:

• Quantum affine spaces and quantum affine toric varieties [Oh-Park-Shin, 2002; Letzter-KG, 2009]

• Quantum symplectic and euclidean spaces [Oh, 2008; Oh-Park, 2002, 2010]

• $\mathcal{O}_q(SL_2(K))$ and $\mathcal{O}_q(GL_2(K))$ [KG, 2010]

• $\mathcal{O}_q(SL_3(K))$ [Fryer, 2017]

Conjecture 2: Let $(A_q)_{q \in K^*}$ = a flat family of quantized coordinate rings for an affine variety V.

Then Spec $A_p \approx$ Spec A_q and Prim $A_p \approx$ Prim A_q $\forall$ non-roots of unity $p, q \in K^*$.

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• Let $\left(A_q\right)_{q\in K^*} = \left(\mathcal{A}/(t-q)\mathcal{A}\right)_{q\in K^*}$ = a flat family of quantized coordinate rings for an affine variety V.

Assume $\mathcal{A}$ is defined over the prime field $K_0 \subset K$, i.e., $\mathcal{A} = \mathcal{A}_0 \otimes_{K_0} K$ for a torsionfree $K_0[t^{\pm 1}]$-algebra $\mathcal{A}_0$.

If $p, q \in K^*$ are transcendental over K_0, then Spec $A_p \approx$ Spec A_q and Prim $A_p \approx$ Prim A_q .

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E.G. $\left(A_q\right)_{q\in K^*} = \left(\mathcal{O}_q(K^2)\right)_{q\in K^*}$.

W.r.t. homeomorphisms $\text{Prim}\, A_p \longrightarrow \text{Prim}\, A_q$ induced by $\widehat{\phi}$,

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Contrast: $\exists$ homeomorphisms $\text{Prim}\, A_p \longrightarrow \text{Prim}\, A_q$ such that

$$\begin{array}{ccccccc} \langle x-\alpha,\, y\rangle & \longmapsto & \langle x-\alpha,\, y\rangle & \qquad & \langle x, y\rangle & \longmapsto & \langle x, y\rangle \\ \langle x,\, y-\beta\rangle & \longmapsto & \langle x,\, y-\beta\rangle & \qquad & \langle 0\rangle & \longmapsto & \langle 0\rangle \end{array}$$

Catenarity in Spec A :

• $\mathcal{O}_{\mathbf{q}}(K^n)$, $\mathcal{O}_q(SL_n(K))$, $\mathcal{O}_q(GL_n(K))$ [Lenagan-KG, 1996]

• quantized Weyl algebras [Lenagan-KG, 1996; Oh, 1997]

• quantum symplectic and euclidean spaces [Oh, 1997; Horton, 2003]

• $\mathcal{O}_q(M_{m,n}(K))$ [Cauchon, 2003]

• quantum semisimple groups [Zhang-KG, 2007; Yakimov, 2014]

• quantum Grassmannians [Launois-Lenagan-Rigal, 2008]

• quantum Schubert cells [Yakimov, 2013]

• quantum nilpotent algebras [Launois-KG, 2020]

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Catenarity in Pspec R :

- Poisson nilpotent algebras [Launois-KG, 2021]
- E.g., semiclassical limits of above A.

Stratification. [Letzter-KG, 2000; Stafford-KG, 2000]

A = a noetherian K-algebra satisfying the NC Nullstellensatz,
H = a torus $(K^*)^r$ acting rationally on A, with $H\text{-Spec}\, A$ finite.

Then $\operatorname{Spec} A = \bigsqcup_{J \in H\text{-Spec}\, A} \operatorname{Spec}_J A$ where

$$\operatorname{Spec}_J A := \{\, P \in \operatorname{Spec} A \mid \bigcap_{h \in H} h(P) = J \,\}$$

and $\operatorname{Spec}_J A \approx \operatorname{Spec} Z_J$ where

Z_J = center of a localization of A/J,

$\cong$ a Laurent polynomial ring over K.

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$\exists$ corresponding partition $\operatorname{Prim} A = \bigsqcup_{J \in H\text{-}\operatorname{Spec} A} \operatorname{Prim}_J A$

and each $\operatorname{Prim}_J A \approx \max Z_J$.

For $J \subset J'$ in $H\text{-}\mathrm{Spec}\, A$, define

$$\phi^{s}_{JJ'} : \mathcal{C}l(\mathrm{Spec}_J\, A) \longrightarrow \mathcal{C}l(\mathrm{Spec}_{J'}\, A), \qquad Y \longmapsto \overline{Y} \cap \mathrm{Spec}_{J'}\, A$$

$$\phi^{p}_{JJ'} : \mathcal{C}l(\mathrm{Prim}_J\, A) \longrightarrow \mathcal{C}l(\mathrm{Prim}_{J'}\, A), \qquad Y \longmapsto \overline{Y} \cap \mathrm{Prim}_{J'}\, A$$

where $\mathcal{C}l(T) = \{\text{closed subsets of } T\}$.

For $J \subset J'$ in H-Spec A, define

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where $\mathcal{C}l(T) = \{\text{closed subsets of } T\}$.

The closed subsets of Spec A are the subsets X such that

- $X \cap \text{Spec}_J A \in \mathcal{C}l(\text{Spec}_J A)$ for all J;
- $\phi^s_{JJ'}(X \cap \text{Spec}_J A) \subseteq X \cap \text{Spec}_{J'} A$ for all $J \subseteq J'$.

For $J \subset J'$ in H-Spec A, define

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Similarly for the closed subsets of Prim A.

Conjecture 3: [Brown-KG, 2015]

For $J \subset J'$ in H-Spec A, $\exists$ an affine variety $V_{JJ'}$ and morphisms

$$\begin{array}{ccc} \mathrm{Prim}_{J'} A & & \\ & \searrow^{f_{JJ'}} & \\ \mathrm{Prim}_J A & \xrightarrow{g_{JJ'}} & V_{JJ'} \end{array}$$

such that $\phi^{p}_{JJ'}(Y) = f^{-1}_{JJ'}(\overline{g_{JJ'}(Y)})$ for $Y \in \mathcal{C}l(\mathrm{Prim}_J A)$.

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such that $\phi^{p}_{JJ'}(Y) = f_{JJ'}^{-1}\left(\overline{g_{JJ'}(Y)}\right)$ for $Y \in \mathcal{C}l(\operatorname{Prim}_J A)$.

Similarly for $\phi^{s}_{JJ'}$, with morphisms of affine schemes.

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Similarly for $\phi^{s}_{JJ'}$, with morphisms of affine schemes.

Known cases:

- $\mathcal{O}_q(M_2(K))$, $\mathcal{O}_q(GL_2(K))$, $\mathcal{O}_q(SL_2(K))$ $\mathcal{O}_q(SL_3(K))$ [Brown-KG, 2015]

- Poisson analog for $\mathcal{O}(SL_3(K))$ [Fryer, 2017]

E.G. Types of auxiliary data for $\operatorname{Prim} \mathcal{O}_q(GL_2(K))$:

$$\begin{array}{ccc} K^* & & \\ & \searrow^{(0,-)} & \\ (K^*)^2 & \xrightarrow{\text{incl}} & K \times K^* \end{array}$$

$$\begin{array}{ccc} (K^*)^2 & & \\ & \searrow^{\text{mult}} & \\ (K^*)^2 & \xrightarrow{\text{pr}_2} & K^* \end{array}$$

$$\begin{array}{ccc} (K^*)^2 & & \\ & \searrow^{\text{mult}} & \\ K^* & \xrightarrow{\text{id}} & K^* \end{array}$$

THANK YOU !

Prim $\mathcal{O}_q(SL_2(K)) \approx$ Pprim $\mathcal{O}(SL_2(K))$: